Generalization of Harris current sheet model for non-relativistic, relativistic and pair plasma

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(Received 11 March 2008)

Abstract. Reconnection is believed to be responsible for plasma acceleration in a large number of space and astrophysical objects. Onset of reconnection is usually related to instabilities of current sheet equilibria. Analytical self-consistent models of an equilibrium current sheet (Harris equilibrium) are known for non-relativistic plasmas and some special cases of relativistic plasmas. We develop a description of generalized Harris equilibria in collisionless plasmas, non-relativistic and relativistic as well. Possible shapes of the magnetic field are analyzed.

1. Introduction

Reconnection is believed to be one of the most powerful mechanisms of plasma energization. It is well-accepted that reconnection is the main factor causing solar flares (Parker, 1957) and magnetospheric substorms (Dungey, 1961). Magnetic reconnection is also important in many high-energy astrophysical objects, like the relativistic jets from active galactic nuclei (Di Matteo, 1998; Lesch and Birk, 1998; Larrabee et al., 2003), and pulsar winds (Coroniti, 1990; Kirk and Skjørraasen, 2003). The reconnection is associated with a disruption of a current sheet between two anti-parallel magnetic fields, either due to external driving or due to internal instabilities. Studies of current sheet instabilities require, as a first step, knowledge of the equilibrium state. Unless such an equilibrium is properly established, it is not possible to separate instability from dynamics of an initially non-steady structure. In particular, a double current sheet equilibrium model has been constructed using a mixture of exact and adiabatic integrals of motion (Sitnov et al., 2003). Since most space plasma environments are collisionless, establishing properties of such an equilibrium (Harris equilibrium (Harris, 1962)) requires kinetic approach, that is, determination of the distribution functions for all species participating in the structure formation, from the system of Maxwell and Vlasov equations. The procedure of deriving the distribution functions is well known (Harris, 1962). However, it is developed only for non-relativistic plasmas. A relativistic Harris equilibrium was derived for the special case of a relativistic Maxwellian distribution (Hoh, 1966) and is the only one which has been used since (Zelenyi and Krasnoselskikh, 1979; Kirk and Skjørraasen, 2003). However, Maxwellian distributions in collisionless plasmas are rather an exception than a rule, and a more general analysis is necessary. In the
present paper we develop generalized Harris equilibria for non-relativistic and relativistic plasmas as well, without limiting ourselves with Maxwellian distributions. We derive the equations for the magnetic field across the current sheet and analyze possible shapes.

2. General equations

We consider a two-dimensional current sheet in $y-z$ plane with all parameters depending only on $x$. A variety of two dimensional non-relativistic current sheet equilibria have been proposed recently Ceccherini et al. (2005); Yoon and Lui (2005) for special shapes of distributions with additional constraints. We defer consideration of such equilibria for elsewhere.

The main component of the magnetic field in our geometry is $B_y(x)$ and the potential electric field $E_x(x)$ depend only upon $x$. There may exist a constant guide field along the current, $B_z = \text{const}$, all other components are zero. This field configuration can be described in terms of the following potentials:

\[ E_x = -\frac{d\phi}{dx}, \]
\[ B_z = \frac{dA_y}{dx}, \quad A_y = B_z x, \]
\[ B_y = -\frac{dA_z}{dx}, \quad A_z = -\int B_y dx \]

In such a geometry Hamiltonian of a charge particle does not depend upon $y, z$ and $t$, therefore the corresponding canonical momenta and energy are integrals of motion:

\[ P_y = p_y + qA_y = \text{const}, \quad (2.4) \]
\[ P_z = p_z + qA_z = \text{const}, \quad (2.5) \]
\[ H = \sqrt{p_x^2 + p_y^2 + p_z^2 + m^2 + q\phi}, \quad (2.6) \]

where the velocity is normalized on the speed of light and so $c \equiv 1$ for convenience.

The equilibrium distribution function should depend only on the integrals of motion:

\[ f = f(P_y, P_z, H) \quad (2.7) \]

The charge and current density can be calculated as:

\[ \rho = \sum q \int f(P_y, P_z, H) dp_x dp_y dp_z, \quad (2.8) \]
\[ j = \sum q \int \frac{P}{H - q\phi} f(P_y, P_z, H) dp_x dp_y dp_z \quad (2.9) \]

where we used

\[ v = \frac{P}{mv\gamma} = \frac{P}{H - q\phi} \quad (2.10) \]

The obtained charge and current densities should be substituted to Maxwell equa-
Generalized Harris equilibria

\[
\frac{d^2 \phi}{dx^2} = -4\pi \rho, \tag{2.11}
\]
\[
\frac{d^2 A}{dx^2} = -4\pi j, \tag{2.12}
\]
in order to find the potentials.

3. Special case: pair plasma without guide field

We start with the analysis of the special case of the two-component pair plasma where \(m_+ = m_- = m\) and \(q_+ = -q_- = q\). For simplicity we shall also assume that the guide field \(B_z = 0\) and \(A_y = 0\). In such a case canonical momentum coincides with ordinary momentum: \(P_y = p_y\). Let us denote \(P_z = qP\), so that \(p_z = q(P - A_z)\).

Pair plasmas allow purely electromagnetic configurations where the electrostatic potential vanishes. The charge density is

\[
\rho = \sum q|q| \int f(p_y, qP, \sqrt{p_z^2 + p_y^2 + q^2(P - A_z)^2}) dp_x p_y dP \tag{3.1}
\]

and vanishes \(\rho = 0\) if

\[
f_+(p_y, qP, \sqrt{p_z^2 + p_y^2 + q^2(P - A_z)^2}) = f_-(p_y, -qP, \sqrt{p_z^2 + p_y^2 + q^2(P - A_z)^2}) \tag{3.2}
\]

With this condition \(j_y = 0\) and

\[
j_z = 2q^3 \int \frac{P - A_z}{H} f dp_z dp_y dP \tag{3.3}
\]

3.1. Nonrelativistic case

Before we start treating the fully relativistic case let us consider the nonrelativistic case where \(H = m + p^2/2m\) so that the current density takes the form

\[
j_z = \frac{2q}{m} \int p_z f(p_y, p_z + qA_z, p_z^2 + p_y^2 + p_z^2) dp_x dp_y dp_z \tag{3.4}
\]

A simple choice of the distribution function would be

\[
f = C \exp(-\beta H + 2kP) \tag{3.5}
\]
\[
= C \exp[-\beta(p_z^2 + p_y^2 + p_z^2)/2m + k(p_z + qA_z)/m] \tag{3.6}
\]
\[
= C \exp[-\beta(p_z^2 + p_y^2)/2m - \beta(p_z - p)^2/2m + V], \tag{3.7}
\]
\[
p = k/\beta, \tag{3.8}
\]
\[
V = p^2/2m = (k/\beta)^2/2m + kqA_z/m \tag{3.9}
\]

We shall normalize the distribution function so that for \(A_z = 0\) the density will be \(n_0\), thus

\[
f = n_0(\beta/2m\pi)^{3/2} \exp[-\beta(p_z^2 + p_y^2)/2m - \beta(p_z - p)^2/2m + kqA_z/m] \tag{3.10}
\]

which is a shifted Maxwellian with the constant shift velocity \(u = p/m = k/\beta m\) and density depending on \(x\):

\[
n(x) = n_0 \exp(kqA_z/m) \tag{3.11}
\]
The current density is easily calculated

\[ j_z = 2qu_n = (2n_0 qk/\beta m) \exp(kqA_z/m) \]  

(3.12)

and by substituting it into (2.12)

\[ \frac{d^2 A_z}{dx^2} = -4\pi (2n_0 qk/\beta m) \exp(kqA_z/m) \]  

(3.13)

Denoting \( A = kqA_z/m, \kappa = 4\pi (2n_0 q^2 k^2/\beta m^2) \), one finally arrives at the equation of the form

\[ \frac{d^2 A}{dx^2} = -\kappa e^A \]  

(3.14)

Integrating this equation once one has

\[ \frac{1}{2} \left( \frac{dA}{dx} \right)^2 + U(A) = E = \text{const}, \]  

(3.15)

\[ U(A) = \kappa e^A \]  

(3.16)

and

\[ \int \frac{dA}{\sqrt{E - U(A)}} = \pm x \]  

(3.17)

Writing \( E = \kappa e^{A_0} \) and switching to \( A - A_0 \) in the integral one immediately finds

The gauge invariance allows to choose \( A_0 = 0 \) and straightforward integration leads to the dependence of vector potential upon \( x \):

\[ A - A_0 = -\ln \cosh(\sqrt{\kappa}x), \]  

(3.18)

which is the standard shape of a simple single current sheet. The field reversal occurs at \( x = 0 \) where the magnetic field passes through zero, \( B_y = -dA_z/dx = 0 \), and changes its sign.

3.2. Relativistic case: an example

In the relativistic case under the same conditions the current density is

\[ j_z = 2q \int \frac{p_z}{\sqrt{p^2 + m^2}} f(p_y, p_z + qA_z, \sqrt{p^2 + m^2}) dp_x dp_y dp_z \]  

(3.19)

As an example let us consider the distribution function of the form

\[ f = C \sqrt{p^2 + m^2} \exp[-\beta p^2 + 2k(p_z + qA_z)] \]  

(3.20)

so that

\[ j_z = (2qNk/\beta)e^{2kqA_z} \]  

(3.21)

The structure of the current is the same as in the previous case so that we arrive at the equation (3.14) with \( A = 2kqA_z \) and \( \kappa = 4\pi (4q^2 Nk^2/\beta) \).

4. Relativistic case: general factored distributions

Remaining with the quasineutral case, \( \phi = 0 \), let us now assume that \( f = F_1(H, p_y)F_2(p_z) \), where \( H = \sqrt{p^2 + m^2} \) and does not contain dependence on \( x \). Let us also seek for the distributions where the dependence on \( x \) can be factored out from the
dependence on \( p \), that is, such that the distribution function could be eventually written as \( f = F_0(p)f_3(x) \). Since \( P_z = p_z + qA_z \), it is clear that this is achieved when \( F_2(P_z) = \exp(kP_z) \). Thus, we try the distribution of the kind \( f_s = F_s(H,p_y)\exp(k_sP_z)\exp(k_sq_sA_z) \), where \( s \) denotes electrons and positrons. It is easy to see that if we choose \( k_s = q_sk \), where \( q = e \) for positrons and \( q = -e \) for electrons, respectively, and \( F_+(H,p_y) = F_-(H,p_y) = F(H,p_y) \), then \( \rho = 0 \), \( j_x = j_y = 0 \) and

\[
\begin{align*}
j_z &= \left[ \int \frac{2e\sinh(ekp_z)p_z}{\sqrt{p^2 + m^2}}F(\sqrt{p^2 + m^2})dp_zdp_ydp_z \right] e^{e^2kA_z} \quad (4.1) \\
&= \left[ \int_1^1 \mu dp \int_0^\infty \frac{p^3\sinh(ekp\mu)}{\sqrt{p^2 + m^2}}F(\sqrt{p^2 + m^2}) \right] e^{e^2kA_z} \quad (4.2) \\
&= Ge^{e^2kA_z} \quad (4.3)
\end{align*}
\]

where \( \mu = \cos \theta \), \( p_z = p \cos \theta \), and \( G \) does not depend on \( x \). As a result we obtain an equation of the same type (3.14) as above with \( A = e^2kA_z \) and \( \kappa = 4\pi Ge^{e^2k} \), and the same shape of solution. The only difference is the scale which depends on \( G \).

5. Guide field

In the previous section we limited ourselves with a simple separable distribution function. We have seen that the choice of the distribution in the form \( f_s = F_1(H,p_y)F_2(sP_z) \), where \( P_s = p_s + sA_z \), ensured that \( \rho = 0 \), \( j_x = j_y = 0 \). Presence of the term \( sA_z \) breaks down the symmetry and results in a nonzero current \( j_z \), while still maintaining zero charge density. The choice of the distribution function ensures that electrons and positrons contribute equally to the current by drifting in the opposite directions of \( z \) with equal speeds. The more specific choice \( F_2(sP_z) = \exp(kSP_s) \) allowed us to factor out the dependence on \( p_z \) and \( x \) (the latter through \( A_z \)).

Being interested in determination of a wide class of distributions suitable for Harris equilibria, let us first discuss the inclusion of a constant guide field \( B_z \). In the presence of this field the distribution functions for electrons and positrons should be \( f_s = F_{s1}(H)F_{s2}(sP_y)F_{s3}(P_z) \), where \( P_{sy} = p_y + sA_y \), \( P_{sz} = p_z + sA_z \). Following the ideas outlined above we shall seek a class of distribution which would maintain \( \rho = 0 \), and \( j_y = 0 \) automatically. It is easy to see that the choice \( f_s = F_1(H)F_2(sP_y)F_3(sP_z) \) preserves \( \rho = 0 \). In this case, however, presence of \( sA_y \) breaks the symmetry along \( y \) axis, and \( j_y \neq 0 \) unless \( F_2 \) = const. Thus, the requirement \( j_y = 0 \) removes the dependence of the distribution function on \( P_y \). It is worth noting that our analysis does not exclude existence of some special distributions satisfying all requirements. However, it precludes symmetric distributions for pair plasma which would depend on \( P_y \).

Since a constant component \( B_z \) is not allowed together with the dependence on \( P_y \), we shall analyze whether a non-trivial variable \( B_z(x) \) can be added, such that the magnetic field vector is allowed to rotate, that is, \( B_z/B_y \neq \text{const} \). As above, we seek to separate the dependence on coordinates from the dependence on momenta,
so that we choose the distribution functions as follows:

\[ f_s = F_1(H)e^{sk_1z_s + sk_2p_y}e^{k_1x_A + k_2x_A^y} \]  

(5.1)

The corresponding currents are

\[ 4\pi j_y = G_1 e^{k_1x_A + k_2x_A^y}, \]  

(5.2)

\[ G_1 = \sum_s \frac{4\pi se_{m}}{m} \int (p_z/H)F_1(H)e^{sk_1z_s + sk_2p_y}dp_ydp_z, \]  

(5.3)

\[ 4\pi j_x = G_2 e^{k_1x_A + k_2x_A^y}, \]  

(5.4)

\[ G_2 = \sum_s \frac{4\pi se_{m}}{m} \int (p_y/H)F_1(H)e^{sk_1z_s + sk_2p_y}dp_ydp_z, \]  

(5.5)

and the equations for the two components of the vector potential are

\[ \frac{d^2}{dx^2}A_x = -G_1 e^{k_1x_A + k_2x_A^y}, \]  

(5.7)

\[ \frac{d^2}{dx^2}A_y = -G_2 e^{k_1x_A + k_2x_A^y}. \]  

(5.8)

A solution including a single current sheet would read

\[ A_x = -\frac{G_1}{K^2} \ln \cosh(Kx) + \frac{ek_2}{K^2}(a + bx), \]  

(5.10)

\[ A_y = -\frac{G_2}{K^2} \ln \cosh(Kx) - \frac{ek_1}{K^2}(a + bx), \]  

(5.11)

\[ K^2 = ek_1G_1 + ek_2G_2, \]  

(5.12)

where \( a \) and \( b \) are arbitrary constants. The magnetic field takes the form

\[ B_y = \frac{G_1}{K} \tanh(Kx) - \frac{ek_2b}{K^2}, \]  

(5.13)

\[ B_z = -\frac{G_2}{K} \tanh(Kx) - \frac{ek_1b}{K^2}, \]  

(5.14)

with \( B_y/B_z \neq \text{const} \). The magnetic field components vanish at different positions:

\[ \tanh(Kx)_y = \frac{ek_2b}{G_1K}, \]  

(5.15)

\[ \tanh(Kx)_z = -\frac{ek_1b}{G_2K}, \]  

(5.16)

while the magnitude of the magnetic field does not vanish anywhere. The maximum of the current (current sheet) resides at \( x = 0 \), between the two partial field reversals. The magnetic field at the current sheet is

\[ B = -\frac{ek_2b}{K^2} \hat{y} - \frac{ek_1b}{K^2} \hat{z} \]  

(5.17)

It is easily seen that

\[ B_1 = \frac{G_2B_y + G_1B_z}{\sqrt{G_1^2 + G_2^2}} = \text{const}, \]  

(5.18)
Generalized Harris equilibria

\[ B_z = \frac{k_1 B_y - k_2 B_z}{\sqrt{k_1^2 + k_2^2}} \propto \tanh(Kx) \] (5.19)

behaves like the magnetic field of a current sheet with only component of the magnetic field present. However, the transformation from \((B_y, B_z)\) to \((B_1, B_2)\) is not orthogonal.

6. Multi-separable distributions

Having established a general shape of symmetric factored distributions for pair plasma, we can now proceed and consider the distributions of the type

\[ f_s = \sum_k F_k(H) \exp(skP_{sz}) = \sum_k F_k(H)e^{skp_z} \exp(ekA_z). \] (6.1)

The only relevant current density component will take the form

\[ j_z = 2e \sum_k G_ke^{kA_z}, \] (6.2)

\[ G_k = 2e \int \left( \frac{p_z}{H} \right) F_k(H) \sinh(kp_z) dp_x dp_y dp_z \] (6.3)

The equation for the vector potential is now

\[ \frac{d^2}{dx^2} A_z = -\sum_k (4\pi eG_k)e^{kA_z}, \] (6.4)

\[ \frac{1}{2} \left( \frac{dA_z}{dx} \right)^2 + U(A_z) = E = \text{const}, \] (6.5)

\[ U(A_z) = \sum_k (4\pi eG_k/k)e^{kA_z} \] (6.6)

It is easy to see that \(kG_k > 0\), so that all terms in the sum are positive. Behavior of the solutions depends on the shape of the pseudo-potential \(U(A_z)\). If, e.g., \(U(-\infty) = 0\) and \(U(\infty) = \infty\) then Harris-like solutions with \(A_z(\pm\infty) = -\infty\) and, respectively, finite \(B_y\) at \(x = \pm\infty\), exist for sufficiently large \(E\). If there is a local maximum of \(U\), solutions with \(B_y\) at one of both spatial infinities are also possible for which \(B_y\) does not change sign. Potentials with \(U(\pm\infty) = \infty\) generate periodically spaced current sheets. This is even more readily seen from

\[ \frac{d}{dx}j_z = -\frac{1}{4\pi} \frac{d^3}{dx^3} A_z = \left[ \sum_k (4\pi eG_k)e^{kA_z} \right] \frac{dA_z}{dx} \] (6.7)

which means that maximum current is achieved at the field reversal \(B_y = -(dA_z/dx) = 0\), which is also a turning point of a trajectory in the pseudo-potential \(U(A_z)\).

7. Electron-ion plasma

When \((q/m)_+ \neq (q/m)_-\) one can no longer assume that \(\phi = 0\). Thus, the dependence on \(x\) now enters also via \(H = \sqrt{p^2 + m^2 + q\phi}\).
7.1. Nonrelativistic case

In order to factor out the dependence on $x$ from dependence on $p$ it is natural to consider the distributions of the kind

$$f = C \exp(-\beta H + kP_z)$$

(7.1)

where the nonrelativistic energy $H = m + p^2 / 2m$. The rest mass can be moved into the constant coefficient, so that eventually the distribution will take the form

$$f = C \exp(-\beta p^2 / 2m + kP_z) \exp(-\beta q\phi + kqA_z)$$

(7.2)

It is clear that $\beta > 0$. In fact, $T = 1 / \beta$ is the temperature. This distribution function can be written as

$$f = n(\beta / 2\pi m)^{3/2} e^{-\beta q\phi + \beta qP_z / m} e^{-\beta q\phi + \beta qA_z}$$

(7.3)

where $p = km / \beta$. It is now straightforward to find

$$\rho = \sum_s q_s n_s e^{-\beta q\phi + \beta qP_z / m_s}$$

(7.4)

$$j_z = \sum_s (q_s n_s p_s / m_s) e^{-\beta q\phi + \beta qP_z / m_s}$$

(7.5)

Respectively, the equations will read

$$\frac{d^2}{dx^2} \phi = -\sum_s 4\pi q_s n_s e^{-\beta q\phi + \beta qP_z / m_s}$$

(7.6)

$$\frac{d^2}{dx^2} A_z = -\sum_s 4\pi (q_s n_s p_s / m_s) e^{-\beta q\phi + \beta qP_z / m_s}$$

(7.7)

There is no chance that we will be able to solve these equations in the general case. Instead, following Harris, we shall seek for a solution where the potential field $E_x = 0$, that is, $\phi = 0$ and the charge density should vanish. This is possible if $\beta q_s p_s / m_s = \alpha$ does not depend on $s$, and $\sum_s q_s n_s = 0$. Substituting $p_s = \alpha m_s / \beta q_s$ one gets

$$\frac{d^2}{dx^2} A_z = -\sum_s 4\pi \alpha n_s / \beta_s e^{\alpha A_z}$$

(7.8)

or

$$\frac{d^2}{dx^2} A = -\kappa e^A, \quad A = \alpha A_z$$

(7.9)

$$\kappa = \sum_s 4\pi \alpha^2 n_s / \beta_s > 0.$$

(7.10)

The solution of this equation is already known. In this frame the ions are moving with the velocity $u_z = p_z / m_i = \alpha / \beta q_i$. Transformation into the ion frame would result in the appearance of the potential $\phi$.

7.2. Nonrelativistic case: generalization

If we are seeking for a solution with $\phi = 0$. We do not have to specify the dependence on the energy, since now $H$ does not depend explicitly on $x$. Coming back to the distributions of the kind

$$f_s = F_s(H) e^{k_s q_s A_z}$$

(7.11)
we only require that \( k_i q_i = k_e q_e = \alpha \). Then

\[
\frac{d^2}{dx^2} \phi = -\left[ \sum_s 4\pi q_s n_s \int F_s(H) e^{k_s p_z} dp_x dp_y dp_z \right] e^{\alpha A_z},
\]

(7.12)

\[
\frac{d^2}{dx^2} A_z = -\left[ \sum_s 4\pi q_s n_s \int p_z F_s(H) e^{k_s p_z} dp_x dp_y dp_z \right] e^{\alpha A_z},
\]

(7.13)

and adding the quasineutrality condition (which does not depend on \( x \) either) that

\[
\sum_s 4\pi q_s n_s \int F_s(H) e^{k_s p_z} dp_x dp_y dp_z = 0
\]

(7.14)

we arrive at the same equation as above

\[
\frac{d^2}{dx^2} A = -\kappa e^A,
\]

(7.15)

\[
\kappa = \sum_s 4\pi \alpha q_s n_s \int p_z F_s(H) e^{k_s p_z} dp_x dp_y dp_z
\]

(7.16)

The difference with the pair plasma is that the charge density does not vanish automatically. Instead, eliminating charge density poses a constraint on the distributions. This constraint is easily fulfilled by adjusting the normalization constants \( n_s \). Since the quasineutrality condition is independent of \( x \), one it is satisfied in the asymptotic region it is satisfied everywhere. In what follows we implicitly assume that the quasineutrality condition is satisfied.

7.3. Relativistic case

The generalization of the above onto the relativistic case is straightforward:

\[
\frac{d^2}{dx^2} \phi = -\left[ \sum_s 4\pi q_s n_s \int F_s(H) e^{k_s p_z} dp_x dp_y dp_z \right] e^{\alpha A_z},
\]

(7.17)

\[
\frac{d^2}{dx^2} A_z = -\left[ \sum_s 4\pi q_s n_s \int \frac{p_z}{H} F_s(H) e^{k_s p_z} dp_x dp_y dp_z \right] e^{\alpha A_z},
\]

(7.18)

\[
\sum_s 4\pi q_s n_s \int F_s(H) e^{k_s p_z} dp_x dp_y dp_z = 0.
\]

(7.19)

with \( k_s = \alpha/q_s \) and \( H = (p^2 + n_s^2)^{1/2} \). Eventually we get again (3.14) with \( A = \alpha A_z \) and

\[
\kappa = \sum_s 4\pi \alpha q_s n_s \int (p_z/H) F_s(H) e^{k_s p_z} dp_x dp_y dp_z
\]

(7.20)

(7.21)

Let us consider \( \kappa \) separately. Since \( H = H(p^2) \) it is convenient to write the integral in the form

\[
\kappa = \sum_s 8\pi^2 \alpha q_s n_s \int_0^{\infty} \frac{p^3 dp}{H} F_s(H) \int_{-1}^{1} \mu e^{k_s \mu p} d\mu
\]

(7.22)
where \( p_z = pm = p \cos \theta \). Further transformations give

\[
\kappa = \sum_s 8\pi^2 n_s \int_0^\infty \frac{p^3 dp}{H} F_s(H) \frac{d}{dp} \left[ \int_{-1}^1 e^{k_s p} dp \right]
\]

(7.23)

\[
= \sum_s 16\pi^2 n_s \int_0^\infty \frac{p^3 dp}{H} F_s(H) \frac{d}{dp} \frac{\sinh(k_s p)}{k_s p}
\]

(7.24)

\[
= \sum_s 16\pi^2 n_s \int_0^\infty \frac{p^2 dp}{H} F_s(H) \left[ \cosh(k_s p) - \frac{\sinh(k_s p)}{k_s p} \right]
\]

(7.25)

\[
= \sum_s 16\pi^2 n_s \int_0^\infty \frac{p^2 dp}{H} F_s(H) \cosh(k_s p) \left[ 1 - \frac{\tanh(k_s p)}{k_s p} \right]
\]

(7.26)

This expression shows that \( \kappa > 0 \), which is necessary to ensure that the density does not diverge far from the sheet.

7.4. Two field components

Generalization of the equilibria with two field components, \( B_y \) and \( B_z \), on electron-ion plasmas is straightforward. We choose the distributions in the form

\[
f_s = F_s(H) \exp[\alpha_1 P_z/q_s + \alpha_2 P_y/q_s]
\]

(7.27)

\[
= F_s(H) \exp[(\alpha_1 P_z + \alpha_2 P_y)/q_s] \exp(\alpha_1 A_z + \alpha_2 A_y)
\]

(7.28)

As a result, \( j_y, j_z \propto \exp(\alpha_1 A_z + \alpha_2 A_y) \), exactly as in the case of a pair plasma. Respectively, the solutions are the same, with the parameters depending on the distributions of electrons and ions.

7.5. Multi-separable distributions

Relativistic generalization of multi-separable distributions onto electron-ion plasmas is straightforward. Namely, we assume the distributions in the form similar to (6.1) (for simplicity single-charged ions are considered, so that \( q_s = se \)):

\[
f_s = \sum_k F_{sk}(H) e^{ksp_z} = \sum_k F_{sk}(H) e^{ksp_z} e^{kA_z}.
\]

(7.29)

Dependence on \( P_y \) is absent to ensure \( y_y = 0 \). The charge and current density read

\[
\rho = \sum_k \sum_s se \int F_{sk}(H)e^{ksp_z} dp_z dp_y dp_z e^{kA_z},
\]

(7.30)

\[
j_z = \sum_k \sum_s se \int (p_z/H) F_{sk}(H)e^{ksp_z} dp_z dp_y dp_z e^{kA_z}.
\]

(7.31)

Requiring quasi-neutrality one gets

\[
\forall k : \sum_s se \int F_k(H)e^{ksp_z} dp_z dp_y dp_z = 0,
\]

(7.32)

while for the vector potential one has

\[
\frac{d^2}{dx^2} A_z = - \sum_k G_k e^{kA_z},
\]

(7.33)

\[
G_k = \sum_s 4\pi se \int (p_z/H) F_{sk}(H)e^{ksp_z} dp_z dp_y dp_z.
\]

(7.34)
It is easy to verify that \( G_k/k > 0 \), and the equation can be integrated once to

\[
\frac{1}{2} \left( \frac{d}{dx}A \right)^2 + \Psi(A) = E = \text{const}, \tag{7.35}
\]

\[
\Psi(A) = \sum_k (eG_k/k)e^{kA}, \quad A = eA_z \tag{7.36}
\]

The pseudo-potential \( \Psi(A) \) determines the shapes of the solutions.

### 8. Applications

#### 8.1. Spatially periodic system of current sheets

Let us consider a pseudo-potential of the form \( \Psi(A) = b_1e^{k_1A} + b_2e^{-k_2A} \), where \( k_1, k_2, b_1, \) and \( b_2 \) are positive. Then \( \Psi(A \rightarrow \infty) \rightarrow \infty \) and has a minimum at \( A = \ln(k_2b_2/k_1b_1)/(k_1 + k_2) \). For \( E > \Psi(A_c) \) the pseudopotential has two turning points (field reversals), therefore, the solution is a periodic function of \( x \). For a more quantitative analysis let \( b_2 = b_1 \) and \( k_2 = k_1 \) (pair plasma), so that

\[
\frac{dA}{dx} = \pm c\sqrt{\cos A_0 - \cosh A}, \tag{8.1}
\]

where \( c \) and \( A_0 \) are easily expressed in terms of the original constants. The solution can be expressed in terms of elliptic functions and represents a periodic sequence of current sheets. For large \( A \) the ratio of the current sheet width \( d \) to the distance \( l \) between the sheets is \( d/l \sim A/e^A \ll 1 \).

#### 8.2. Non-Maxwellian relativistic current sheet

The relativistic Maxwellian distribution is given by \( f \propto \exp(-\beta H + kp_\perp) \). Here \( |k| < \beta \) is required to ensure that \( f \rightarrow 0 \) when \( p \rightarrow \infty \). We shall consider a non-Maxwellian distribution of the form

\[
f = Ce^{-p^2/2p_\perp^2 + kp_\perp} \Rightarrow f = Ce^{-p^2/2p_\perp^2 + kp_\perp} \tag{8.2}
\]

where \( k \) is no longer limited. The normalization conditions \( \int f d^3p = n \) reads

\[
\frac{4\pi C}{k} \int_0^\infty pe^{-p^2/2p_\perp^2} \sinh(kp)dp = n, \tag{8.3}
\]

while the expression for \( G_k \) takes the form

\[
G_k = \sum_s 16\pi^2C_s \int_0^\infty \frac{p^2dp}{p^2 + m^2} e^{-p^2/2p_\perp^2} \cosh(skp) \left[ 1 - \frac{\tanh(skp)}{skp} \right] \tag{8.4}
\]

For simplicity, let us consider a pair plasma. We shall derive the asymptotic expressions for \( kp_0 \ll 1 \) and \( kp \gg 1 \).

**Case** \( kp_0 \ll 1 \). Expanding to the lowest nonzero order, one has

\[
G_k = (32\pi^2Ch^2/3) \int_0^\infty \frac{p^4dp}{p^2 + m^2} e^{-p^2/2p_\perp^2} \sim (32\pi^2Ch^2/3)p_0^4 \tag{8.5}
\]

if \( p_0 \gg m \). The normalization constant is obtained from

\[
4\pi C \int_0^\infty p^2e^{-p^2/2p_\perp^2}dp = n \rightarrow C = n(2\pi)^{-3/2}p_0^{-3} \tag{8.6}
\]
12 Balikhin and Gedalin

Case $kp_0 \gg 1$. One has

$$G_k = 16\pi^2 C \int_0^\infty \frac{p^2 dp}{p^2 + m^2} e^{-p^2/2p_0^2 + kp} \sim 16\pi^{5/2} p_0 C$$

(8.7)

while

$$\frac{2\pi C}{k} \int_0^\infty p e^{-p^2/2p_0^2 + kp} dp = n \rightarrow C = n (2\pi)^{-1/2} p_0^{-3}$$

(8.8)

8.3. Power-law current sheet

Often distributions of relativistic particles have a power-law shape, $F_s(H) = CH^{\alpha}(p_0^2 - p^2)$, where $\theta(x)$ is the step-function. In this case it is easier to use the expression (7.20):

$$\kappa \propto \int p z (p^2 + m^2)^{-\alpha/2} e^{kp_0}(p_0^2 - p^2) dp_x dp_y dp_z$$

(8.9)

$$= \pi \int_{-p_0}^{p_0} p z e^{kp_0} dp_z \int_0^{p_0^2} (p^2 x + m^2)^{-\alpha/2} dx$$

(8.10)

$$= \frac{2\pi}{\alpha} \int_{-p_0}^{p_0} p z e^{kp_0} dp_z [p_z^2 + m^2]^{-\alpha/2} - (p_0^2 + m^2)^{-\alpha/2}]$$

(8.11)

If $p_0 \gg m$ and $kp_0 \gg 1$, the major contribution into the integral comes from $p_z > 0$ and $p_z \gg m$, so that we can estimate

$$\kappa \propto e^{kp_0} p_0^{2-\alpha}$$

(8.12)

Thus, for a wide class of power-law ultrarelativistic distributions with a high-energy cutoff the width of the current sheet is determined by the cutoff particles.

9. General properties

Having established that presence of a constant guide field $B_z$ requires that the distribution functions be independent of $P_y$ and assuming quasineutrality, so that $\phi = 0$, we write the distributions in the form $f = f(H, p_z/q + A_z)$, where $H = \sqrt{p^2 + m^2}$. Let $f = \partial F(H, p_z/q + A_z)/\partial A_z$, then

$$j_i = \sum q \int (p_i/H) f dp_x dp_y dp_z$$

(9.1)

$$= \frac{d}{dA_z} \left[ \sum q \int (p_i/H) F dp_x dp_y dp_z \right] = \frac{d}{dA_z} \Psi_i(A_z)$$

(9.2)

where

$$\Psi_i(A_z) = \sum q \int (p_i/H) F(H, p_z/q + A_z) dp_x dp_y dp_z$$

(9.3)

Because of the symmetry $p_x \leftrightarrow -p_x, p_y \leftrightarrow -p_y$ one has $j_x = j_y = 0$. Quasineutrality requires

$$\frac{d}{dA_z} \left[ \sum q \int F(H, p_z/q + A_z) dp_x dp_y dp_z \right] = 0,$$

(9.4)
then eventually
\[
\frac{d^2 A_z}{dx^2} = -4\pi \frac{d}{dA_z} \Psi_z, \tag{9.5}
\]
\[
\Psi_z(A_z) = \sum q \int \frac{p_z}{H} F(H, p_z/q + A_z) dp_z dp_y dp_z. \tag{9.6}
\]
The first integral of the derived equation is
\[
\frac{1}{2} \left( \frac{dA_z}{dx} \right)^2 + 4\pi \Psi_z(A_z) = E = \text{const} \tag{9.7}
\]
Thus, the magnetic profile is determined by the energy level in the pseudo-potential \( \Psi \).

Equation (9.7) is the general equation for a generalized current sheet without a potential but including a constant guide field. Therefore, possible magnetic field profiles are determined by the shape of the pseudo-potential \( \Psi(A) \). From the theory of orbits in a one-dimensional potential we know that there are orbits a) without a turning point, where \( dA/dx = 0 \), that is, without a current sheet with vanishing magnetic field, b) with one turning point, that is, a single current sheet, and c) with two turning points, which corresponds to a spatially periodic system of current sheets. A particular solution with a current sheet and zero asymptotic \( B_z \) would be possible if \( A_z \) had a maximum. It is not clear how such shape of the potential can be achieved if at all. There is no a double field reversal solution in this one-dimensional electrostatic potential free configuration, since infinite trajectories with two turning points do not exist. Apparently, double field reversal solutions require charge-separation, a guide field of a more general form, or second dimension (cf. Schindler and Birn, 2002; Sitnov et al., 2003), or even are not stationary.

With a less conservative definition of a current sheet as a local maximum of the current \( j_z(x) \) it is easy to see that it corresponds to the points where
\[
\frac{dj_z}{dx} = \frac{dA_z}{dx} \frac{dA_z}{dx} = 0 \tag{9.8}
\]
On a semi-infinite trajectory, corresponding to a non-periodic structure, the point \( dA_z/dx = 0 \) appears only once (being a single turning point on this trajectory), while each point with \( dj_z/dA_z = 0 \) is crossed twice. Thus, even number of current sheets is impossible.

10. General integrable equilibria with two variable magnetic field components

Here we consider more general distributions of the kind \( f(H, Q) \), where \( Q = a_1(p_z/q + A_z) + a_2(p_y/q + A_y) \). Let \( f = \partial F/\partial A \), where \( A = a_1 A_z + a_2 A_y \), then
\[
4\pi j_z = \sum_s 4\pi n_s q_s \int (p_z/H) f_s(H, a_1 p_z/q + a_2 p_y/q + A) dp_z dp_y dp_z \tag{10.1}
\]
\[
= \frac{d}{dA} \Psi_s(A) \tag{10.2}
\]
\[
\Psi_s(A) = \sum_s 4\pi n_s q_s \int (p_z/H) F_s(H, a_1 p_z/q + a_2 p_y/q + A) dp_z dp_y dp_z. \tag{10.3}
\]
Now the equations for the vector potential read

\[
\frac{d^2}{dx^2} A_i = -\frac{d}{dA} \Psi_i(A) \quad (10.4)
\]

and immediately

\[
\frac{d^2}{dx^2} A = -\frac{d}{dA} \Psi(A), \quad (10.5)
\]

\[
\Psi(A) = (a_1 \Psi_z(A) + a_2 \Psi_y(A)) \quad (10.6)
\]

Eq. (10.5) is immediately integrable using the pseudo-potential method. Once \( A(x) \) is obtained both \( A_i \) can be found by direct integration

\[
A_i = -\int dy \int dz \frac{d}{dA} \Psi_i(A(z)) + C_i x + D_i. \quad (10.7)
\]

The solution should satisfy the condition \( a_1 A_z(x) + a_2 A_y(x) = A(x) \) which, in principle, may place a severe constraint.

11. Summary

In summary, the generalization of Harris equilibrium for cases of non-Maxwellian non-relativistic, relativistic electron-ion, and relativistic pair plasma has been developed. It is shown that for a wide class of separable distributions the shape of the current sheet is not sensitive to the details and is similar to the simple Harris equilibrium. A spatially periodic pattern of current sheets is constructed, as well as a current sheet with two components of a rotating magnetic field. It is shown that an one-dimensional double field reversal without charge separation or a variable guide field cannot exist. An example of ultra-relativistic current sheet with power-law distribution is given. The developed formalism provides the basis for further analysis of the stability of current sheets with non-Maxwellian and relativistic plasmas.

Acknowledgements

The authors acknowledge support by Royal Society. M. Gedalin was partially supported by BSF Grant no. 2006095 and ISF Grant no. 275/07.

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Generalized Harris equilibria


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