

# 1 Arithmetic calculations (calculator is not allowed)

## 1.1 Check the result

Problem 1.1.

$$78\frac{5}{6} + 24\frac{3}{4} - 99\frac{1}{3} = 4\frac{1}{4}$$

Problem 1.2.

$$\frac{3}{4} : \frac{5}{6} + 2\frac{1}{2} \cdot \frac{2}{5} - 1 : 1\frac{1}{9} = 1$$

Problem 1.3.

$$\left(\frac{2}{15} + 1\frac{7}{12}\right) \cdot \frac{30}{103} - \left(2 : 2\frac{1}{4}\right) \cdot \frac{9}{32} = \frac{1}{4}$$

Problem 1.4.

$$\frac{\left(5\frac{4}{45} - 4\frac{1}{15}\right) \cdot 30}{1\frac{1}{3}} - \frac{4.25 : 0.85 + 1 : 0.5}{(5.56 - 4.06) : 3} = 9$$

Problem 1.5.

$$\frac{(82.15 - 5.7) \cdot 0.05}{2.23 - 1\frac{49}{50}} + \left(0.81 + \frac{1}{2}\right)\left(0.81 - \frac{1}{2}\right) = 15.6961$$

Problem 1.6.

$$\frac{(2.4 + 1\frac{1}{2}) \cdot 2.5 + \left(5\frac{1}{12} : 6 - 1\frac{1}{72}\right) : \left(8\frac{5}{7} - 1\frac{5}{21}\right)}{54.75 - 4.5 : 0.1} = 1$$

Problem 1.7.

$$\left(\frac{(11 - 9\frac{1}{2})/0.003}{(4.05 - 3\frac{13}{20}) \cdot 20} - \frac{0.45 - \frac{9}{40}}{13\frac{5}{8} : \left(2\frac{3}{5} + \frac{1}{8}\right)}\right) : 62\frac{91}{200} = 1$$

Problem 1.8.

$$\left(\sqrt{3 - \sqrt{5}} - \sqrt{3 + \sqrt{5}}\right)^2 = 2$$

Problem 1.9.

$$(12\sqrt[3]{3} + \sqrt[3]{16} - 3\sqrt[3]{2})(5\sqrt[3]{4} - 2\sqrt{\frac{1}{2}}) = 84$$

## 1.2 Calculate

**Problem 1.10.**

$$\left( \frac{(1\frac{1}{4} : 3\frac{7}{12}) \cdot 5\frac{1}{60} - 3\frac{13}{15} : \frac{42}{45} + (6\frac{53}{56} - 2.375)}{5.225 - \frac{5}{9} - 3\frac{5}{6}} - \frac{3\frac{13}{15} : \frac{42}{45} + (6\frac{53}{56} - 2.375)}{2.25 + 0.25 \cdot 8\frac{3}{7}} \right)$$

**Problem 1.11.**

$$\left( \frac{928 \cdot \frac{1}{100} - 0.6}{0.8} \right) \cdot \left( \frac{(42 \cdot 3\frac{5}{6} - 3.3 : 0.003) : \frac{1}{15}}{(3\frac{3}{4} : 0.625 - 0.84/0.8) : 0.03} \right)$$

**Problem 1.12.**

$$\frac{99}{4}(10 : 2.6 + 7.5 : 10)\left(\frac{3}{40} - 0.23 \cdot \frac{1}{4} + \frac{157}{360}\right) \cdot (0.17 - 0.13)$$

**Problem 1.13.**

$$\frac{((58.26 - 56\frac{7}{24}) : \frac{4}{5} + 2.1 \cdot 0.225) \cdot 1.6}{\frac{125000}{125403} \cdot 8.75 \cdot (1.001001 + 0.013 \cdot 0.171)}$$

## 2 Algebraic calculations

### 2.1 Factorize

EXAMPLE 2.1. [ ]



$$2x - x^3 = x(\sqrt{2} - x)(\sqrt{2} + x)$$

**Problem 2.1.**

$$5a^3 + 20a^2b + 20ab^2$$

**Problem 2.2.**

$$a^2(x - 1) - b^2(x - 1)$$

**Problem 2.3.**

$$(x + y)^2 - 4xy$$

**Problem 2.4.**

$$(a + b)^3 - a(a + b)^2$$

**Problem 2.5.**

$$x^3 + 3x^2 - 9x - 27$$

**Problem 2.6.**

$$a^5 - a^3 - a^2 + 1$$

**Problem 2.7.**

$$x^n - x^n y^2$$

**Problem 2.8.**

$$a^{n+1} + a^n$$

**Problem 2.9.**

$$\frac{144}{169}a^{12} - 0.0004b^{14}$$

**Problem 2.10.**

$$\frac{1}{4}a^2b^4 + \frac{3}{4}ab^2c^3d^4 + \frac{9}{16}c^6d^8$$

**Problem 2.11.**

$$2a - a^2 - a^3$$

**Problem 2.12.**

$$(n^2 - 2nx)^2 + 2(n^2x - 2nx^2) + x^2$$

**Problem 2.13.**

$$a^2 - n^2 + 2np - p^2$$

**Problem 2.14.**

$$xy - xz - (y^2 - 2yz + z^2)$$

**Problem 2.15.**

$$2a^2 - a^2n + (n - 2)(an - a)^2$$

**Problem 2.16.**

$$27c^4 + 54c^6 + 36c^8 + 8c^{10}$$

**Problem 2.17.**

$$x^{6n} + 12x^{4n} + 48x^{2n} + 64$$

**Problem 2.18.**

$$x^8 - 2x^7 + x^6 - x^5 + 2x^4 - x^3$$

**Problem 2.19.**

$$(4a + 3b)^2 - 16(a - b)^2$$

**Problem 2.20.**

$$xyz + x^2y^2 + 3x^4y^5 + 3x^3y^4z - xy - z$$

**Problem 2.21.**

$$x^3 + x^2 + 4$$

**Problem 2.22.**

$$x^4 - 4$$

**Problem 2.23.**

$$(x + y + z)^3 - x^3 - y^3 - z^3$$

**Problem 2.24.**

$$(a - b)^3 + (b - c)^3 + (c - a)^3$$

**Problem 2.25.**

$$x^3 + 8x^2 + 17x + 10$$

**Problem 2.26.**

$$x^4 - 12x^3 + 47x^2 - 60x$$

**Problem 2.27.**

$$(ax - by)^2 + (bx + ay)^2$$

**Problem 2.28.**

$$xyz^2 - (x - y)x^3 - z^4$$

**Problem 2.29.**

$$(x + 1)^4 - 1$$

**Problem 2.30.**

$$x^4 + 1$$

**Problem 2.31.**

$$(n - x)(5n^2 - 4x^2) - (3x^2 - 4n^2)(x - n)$$

**Problem 2.32.**

$$3x^{11} - 48x^3 - 6x^9 + 96x$$

**Problem 2.33.**

$$a^6 - a^6z^4 + 3a^4z^2 - a^4z^6 + 3a^2z^4 + z^6$$

**Problem 2.34.**

$$x^3 + (a - 1)x + a$$

**Problem 2.35.**

$$x^6 + 1$$

**Problem 2.36.**

$$x^4 + x^2 + 1$$

**Problem 2.37.**

$$(a + 1)^4 + 1$$

**Problem 2.38.**

$$(a + n)^6 + (a - n)^6$$

**Problem 2.39.**

$$x^4 + x^2 + \sqrt{2}x + 2$$

## 2.2 Simplify

**Problem 2.40.**

$$\frac{ab}{bc - c^2 + ac - ab} + \frac{ac}{ab - ac - b^2 + bc} + \frac{bc}{ab + ac - bc - a^2}$$

**Problem 2.41.**

$$\frac{x}{x^2 - 1} + \frac{x^2 + x - 1}{x^3 - x^2 + x - 1} + \frac{x^2 - x - 1}{x^3 + x^2 + x + 1} - \frac{2x^3}{x^4 - 1}$$

**Problem 2.42.**

$$\frac{n^2 + nx + x^2}{n^3 + x^3} : \frac{n^3 - x^3}{n^2 - nx + x^2}$$

**Problem 2.43.**

$$\left(\frac{a}{a-1} + 1\right) : \left(1 - \frac{3a^2}{1-a^2}\right)$$

**Problem 2.44.**

$$\frac{3}{5x} - \frac{3}{x+y} \cdot \left(\frac{x+y}{5x} - x - y\right)$$

**Problem 2.45.**

$$\left(a + n^2 - 3n - \frac{n^2(3n+a)}{2a}\right) : \left(\frac{1}{2}a^2 + 4.5n^2 - 3an\right)$$

**Problem 2.46.**

$$\left(\frac{1+x}{1-x} - \frac{1-x}{1+x} + \frac{4x^2}{x^2-1}\right) : \left(\frac{1}{x^3+x^2} - \frac{1-x}{x^2} - 1\right)$$

**Problem 2.47.**

$$\left(\frac{2x^2 - 5nx + 2n^2}{n - 2x} - (2n + x)\right) \cdot \left(\frac{n-x}{4x^{n+1}}\right)$$

**Problem 2.48.**

$$\left(\frac{x^4}{x+1} - \frac{1}{x^4+x^5}\right) : \left(x^3 + x + \frac{1}{x} + \frac{1}{x^3}\right)$$

**Problem 2.49.**

$$\left(\frac{4x^n - 4}{x^{2n} - 2} + \frac{x^{2n} - 2}{x^n + 1}\right) : \left(\frac{3x^{2n-2} - 12x^{n-4}}{x^{3n} + 5x^{2n} - 2x^n - 10} \cdot \frac{x^{4n+1} + 5x^{3n+1}}{x^{n+2} - 4}\right)$$

**Problem 2.50.**

$$\left(x^n - 1 - \frac{7-x^n}{3+x^n}\right) : \left(\frac{6x^{2n} - 24}{x^{2n+3} + 6x^{n+3} + 9x^3} \cdot \frac{2x}{3x^n + 6} : \frac{4}{x^{n+2} + 3x^2}\right)$$

**Problem 2.51.**

$$\frac{\left(\frac{(a+x)^2}{ax} - 4\right)\left(\frac{(a-x)^2}{ax} + 4\right) : (a^6 - x^6)}{(a^2x - ax^2) : (((a+x)^2 - ax)((a-x)^2 + ax))} \cdot \frac{a - \frac{ax}{a+x}}{a - \frac{ax}{a-x}}$$

**Problem 2.52.**

$$\left(\frac{a+2b}{ab^4} - \frac{2a+b}{a^4b}\right) : \left(\frac{b^2+c^2}{b^2c^2}\left(\frac{1}{b^2} - \frac{1}{c^2}\right) - \left(\frac{1}{a^2} - \frac{1}{c^2}\right)\frac{a^2+c^2}{a^2c^2}\right)$$

**Problem 2.53.**

$$(\sqrt{8x^2y} - 2y\sqrt{x} - x\sqrt{x}) : (\sqrt{2y} - \sqrt{x})$$

**Problem 2.54.**

$$(\sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2})(\sqrt[3]{a} + \sqrt[3]{b})$$

**Problem 2.55.**

$$\sqrt{\frac{a+x}{a-x}} + \sqrt{\frac{a-x}{a+x}} - \sqrt{\frac{4a^2}{a^2-x^2}}$$

**Problem 2.56.**

$$(\sqrt{m} + \sqrt{n})(\sqrt[4]{m} - \sqrt[4]{n})(\sqrt[4]{m} + \sqrt[4]{n})$$

**Problem 2.57.**

$$\frac{\sqrt{12} - \sqrt{6}}{\sqrt{30} - \sqrt{15}} + \frac{5 + \sqrt{10}}{2 + \sqrt{10}}$$

**Problem 2.58.**

$$\sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}$$

**Problem 2.59.**

$$\sqrt[3]{6\sqrt{3} + 10} - \sqrt[3]{6\sqrt{3} - 10}$$

**Problem 2.60.**

$$\frac{1}{7 + 4\sqrt{3}} + \frac{1}{7 - 4\sqrt{3}}$$

**Problem 2.61.**

$$\frac{\sqrt{x+y} + \sqrt{x-y}}{\sqrt{x+y} - \sqrt{x-y}}$$

**Problem 2.62.**

$$\frac{\sqrt{\sqrt{a} - \sqrt{b}}}{\sqrt{\sqrt{a} + \sqrt{b}}}$$

**Problem 2.63.**

$$\frac{14}{\sqrt[4]{3} + \sqrt[8]{2}}$$

**Problem 2.64.**

$$\sqrt[3]{\frac{2}{3}\sqrt[3]{\frac{3}{2}}\sqrt{\frac{2}{3}}}$$



Problem 2.65.

$$\frac{\sqrt{x\sqrt{x\sqrt{x\sqrt{x}}}}}{x^{11/16}}$$

Problem 2.66.

$$\frac{1}{(ab)^{mn}} \sqrt[m+n]{a^{2m+n}b^{m+2n} - a^{m+2n}b^{2m+n}}$$

Problem 2.67.

$$\sqrt{\frac{x}{y}\sqrt{\frac{y}{x}\sqrt[3]{\frac{x}{y}}}}x^{-1/3}y^{1/3}$$

## 3 Equations

### 3.1 Solve sets

Problem 3.1.

$$\begin{cases} 7x - 3y = 15 \\ 5x + 6y = 27 \end{cases}$$

Problem 3.2.

$$\begin{cases} 25x - 4y + 1 = 0 \\ 31x - 5y + 16 = 0 \end{cases}$$

Problem 3.3.

$$\begin{cases} ax - 3y = 4 \\ x - y = \frac{4}{3} \end{cases}$$

Problem 3.4.

$$\begin{cases} 2x + ay = 8 \\ 3x - 5y = 6 \end{cases}$$

Problem 3.5.

$$\begin{cases} x - y = 2 \\ 2x - 2y = a \end{cases}$$

**Problem 3.6.**

$$\begin{cases} x + y = 1 \\ y = \sqrt{2}(x + 1) \end{cases}$$

**Problem 3.7.**

$$\begin{cases} 2\sqrt{3}(1 + y) = -(\sqrt{3} + x) \\ \sqrt{3}(y + 1) = x \end{cases}$$

**Problem 3.8.**

$$\begin{cases} \frac{2\sqrt{2}}{x+y\sqrt{2}} - \frac{1}{x-y\sqrt{2}} = 1 \\ \frac{10\sqrt{2}}{x+y\sqrt{2}} + \frac{3}{x-y\sqrt{2}} = 1 \end{cases}$$

**Problem 3.9.**

$$\begin{cases} (x - 2)(y + 6) = xy + 13 \\ (y - 2)(x + 4) = xy - 13 \end{cases}$$

**Problem 3.10.**

$$\begin{cases} \frac{1}{x+y-1} + \frac{1}{x-y+1} = 1 \\ \frac{5}{x+y-1} - \frac{3}{x-y+1} = 1 \end{cases}$$

**Problem 3.11.**

$$\begin{cases} x^2 - y^2 = 3(x + y) \\ x + y = 4(x - y) + 1 \end{cases}$$

**Problem 3.12.**

$$\begin{cases} x + y + z = 6 \\ x + y - z = 10 \\ x - y + z = 0 \end{cases}$$

**Problem 3.13.**

$$\begin{cases} ax + by + cz = d \\ a^2x + b^2y + c^2z = d^2 \\ a^3x + b^3y + c^3z = d^3 \end{cases}$$

**Problem 3.14.**

$$\left\{ \begin{array}{l} \frac{a-b}{x} + \frac{b-c}{y} + \frac{c-a}{z} = a + b + c \\ \frac{a^2-b^2}{x} + \frac{b^2-c^2}{y} + \frac{c^2-a^2}{z} = 2(ab + ac + bc) \\ \frac{(a-b)^2}{x} + \frac{(b-c)^2}{y} - \frac{(a-c)^2}{z} = 2(a-c)b \end{array} \right.$$

## 3.2 Solve equations

**Problem 3.15.**

$$\frac{(2x+1)^2 - 8x}{4x^2 - 1} = \frac{3(2x-1)}{7(2x+1)}$$

**Problem 3.16.**

$$\frac{x}{x^2 - 9} = \frac{1}{x^2 + 3x} - \frac{3}{6x + 2x^2}$$

**Problem 3.17.**

$$\frac{2x-1}{14x^2+7x} + \frac{8}{12x^2-3} = \frac{2x+1}{6x^2-3x}$$

**Problem 3.18.**

$$\frac{x-3}{x+1} = \frac{8-x}{x-4} + \frac{20}{4+3x-x^2}$$

**Problem 3.19.**

$$\frac{3}{x^2-9} - \frac{1}{9-6x+x^2} = \frac{3}{2x^2+6x}$$

**Problem 3.20.**

$$\frac{x^3+3x^2+5x+3}{x^2+3x+2} = \frac{2x^2+x+3}{2x+1}$$

**Problem 3.21.**

$$\frac{25x-21}{2x^2+5x-12} + \frac{2x-3}{x+4} = \frac{x+4}{2x-3}$$

**Problem 3.22.**

$$3x^2 - 5ax - 2a^2 = 0$$

**Problem 3.23.**

$$x^2 - 2nx + 4n - 4 = 0$$

**Problem 3.24.**

$$x + \frac{1}{x} = 2\frac{a^2 + b^2}{a^2 - b^2}$$

**Problem 3.25.**

$$\frac{x - a}{x - b} + \frac{x - b}{x - a} = 2.5$$

**Problem 3.26.**

$$\frac{ax^2}{x - 1} = (a + 1)^2$$

**Problem 3.27.**

$$\frac{1}{x - a} + \frac{1}{x - b} = \frac{1}{a} + \frac{1}{b}$$

**Problem 3.28.**

$$\frac{(x - a)^2 + x(x - a) + x^2}{(x - a)^2 - x(x - a) + x^2} = \frac{19}{7}$$

**Problem 3.29.**

$$\frac{1}{x} + \frac{1}{a} + \frac{1}{b} = \frac{1}{x + a + b}$$

**Problem 3.30.**

$$\frac{x}{\sqrt{3} + \sqrt{2}} + \frac{2\sqrt{3} - x}{\sqrt{3} - 2} - \frac{\sqrt{3} + \sqrt{2}}{x} = 1$$

**Problem 3.31.**

$$4x^4 - 5x^2 + 1 = 0$$

**Problem 3.32.**

$$4x^4 + a^2 = x^2 + 4a^2x^2$$

**Problem 3.33.**

$$m^2n^2x^4 - (m^4 + n^4)x^2 + m^2n^2 = 0$$

**Problem 3.34.**

$$x^4 - 2(a^2 + b^2)x^2 + (a^2 - b^2)^2 = 0$$

**Problem 3.35.**

$$(x^2 - 8)^2 + 4(x^2 - 8) - 5 = 0$$

**Problem 3.36.**

$$\left(x + \frac{1}{x}\right)^2 - 4.5\left(x + \frac{1}{x}\right) + 5 = 0$$

**Problem 3.37.**

$$\frac{1}{x^2 + 2x - 3} + \frac{18}{x^2 + 2x + 2} = \frac{18}{x^2 + 2x + 1}$$

**Problem 3.38.**

$$(x + 3)^3 - (x + 1)^3 = 56$$

**Problem 3.39.**

$$x^4 = 2(2 + \sqrt{3})x^2 - 8\sqrt{3}$$

**Problem 3.40.**

$$(x^2 - 6x)^2 - 2(x - 3)^2 = 81$$

**Problem 3.41.**

$$(x + 5)^4 = 13(x + 5)^2 x^2 - 36x^4$$

**Problem 3.42.**

$$x^3 + 2x^2 = 2x - 3$$

**Problem 3.43.**

$$x^3 + 3\sqrt{x^3} = 32$$

**Problem 3.44.**

$$x^5 - x^3 + 4x^2 = 4x$$

**Problem 3.45.**

$$x^6 - 64 = 0$$

Problem 3.46.

$$\frac{1}{x+1} + \frac{2}{x-2} = 1$$

Problem 3.47.

$$\frac{x^3}{5} + \frac{5}{x} = 2.8x$$

Problem 3.48.

$$\frac{x^2 - 7}{x^2 - 9} = \frac{x^2}{7}$$

Problem 3.49.

$$\frac{20}{x^2 + 1} = 1 + \frac{1}{x^2}$$

Problem 3.50.

$$\frac{x}{3(x^2 - 1)} + \frac{2x}{3(1 - x^4)} = \frac{1}{x(1 + x^2)}$$

Problem 3.51.

$$\frac{24}{x^2 + 2x - 8} - \frac{15}{x^2 + 2x - 3} = 2$$

Problem 3.52.

$$\frac{x^2 - x}{x^2 - x + 1} - \frac{x^2 - x + 2}{x^2 - x - 2} = 1$$

Problem 3.53.

$$7\left(x + \frac{1}{x}\right) - 2\left(x^2 + \frac{1}{x^2}\right) = 9$$

Problem 3.54.

$$1 + \frac{1014}{x^4 - 3} = \frac{1106}{x^4 - 2}$$

Problem 3.55.

$$\frac{4x}{x^2 + x + 3} + \frac{5x}{x^2 - 5x + 3} = -\frac{3}{2}$$

Problem 3.56.

$$\sqrt{x^2 - 2} = \sqrt{x}$$

**Problem 3.57.**

$$\sqrt{x-1} \cdot \sqrt{x+4} = 6$$

**Problem 3.58.**

$$x+3 = \sqrt{(3x+1)(x-1)}$$

**Problem 3.59.**

$$\sqrt[3]{x^2+5} = 2\sqrt[3]{x+1}$$

**Problem 3.60.**

$$\sqrt{7 + \sqrt[3]{x^2+7}} = 3$$

**Problem 3.61.**

$$\sqrt{6x-11} - \sqrt{x-2} = \sqrt{x+3}$$

**Problem 3.62.**

$$\sqrt[3]{8x+4} - \sqrt[3]{8x-4} = 2$$

**Problem 3.63.**

$$\frac{2-\sqrt{x}}{2-x} = \sqrt{\frac{2}{2-x}}$$

**Problem 3.64.**

$$\frac{10}{x + \sqrt{10-x^2}} + \frac{10}{x - \sqrt{10-x^2}} = 7.5$$

**Problem 3.65.**

$$x^2 - 4x = 3\sqrt{x^2 - 4x + 20} - 10$$

**Problem 3.66.**

$$\sqrt[3]{x}\sqrt[3]{x^2+6} = \sqrt[3]{x^2+1}$$

**Problem 3.67.**

$$\sqrt{x} + \sqrt[3]{x} = 3(\sqrt{x} - \sqrt[3]{x})$$

**Problem 3.68.**

$$\sqrt{x+2} = \sqrt[3]{3x+2}$$

**Problem 3.69.**

$$\sqrt{x+8+2\sqrt{x+7}} + \sqrt{x+1-\sqrt{x+7}} = 4$$

**Problem 3.70.**

$$\sqrt[3]{x+a} + \sqrt[3]{x+a+1} = -\sqrt[3]{x+a+2}$$

## 4 Exponents and logs

Useful formulae:

$$y = a^x \leftrightarrow x = \log_a y$$

$$a^x \cdot a^y = a^{x+y}$$

$$a^x \cdot b^x = (ab)^x$$

$$\log(ab) = \log a + \log b$$

$$\log_a b = \frac{\log_c b}{\log_c a}$$

$$x^y = e^{y \ln x}$$

### 4.1 Find $x$ without calculator

**Problem 4.1.**

$$x = 10^{\log_{10} 3 - \log_{10} 2}$$

**Problem 4.2.**

$$x = 36^{\log_6 2}$$

**Problem 4.3.**

$$x = 81^{0.5 \log_9 7}$$

**Problem 4.4.**

$$\log_x 0.001 = -3$$



**Problem 4.5.**

$$\log_x n = n$$

**Problem 4.6.**

$$x = a^{2+\log_a b}$$

**Problem 4.7.**

$$\log_{\sqrt{2}} x = 4$$

**Problem 4.8.**

$$x = \log_{2\sqrt{2}}\left(\frac{1}{8}\right)$$

**Problem 4.9.**

$$\log_x(2\sqrt{2} - 3) = 2$$

**Problem 4.10.**

$$\log_{(\sqrt{2}+1)}(3 + 2\sqrt{2}) = x$$

## 4.2 Solve equations

**Problem 4.11.**

$$13^{(x-2)(x-3)} = 1$$

**Problem 4.12.**

$$1000 \cdot 0.1^{1/x} = 100^x$$

**Problem 4.13.**

$$2^{x^2+x-0.5} = 4\sqrt{2}$$

**Problem 4.14.**

$$2^x \cdot 5^x = 0.1(10^{x-1})^5$$

**Problem 4.15.**

$$5^x + 125 \cdot 5^{-x} = 30$$

**Problem 4.16.**

$$\frac{2^x + 10}{4} = \frac{9}{2^{x-2}}$$

**Problem 4.17.**

$$13^{2\sqrt{x^2+x-1}} = 17^{2\sqrt{x^2+x-1}}$$

**Problem 4.18.**

$$(\sqrt{3})^{\sqrt{10x-2}+\sqrt{10x-29}} = 27$$

**Problem 4.19.**

$$2^{2x+2\sqrt{x^2-2}} - 5 \cdot 2^{x-1+\sqrt{x^2-2}} = 6$$

**Problem 4.20.**

$$(\sqrt{x} + 2)^{10x^2-3x-1} = 1$$

**Problem 4.21.**

$$(0.4)^{x-1} = (6.25)^{6x-5}$$

**Problem 4.22.**

$$\log_{x-1}(x^2 - 5x + 10) = 2$$

**Problem 4.23.**

$$\log_2 \log_3 \log_4 x = 0$$

**Problem 4.24.**

$$\log_5 \log_{10} \sqrt{x^2 + 19} = 0$$

**Problem 4.25.**

$$\sqrt{\log_x \sqrt{5x}} = -\log_x 5$$

**Problem 4.26.**

$$\log_a(1 + \log_b(1 + \log_c(1 + \log_p x))) = 0$$

**Problem 4.27.**

$$\log_2(4^x + 4) = x + \log_2(2^{x+1} - 3)$$

**Problem 4.28.**

$$7^{\ln x} - 5^{\ln x+1} = 3 \cdot 5^{\ln x-1} - 13 \cdot 7^{\ln x-1}$$

**Problem 4.29.**

$$\log_{\sqrt{5}}(4^x - 6) - \log_{\sqrt{5}}(2^x - 2) - 2 = 0$$

**Problem 4.30.**

$$\left(\sqrt{4 + \sqrt{15}}\right)^x + \left(\sqrt{4 - \sqrt{15}}\right)^x = 8$$

**Problem 4.31.**

$$3(\sqrt{\log_a x} + \sqrt{\log_x a}) - 10 = 0$$

**Problem 4.32.**

$$\log_3 x + \log_{\sqrt{x}} x - \log_{1/3} x = 6$$

**Problem 4.33.**

$$\log_{x^2} 16 + \log_{2x} 64 = 3$$

**Problem 4.34.**

$$\log_x 9x^2 \cdot \log_3^2 x = 4$$

**Problem 4.35.**

$$\log_5(x - 2) + \log_{\sqrt{5}}(x^3 - 2) + \log_{0.2}(x - 2) = 4$$

## 5 Trigonometry

Useful formululae:

$$\sin(\pi/2 - x) = \cos x$$

$$\sin(x + y) = \sin x \cos y + \sin y \cos x$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\tan x = \frac{\sin x}{\cos x}$$

## 5.1 Prove

**Problem 5.1.**

$$\frac{\tan x + \tan y}{\tan x - \tan y} = \frac{\sin(x + y)}{\sin(x - y)}$$

**Problem 5.2.**

$$\frac{\sin 6x}{\tan 3x} - \cos 6x = 1$$

**Problem 5.3.**

$$\frac{\tan(\pi/8 + x) + \tan(\pi/8 - x)}{1 - \tan(\pi/8 + x) \tan(\pi/8 - x)} = 1$$

**Problem 5.4.**

$$1 - 8 \sin^2 x \cos^2 x = \cos 4x$$

**Problem 5.5.**

$$2 \cos^2 x - \cos 2x = 1$$

**Problem 5.6.**

$$\frac{\tan x}{1 + \tan x} + \frac{\tan x}{1 - \tan x} = \tan 2x$$

**Problem 5.7.**

$$2 \sin^2(\pi/4 - x) + \sin 2x = 1$$

**Problem 5.8.**

$$\sin 2x + 2 \sin\left(\frac{5\pi}{12} - x\right) \cos\left(\frac{5\pi}{12} + x\right) = 0.5$$

**Problem 5.9.**

$$\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \cdot \frac{1 - \sin x}{\cos x} = 1$$

**Problem 5.10.**

$$\frac{\sin 6x}{\sin 2x} + \frac{\cos(6x - \pi)}{\cos 2x} = 2$$

**Problem 5.11.**

$$\cos 2x + \sin 2x \tan x = 1$$

**Problem 5.12.**

$$\sin 4x - 2 \cos^2 2x + 1 = \sqrt{2} \sin(4x - \pi/4)$$

**Problem 5.13.**

$$\sin^2(x - \pi/2) - \cos^2(y - 3\pi/2) = \cos(x + y) \cos(x - y)$$

**Problem 5.14.**

$$\sin 2x + \sin 4x + \sin 6x = 4 \sin 3x \cos 2x \cos x$$

**Problem 5.15.**

$$\cos x - \sin x \sin 2x = \cos x \cos 2x$$

**Problem 5.16.**

$$1 + \sin x + \cos x + \tan x = 2\sqrt{2} \cos^2 \frac{x}{2} \cdot \frac{\sin(\pi/4 + x)}{\cos x}$$

**Problem 5.17.**

$$\tan a + \tan b + \tan c - \tan a \tan b \tan c = \frac{\sin(a + b + c)}{\cos a \cos b \cos c}$$

**Problem 5.18.**

$$\frac{\sin(a + b) - 2 \cos a \sin b}{2 \cos a \cos b - \cos(a + b)} = \tan(a - b)$$

**Problem 5.19.**

$$\frac{\tan(a - b) + \tan b}{\tan(a + b) - \tan b} = \frac{\cos(a + b)}{\cos(a - b)}$$

## 5.2 Solve equations

**Problem 5.20.**

$$\sin x \cos 2x = 0$$

**Problem 5.21.**

$$\cos x = \sin 2x \cos x$$

**Problem 5.22.**

$$3 \sin x - \sin^2 x = 2$$

**Problem 5.23.**

$$1 - \cos x = \sin x$$

**Problem 5.24.**

$$\sin^3 2x = \sin 2x$$

**Problem 5.25.**

$$\tan^2 x + \frac{1}{\tan^2 x} = 2$$

**Problem 5.26.**

$$\cos 4x = -\cos x$$

**Problem 5.27.**

$$\sin^2 x + \sin^2 2x + \sin^2 3x = 1.5$$

**Problem 5.28.**

$$\cos^2 x + \cos^2 2x + \cos^2 3x = 1$$

**Problem 5.29.**

$$\sqrt{3} \cos x + \sin x = \sqrt{3}$$

**Problem 5.30.**

$$4 \sin x + 3 \cos x = 2$$

**Problem 5.31.**

$$\sin a \cos(a + x) = \cos a \sin(a + x)$$

**Problem 5.32.**

$$\sin^4 x + \cos^4 x = \sin 2x$$

**Problem 5.33.**

$$\sin x + \sin 2x + \sin 3x + \sin 4x = 0$$

**Problem 5.34.**

$$\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$$

**Problem 5.35.**

$$\sin(x - \pi/3) = \cos(x + \pi/6)$$

**Problem 5.36.**

$$\sin^6 x + \cos^6 x = \frac{7}{16}$$

## 6 Differentiation

Notation: function  $y(x)$ , first derivative  $y'(x)$  or  $dy/dx$ , second derivative (derivative of derivative)  $y''(x)$  or  $d^2y/dx^2$ . Useful formulae:

$$(y(x) + z(x))' = y'(x) + z'(x)$$

$$(yz)' = y'z + yz'$$

Chain rule: let  $z = z(y)$  and  $y = y(x)$ , so that  $z = z(y(x))$ , then  $z'_x = z'_y \cdot y'_x$ .

EXAMPLE 6.1. [ ]  $z = (\ln x)^2$ :  $z = y^2$ ,  $y = \ln x$



$$z'_y = 2y = 2 \ln x$$

$$y'_x = \frac{1}{x}$$

$$z'_x = \frac{2}{x} \ln x$$

Basic functions and their derivatives:

$$y = x^a, \quad y' = ax^{a-1}$$

$$y = \ln x, \quad y' = 1/x$$

$$y = e^x, \quad y' = e^x$$

$$y = \sin x, \quad y' = \cos x$$

$$y = \cos x, \quad y' = -\sin x$$

$$y = \arcsin x, \quad y' = \frac{1}{\sqrt{1-x^2}}$$

## 6.1 Find first and second derivatives

**Problem 6.1.**

$$y = \frac{x^3}{3} - 2x^2 + 4x - 5$$

**Problem 6.2.**

$$y = \frac{1}{x} + \frac{1}{x^2} - \frac{1}{3x^3}$$

**Problem 6.3.**

$$y = \sqrt{x}(x^3 - \sqrt{x} + 1)$$

**Problem 6.4.**

$$y = (\sqrt{a} - \sqrt{x})^2$$

**Problem 6.5.**

$$y = (x^2 - 3x + 3)(x^2 + 2x + 1)$$

**Problem 6.6.**

$$y = \frac{x+1}{x-1}$$

**Problem 6.7.**

$$y = \frac{\sqrt{x}}{\sqrt{x}+1}$$



Problem 6.8.

$$y = 6\sqrt[3]{x}$$

Problem 6.9.

$$y = \frac{8}{\sqrt[4]{x}} - \frac{6}{\sqrt[3]{x}}$$

Problem 6.10.

$$y = \left(1 + \frac{1}{\sqrt[3]{x}}\right)^3$$

Problem 6.11.

$$y = x\sqrt{\frac{1-x}{1+x^2}}$$

Problem 6.12.

$$y = x - \sin x$$

Problem 6.13.

$$y = \sqrt{x} \cos x$$

Problem 6.14.

$$y = \sin^2 x + \sqrt{\sin x}$$

Problem 6.15.

$$y = \sin^2 x^3$$

Problem 6.16.

$$y = \frac{\cos x}{1 - \sin x}$$

Problem 6.17.

$$y = \sin \sqrt{x} + \sin \frac{1}{x}$$

Problem 6.18.

$$y = \sqrt{1-x^2} + \arcsin x$$

Problem 6.19.

$$y = \ln x - 2\sqrt{x}$$

Problem 6.20.

$$y = x \ln x - 1$$

Problem 6.21.

$$y = \ln(1 + \cos x)$$

Problem 6.22.

$$y = \frac{1}{\ln x} + \frac{\ln x}{x^n}$$

Problem 6.23.

$$y = x^n \ln x + \sqrt{1 + \ln^2 x}$$

Problem 6.24.

$$y = \ln(x + \sqrt{a^2 + x^2})$$

Problem 6.25.

$$y = 2\sqrt[3]{x} - 4 \ln \sqrt{2 + \sqrt{x}}$$

Problem 6.26.

$$y = \ln \frac{x^2}{1 - x^2}$$

Problem 6.27.

$$y = 2^x + x^2$$

Problem 6.28.

$$y = xe^x$$

Problem 6.29.

$$y = ae^{-x/a} + xe^{-x/a}$$

Problem 6.30.

$$y = x^x$$

Problem 6.31.

$$y = \cos x^{\sin x}$$

Problem 6.32.

$$y = (x^x)^x$$

## 7 Integrals

### 7.1 Indefinite integrals

If  $y = z'$  then  $\int y dx = z + C$ ,  $C = \text{const}$ .

Substitution: if  $y = f(x)$  then  $\int g(y) dy = \int g(y(x)) y' dx$ .

Integration by parts: if  $u(x)$  and  $v(x)$  are two functions then  $\int v u' dx = v u - \int u v' dx$ .

Attention:  $\int (1/x) dx = \ln |x|$ .

Problem 7.1.

$$\int \sqrt{x} dx$$

Problem 7.2.

$$\int \sqrt[n]{x^n} dx$$

Problem 7.3.

$$\int \frac{dx}{x^2}$$

Problem 7.4.

$$\int 10^x dx$$

Problem 7.5.

$$\int a^x e^x dx$$

Problem 7.6.

$$\int \frac{dx}{2\sqrt{x}}$$

Problem 7.7.

$$\int (1 - 2u)du$$

Problem 7.8.

$$\int (\sqrt{x} + 1)(x - \sqrt{x} + 1)dx$$

Problem 7.9.

$$\int \frac{\sqrt{x} - x^3 e^x + x^2}{x^3} dx$$

Problem 7.10.

$$\int \left(\frac{1-z}{z}\right)^2 dz$$

Problem 7.11.

$$\int \frac{(1-x)^2}{x\sqrt{x}} dx$$

Problem 7.12.

$$\int \frac{1 + \sqrt{x}}{\sqrt[3]{x}} dx$$

Problem 7.13.

$$\int \frac{dx}{\sqrt{3 - 3x^2}}$$

Problem 7.14.

$$\int \frac{3 \cdot 2^x - 2 \cdot 3^x}{2^x} dx$$

Problem 7.15.

$$\int \frac{1 + \cos^2 x}{1 + \cos 2x} dx$$

Problem 7.16.

$$\int \frac{\cos 2x}{\cos^2 x \sin x} dx$$

Problem 7.17.

$$\int \tan^2 x dx$$

Problem 7.18.

$$\int 2 \sin^2 \frac{x}{2} dx$$

Problem 7.19.

$$\int \frac{(1 + 2x^2)}{x^2(1 + x^2)} dx$$

Problem 7.20.

$$\int \frac{(1 + x)^2}{x(1 + x^2)} dx$$

Problem 7.21.

$$\int \frac{dx}{\cos 2x + \sin^2 x}$$

Problem 7.22.

$$\int (\arcsin x + \arccos x) dx$$

Problem 7.23.

$$\int (x + 1)^{15} dx$$

Problem 7.24.

$$\int \frac{dx}{(a + bx)^c}, \quad c \neq 1$$

Problem 7.25.

$$\int \sqrt[5]{(8 - 3x)^6} dx$$

Problem 7.26.

$$\int 2x\sqrt{x^2 + 1} dx$$

Problem 7.27.

$$\int x^2 \sqrt[5]{x^3 + 2} dx$$

Problem 7.28.

$$\int \sin^3 x \cos x dx$$

Problem 7.29.

$$\int \frac{\sin x dx}{\cos^2 x}$$

Problem 7.30.

$$\int \cos^3 x \sin 2x dx$$

Problem 7.31.

$$\int \frac{\sqrt{\ln x}}{x} dx$$

Problem 7.32.

$$\int \frac{(\arctan x)^2 dx}{1 + x^2}$$

Problem 7.33.

$$\int \frac{dx}{\arcsin x \sqrt{1 - x^2}}$$

Problem 7.34.

$$\int e^x \sin e^x dx$$

Problem 7.35.

$$\int \frac{e^x dx}{e^x + 1}$$

Problem 7.36.

$$\int \tan x dx$$

## 7.2 Definite integrals

If  $y = z'$  then  $\int_a^b y dx = z(b) - z(a)$ .

Problem 7.37.

$$\int_0^{\pi/2} \sin x dx$$

Problem 7.38.

$$\int_0^1 e^x dx$$

Problem 7.39.

$$\int_0^1 (1+x) dx$$

Problem 7.40.

$$\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2}$$

Problem 7.41.

$$\int_0^{\pi/4} \sin 2x dx$$

Problem 7.42.

$$\int_0^{\pi/2} \cos x dx$$

Problem 7.43.

$$\int_0^{\pi/4} \tan x dx$$

Problem 7.44.

$$\int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}}$$

Problem 7.45.

$$\int_1^2 \frac{dx}{\sqrt{1+x^2}}$$

Problem 7.46.

$$\int_0^2 |1-x| dx$$

Problem 7.47.

$$\int_0^2 x \ln x dx$$

**Problem 7.48.**

$$\int_0^{\pi/2} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}, \quad a, b > 0$$

**Problem 7.49.**

$$\int_0^{\pi} x \sin x dx$$

**Problem 7.50.**

$$\int_0^{\ln 2} x e^{-x} dx$$

**Problem 7.51.**

$$\int_0^1 x f''(x) dx$$

## 8 Series (progressions)

### 8.1 Arithmetic

Arithmetic progression (series):  $a_{n+1} = a_n + d$ ,  $a_n = a_1 + (n - 1)d$ ,

$$S_n = a_1 + \dots + a_n = \frac{(a_1 + a_n)n}{2} = na_1 + \frac{n(n-1)d}{2}$$



**Problem 8.1.** Given  $a_1 = -1.6$ ,  $d = -0.2$ , find  $a_{23}$ .

**Problem 8.2.** Given  $a_1 = 5.2$ ,  $d = 0.4$ , find  $S_{43}$ .

**Problem 8.3.** Given  $a_1 = a$ ,  $a_n = 9a + 8b$ , find  $d$  and  $S_9$ .

**Problem 8.4.** Given  $d = 1 + q$ ,  $a_n = 28 + 27q$ , find  $a_1$  and  $S_{28}$ .

**Problem 8.5.** Find  $\sum_{k=1}^n k$ .

**Problem 8.6.** Given  $5a_1 + 10a_5 = 0$  and  $S_4 = 14$ , find  $a_1$  and  $d$ .

**Problem 8.7.** Given  $a_n = 55$ ,  $a_2 + a_5 = 32.5$ ,  $S_{15} = 412.5$ , find  $a_1$ ,  $d$ , and  $n$ .

**Problem 8.8.** Given  $a_4^2 + a_{12}^2 = 1170$ ,  $a_7 + a_{15} = 60$ , find  $a_1$  and  $d$ .

**Problem 8.9.** Solve

$$1 + 4 + 7 + \dots + x = 117$$

**Problem 8.10.**

$$(x + 1) + (x + 4) + (x + 7) + \dots + (x + 28) = 155$$

**Problem 8.11.** Find sum of the first  $n$  terms:

$$\frac{x-1}{x} + \frac{x-3}{x} + \frac{x-5}{x} + \dots$$

**Problem 8.12.** Calculate

$$50^2 - 49^2 + 48^2 - 47^2 + \dots + 2^2 - 1$$

## 8.2 Geometric

Geometric progression (series):  $a_{n+1} = a_n q$ ,  $q \neq 1$ ;  $a_n = a_1 q^{n-1}$ ,

$$S_n = a_1 + \dots + a_n = \frac{a_1(q^n - 1)}{q - 1}$$

**Problem 8.13.** Given  $a_3 = 135$ ,  $S_3 = 195$ , find  $a_1$  and  $q$ .

**Problem 8.14.** Given  $a_1 = 2.5$ ,  $q = 1.5$ , find  $S_5$ .

**Problem 8.15.** Given  $q = 3/4$ ,  $a_3 = 1\frac{115}{128}$ , find  $a_1$  and  $S_5$ .

**Problem 8.16.** Given  $q = 2$ ,  $S_{12} = 4095$ , find  $a_1$  and  $a_{12}$ .

**Problem 8.17.** Given  $q = 2$ ,  $a_n = 96$ ,  $S_n = 189$ , find  $n$ .

**Problem 8.18.** Given progression:  $4, -1, 1/4, \dots$ , find  $S_6$ .

**Problem 8.19.** Given  $a_1 = 3$ ,  $q = 1/2$ ,  $a_n = 3/64$ , find  $n$  and  $S_n$ .

**Problem 8.20.** Given  $a_1$  and  $q$ , find  $a_1 \cdot a_2 \cdot \dots \cdot a_n$ .

### 8.3 Geometric with $|q| < 1$

In this case  $S = S_{n \rightarrow \infty} = \frac{a_1}{1-q}$ .

**Problem 8.21.** Given  $a_1 = 1$ ,  $S = 2$ , find  $q$ .

**Problem 8.22.** Given  $S_4 = 33\frac{3}{4}$ ,  $S = 36$ , find  $a_1$ .

**Problem 8.23.** Given  $a_1 = 66$ ,  $S = 110$ , find  $q$ .

**Problem 8.24.**

$$1 - \frac{2}{3} + \frac{4}{9} + \dots = ?$$

**Problem 8.25.**

$$(2 + \sqrt{2}) + (\sqrt{2} + 1) + (1 + \frac{1}{\sqrt{2}}) + \dots = ?$$

**Problem 8.26.**

$$\frac{a+x}{a-x} + \frac{a-x}{a+x} + (\frac{a-x}{a+x})^3 + \dots = ?$$

For which  $x$  the infinite sum exists ?

**Problem 8.27.**

$$\sqrt{2\sqrt{2\sqrt{2\sqrt{2}\dots}}} = ?$$

The source:

V.A. Bachurin, Zadachi po elementarnoi matematike i nachalam matematicheskogo analiza (in Russian).