Conductance of multimode ballistic rings: beyond Landauer and Kubo

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Related references:

- D. Cohen and Y. Etzioni, J. Phys. A **38**, 9699 (2005)
- D. Cohen, T. Kottos and H. Schanz, J. Phys. A 39, 11755 (2006)
- M. Wilkinson, B. Mehlig, D. Cohen, Europhys. Lett. 75, 709 (2006)

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$ISF $GIF

Motivation

Conductance of closed diffusive rings studied

experiment	:	B. Reulet et. al., PRL **75**, 124 (1995)
theoretical review	:	A. Kamenev and Y. Gefen, IJMP B**9**, 751 (1995)

$$G \;=\; G_{\text{Drude}} \;+\; \text{weak-localization corrections}$$

$$G_{\text{Drude}} = \frac{e^2}{2\pi\hbar}\mathcal{M}\frac{\ell}{L}$$

We would like to find
the conductance of **ballistic rings**

Ballistic ring: $\ell \gg L$

Diffusive ring: $\ell \ll L$

L = system size

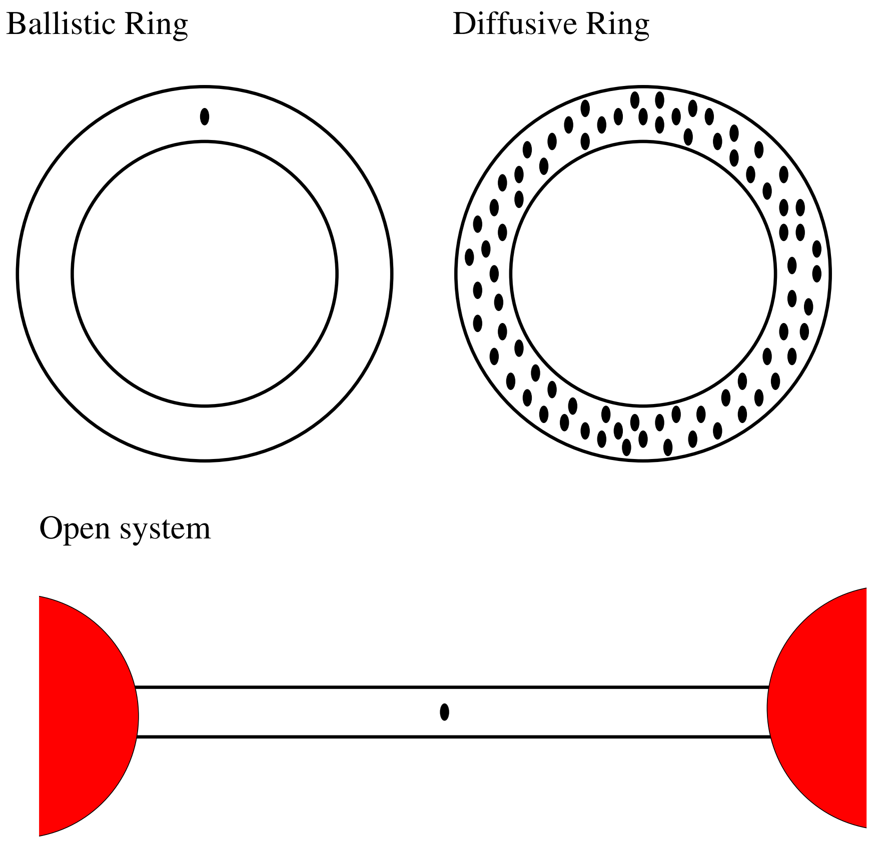
ℓ = mean free path

Take-home messages

- In the mesoscopic regime the leading order result for the conductance of ballistic rings is not Drude.

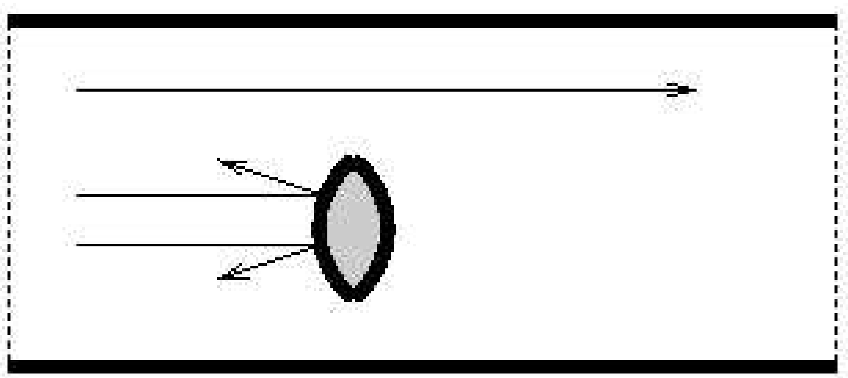
- In classical treatment we do get Drude.

Conductance Scheme

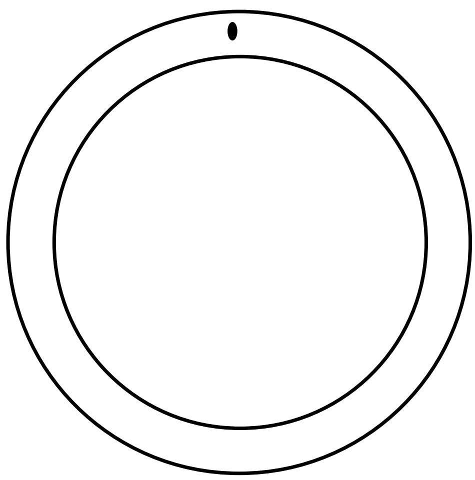


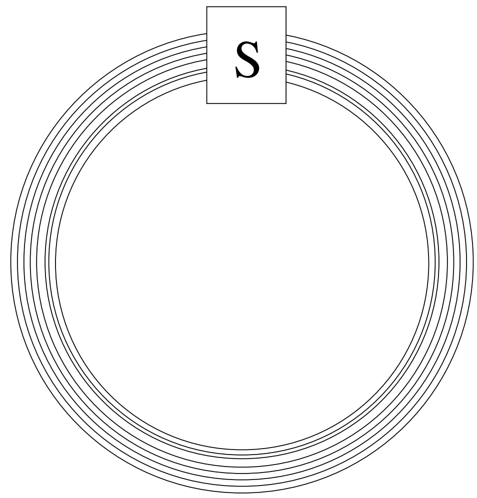
This is an example of a scatterer.

We will choose a corresponding S matrix.



The Model System





In our S matrix:

$$g^R_{ab} = \left|S^R_{ab}\right|^2 = \epsilon^2$$

$$g^T_{ab} = \left|S^T_{ab}\right|^2 = \left(1 - \mathcal{M}\,\epsilon^2\right)\delta_{ab}$$

$$g_T = 1 - \mathcal{M}\,\epsilon^2 \qquad\qquad 0 < g_T < 1$$

$$\mathbf{S} = \begin{pmatrix} \epsilon\exp(2\pi i\frac{ab}{\mathcal{M}}) & \sqrt{1-\mathcal{M}\epsilon^2}\delta_{a,b} \\ \sqrt{1-\mathcal{M}\epsilon^2}\delta_{a,b} & -\epsilon\exp(-2\pi i\frac{ab}{\mathcal{M}}) \end{pmatrix}$$

a, b : mode index

$\mathcal{M}$: number of open modes

The notion of conductance

We define:

Conductance = energy absorption coefficient.

"Joule's law"

$$\frac{dE}{dt} = G\,\dot{\Phi}^2$$

In the mesoscopic regime it is assumed that

Relaxation processes $\ll$ EMF driven transition

This assumption takes us beyond the LRT regime.

Landauer and Drude

The Landauer conductance for open system

$$G_{\text{Landauer}} = \frac{e^2}{2\pi\hbar}\sum_{ab} g_{ab}^T$$

For the opened version of our model we get

$$G_{\text{Landauer}} = \frac{e^2}{2\pi\hbar}\mathcal{M} g_T$$

In a recent work we found

the classical conductance for a close ballistic ring

$$G = \frac{e^2}{2\pi\hbar}\sum_{ab}\left[2g^T/(1-g^T+g^R)\right]_{ab}$$

For the closed version of our model we get

$$G = \frac{e^2}{2\pi\hbar}\mathcal{M}\frac{g_T}{1-g_T}$$

Note that $\ell \approx \frac{L}{1-g_T}$ for $g_T \sim 1$

$$G \approx \frac{e^2}{2\pi\hbar}\mathcal{M}\frac{\ell}{L} = G_{\text{Drude}}$$

Classical Kubo

Kubo formula

$$G = \varrho_F \times \frac{1}{2}\int_{-\infty}^{\infty} \langle\!\langle \mathcal{I}(\tau)\mathcal{I}(0) \rangle\!\rangle \; d\tau$$

The Drude assumption

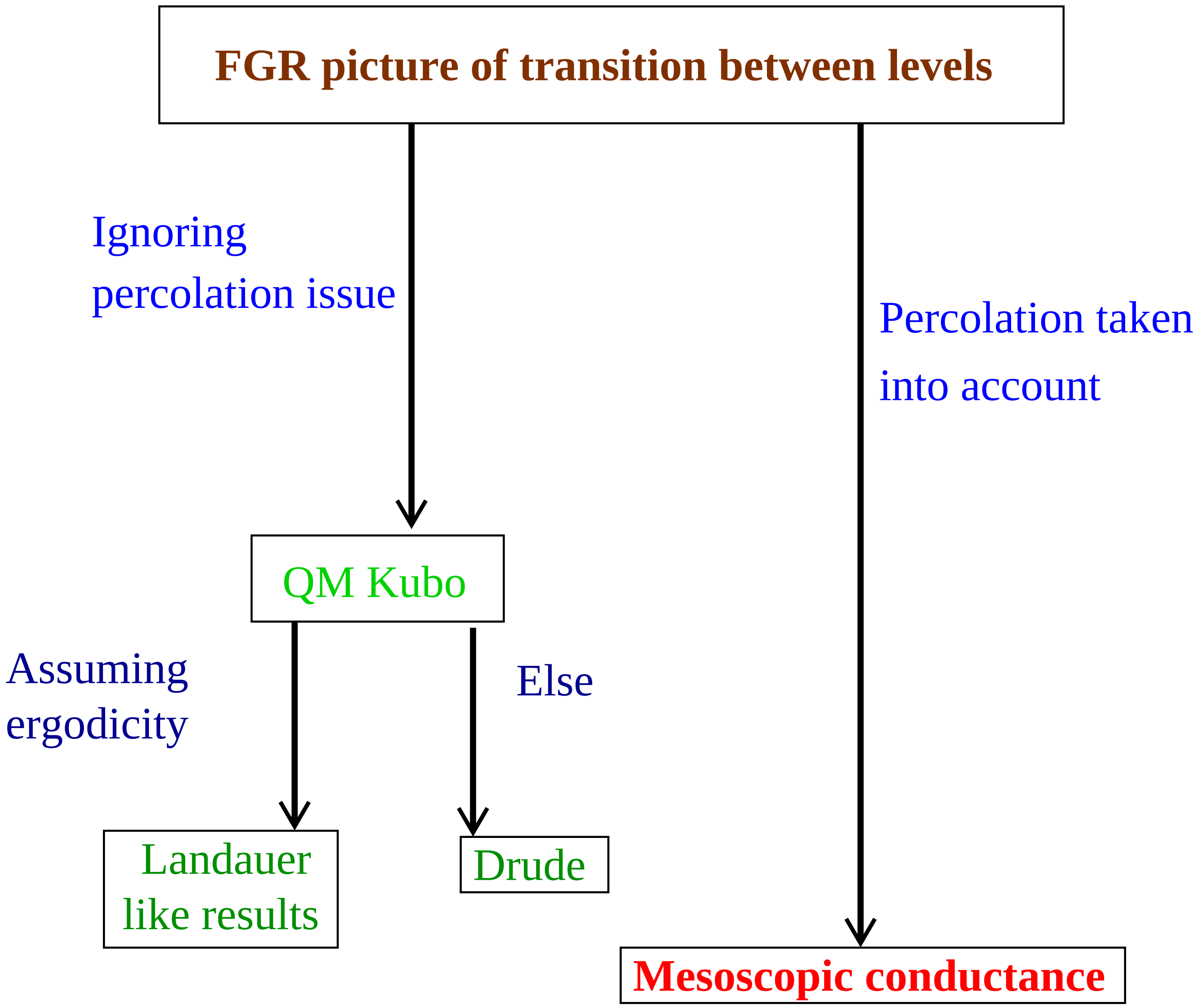
$$\langle\!\langle \mathcal{I}(\tau)\mathcal{I}(0) \rangle\!\rangle = \left(\frac{e}{L}v_E\right)^2 \mathrm{e}^{-(v_E/l)\tau}$$

Using it we get Drude

$$G_{\text{Drude}} = \frac{e^2}{2\pi\hbar}\mathcal{M}\frac{\ell}{L}$$

Do we get the same in the quantum case?

Outline


FGR picture of transition between levels
Ignoring
percolation issue
Percolation taken
into account
QM Kubo
Assuming
ergodicity
Else
Landauer
like results
Drude
Mesoscopic conductance

Quantum Kubo

Quantum version of Kubo

$$G_{\text{Kubo}} \;=\; \pi\hbar\varrho_F^2 \times \left\langle\!\!\left\langle |\mathcal{I}_{nm}|^2 \right\rangle\!\!\right\rangle$$

$$\begin{aligned} \varrho_F &= \mathcal{M}L/(\pi\hbar v_F) \\ \mathcal{I}_{nm} &\equiv \text{current operator elements} \end{aligned}$$

$\langle\!\langle \cdots \rangle\!\rangle$ stands for algebraic average

Kubo derivation **assumes:**

1. FGR transition rates.
2. All elements are comparables.

The latter assumption is problematic!

(we shall see that later)

Landauer?

If all the eigenfunctions were **ergodic**,
all the $\mathcal{I}_{nm}$ elements would be **comparable**.

$$G_{\text{Kubo}} \;=\; \frac{e^2}{2\pi\hbar}\mathcal{M} \;=\; G_{\text{Landauer}}$$

But the ergodic assumption is wrong.

Let us see how the $\mathcal{I}_{nm}$ look like.

Eigenstates of our system

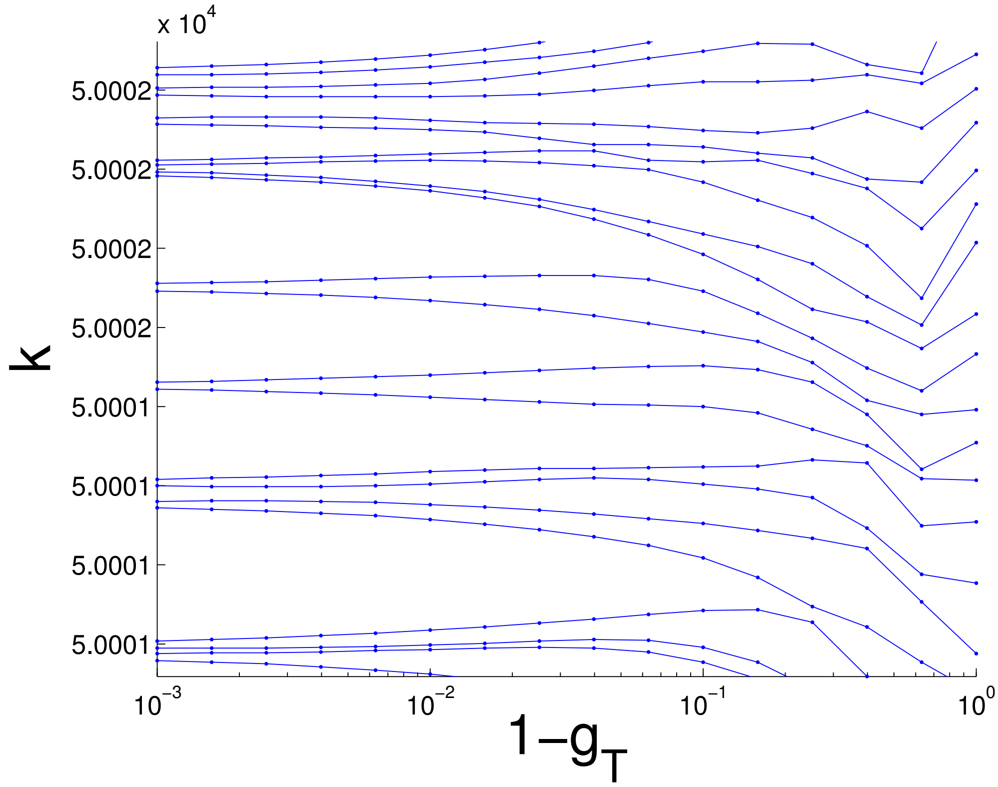
The eigenfunction of the ring

$$|\psi\rangle = \sum_a A_a \sin(kx + \varphi_a) \otimes |a\rangle$$

$a \equiv$ mode index $= 1, \cdots, \mathcal{M}$

For a given g_T we find a set of values

$$(k_n, \varphi_a^{(n)}, A_a^{(n)}) \qquad n = \text{level index}$$

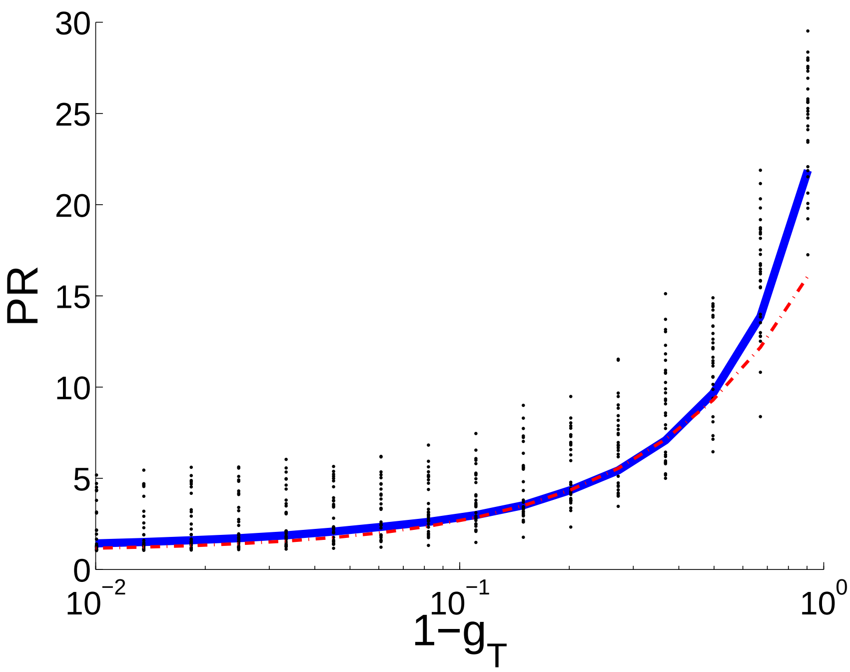


$L_a \sim 1$ and $\mathcal{M} = 50$

Ergodicity measure

Participation ratio

$$\mathrm{PR} \equiv \left[\sum_a \left(\frac{L_a}{2}A_a^2\right)^2\right]^{-1} = \begin{cases} 1 & \text{Localized} \\ \mathcal{M} & \text{Ergodic} \end{cases}$$



$$\mathrm{PR} \approx \ 1 + \tfrac{1}{3}(1 - g_T)\mathcal{M}$$

non-trivial ballistic regime

$$1/\mathcal{M} \quad \ll \quad (1 - g_T) \quad \ll \quad 1$$

No “quantum chaos” ergodicity

Current Operator

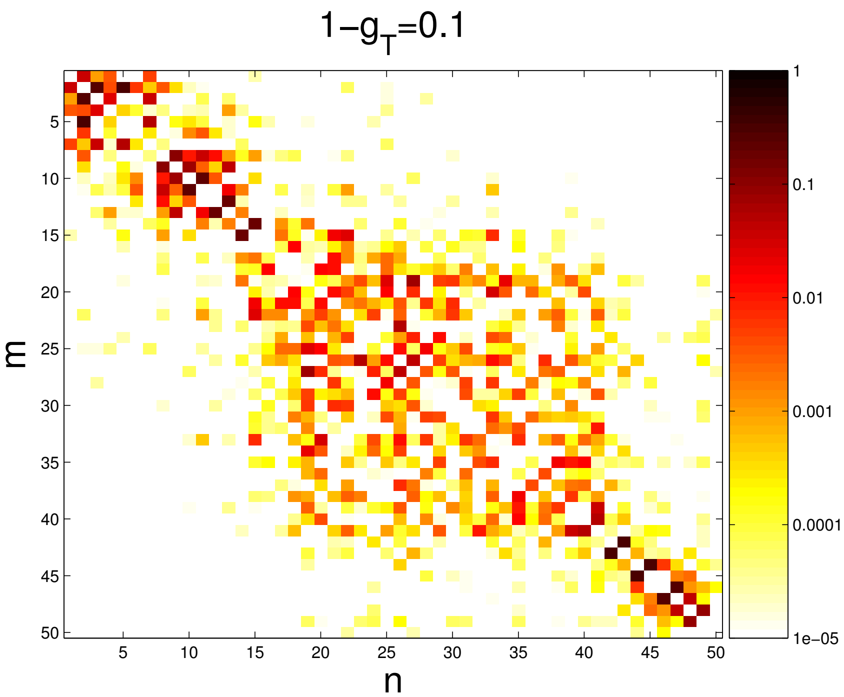
The matrix elements of the current operator are

$$\hat{\mathcal{I}} \quad = \quad e\hat{v}\delta(\hat{x} - x_0) \quad \text{(symmetrized)}$$

$$\mathcal{I}_{nm} \quad \approx \quad \sum_a \frac{L_a}{2} \mathsf{A}_a^{(n)} \mathsf{A}_a^{(m)} \sin(\varphi_a^{(n)} - \varphi_a^{(m)})$$

$a \equiv$ mode index $= 1, \cdots, \mathcal{M}$

Small PR of wavefunctions implies 'sparsity' of $\mathcal{I}_{nm}$



What is the Kubo result?

$$G_{\text{Kubo}} \;=\; \pi\hbar\varrho_F^2 \times \left\langle\left\langle \left|\mathcal{I}_{nm}\right|^2 \right\rangle\right\rangle$$

If we make an **algebraic average** we get

$$G_{\text{Kubo}} \;=\; G_{\text{Drude}}$$

where we assumed

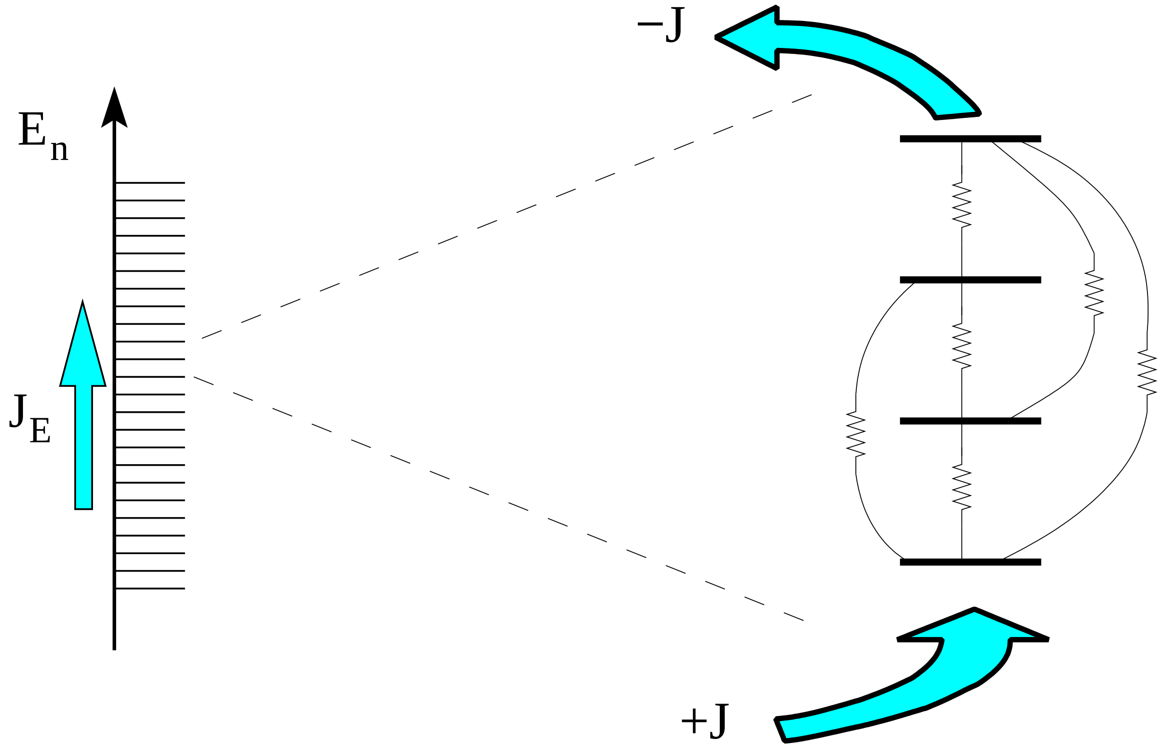
$$\frac{1}{1-g_T} \ll \mathcal{M}$$

mean free time $\ll$ Heisenberg time

But the conductance depends on the possibility to make **a connected sequence of transitions.**

Therefore, **algebraic average is not correct.**

Resistor network



level index n	node n
transition rate w_{nm}	inverse resistor g_{nm}
master equation	Kirchhoff
conductance G	inverse resistivity g

Master equation

$$\frac{\mathrm{d}p_n}{\mathrm{d}t} = \sum_m w_{nm}(p_m - p_n)$$

Kirchhoff

$$J_n = \sum_m \mathrm{g}_{nm}(V_n - V_m)$$

The mesoscopic conductance

The FGR transition rate

$$w_{nm} \;=\; 2\pi\hbar\frac{|\mathcal{I}_{nm}|^2}{(E_n-E_m)^2}\;\dot{\Phi}^2\;\delta_{\Gamma}(E_n-E_m)$$

Dimensionless transition rate

$$\mathsf{g}_{nm} \;=\; \frac{|I_{nm}|^2}{(n-m)^2}\;\frac{1}{\gamma}F\left(\frac{n-m}{\gamma}\right)$$

$\gamma = \Gamma/\Delta \equiv$ hopping range $\qquad 1 < \gamma \ll \mathcal{M}$

$$G = \frac{e^2}{2\pi\hbar} \times 2\mathcal{M}^2\mathsf{g}$$

g^{-1} resistivity of the network.

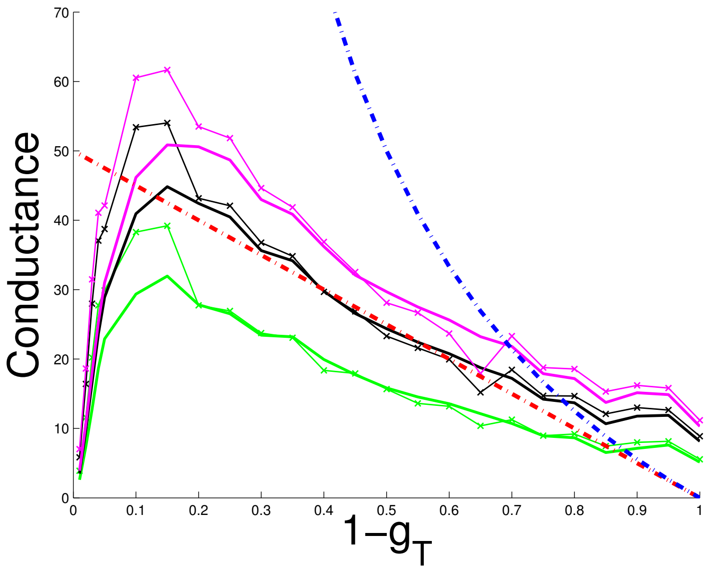
Approximation via harmonic average ("resistor in series")

$$\mathsf{g} = \left[\frac{1}{N}\sum_{n}^{N}\left[\sum_{m}^{n}(m-n)^2\mathsf{g}_{nm}\right]^{-1}\right]^{-1}$$

The results

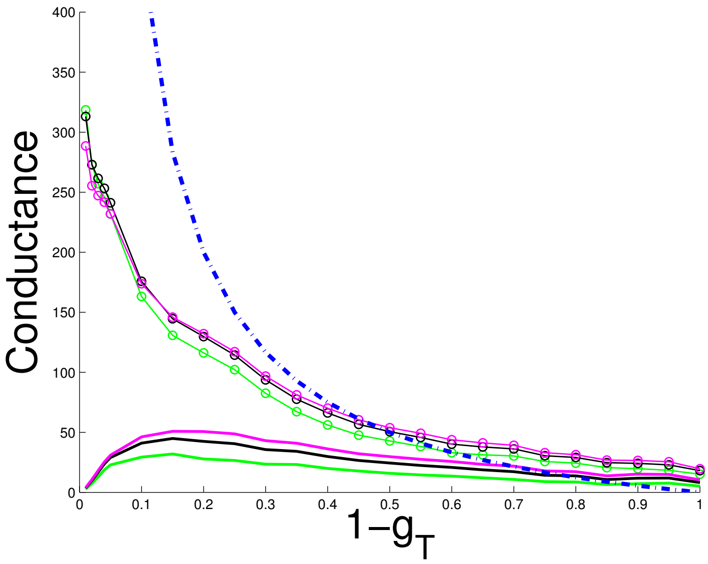
The coarse-grained conductance

$$G = \frac{e^2}{2\pi\hbar} \times 2\mathcal{M}^2 \mathrm{g}$$



G_{Drude} G_{Landauer}

Traditional Kubo (algebraic average)



Conclusion

- Ballistic rings are not quantum ergodic
- The perturbation matrix is sparse
- Kubo formula does not hold in mesoscopic regime
- Therefore, we do not get Drude formula
- Finding the conductance is analogous to solving a resistor network problem
- Conductance is typically not larger than the number of open modes