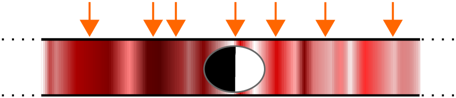
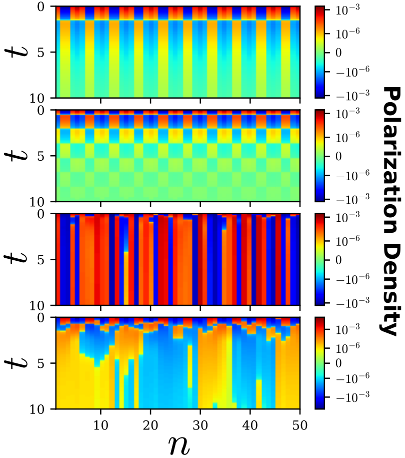
Localization due to topological stochastic disorder in active networks

Dekel Shapira

Ben-Gurion University



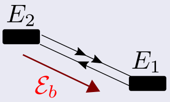


[1] DS, Dganit Meidan and Doron Cohen (Phys. Rev. E 98, 012107)

Active networks

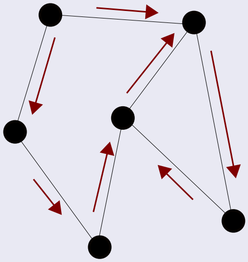
Rate equation: $\dot{\boldsymbol{p}} = \boldsymbol{W}\boldsymbol{p}$ $\quad$ $\boldsymbol{p}(t) = \boldsymbol{p}^{\mathrm{NESS}} + \sum_\lambda \psi_\lambda e^{-\lambda t}$ $\quad$ $\boldsymbol{W}\psi_\lambda = -\lambda\psi_\lambda$

Two sites



b = bond

Network



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Boltzmann $\quad \mathcal{E}_b = \dfrac{E_2 - E_1}{T_b}$

- **Affinity** $\equiv \oint \mathcal{E} dl$
- All **affinities** are 0 $\Leftrightarrow$ $\mathcal{E}$ conservative $\Rightarrow$ λ are real (Example: detailed balance)

Active Network: Non-zero **affinities** $\Rightarrow$ λ might be complex, under-damped relaxation

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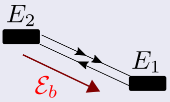

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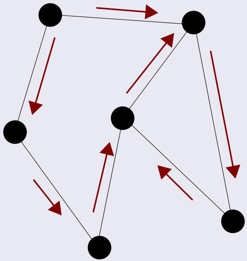

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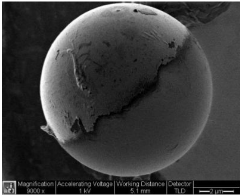
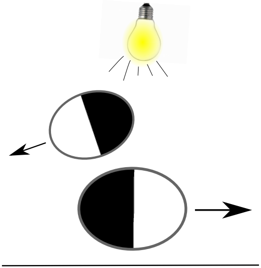
Janus particles

- Spherical-like “nano-particles” (100nm - 10μm), coated at each of their two hemispheres with different materials
- Placed in solution - diffusion
- Due to asymmetry - can be made to self propel (“active particle”)
- Sample mechanism: self-propelled when radiated with light by thermophoresis [1]





[1] Jiang, Hong-Ren, Natsuhiko Yoshinaga, and Masaki Sano. “Active motion of a Janus particle by self-thermophoresis in a defocused laser beam.” PRL 105.26 (2010): 268302.

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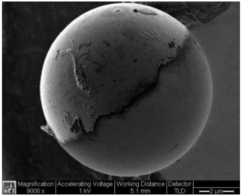

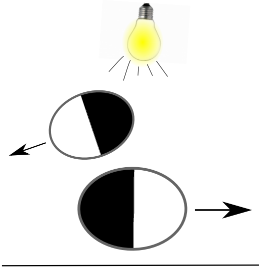

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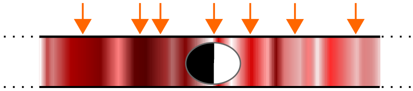
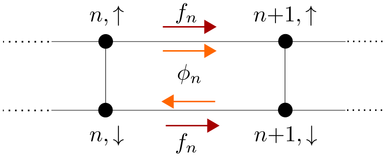
Janus particle - minimal model

- Rate equation - quasi 1D network
- Janus 1D $\Leftrightarrow$ $|n,s\rangle$
  Position: $n = 1, 2, .., N$
  Polarization: $s = \uparrow, \downarrow$



Stochastic field on bond (n, s):

- Drift: $f_n = \bar{f} + [-\sigma_f, \sigma_f]$
  Conservative Stochastic Disorder (CSD)
- Propulsion: $\phi_n = \bar{\phi} + [-\sigma_\phi, \sigma_\phi]$
  Topological Stochastic Disorder (TSD)



$\mathcal{E}_{n,\uparrow} = f_n + \phi_n$
$\mathcal{E}_{n,\downarrow} = f_n - \phi_n$

Affinity in unit cell = $2\phi_n$

$W = \boldsymbol{\sigma_x} + W_{\text{hop}} - \sum_{n,s} |n,s\rangle\, \gamma_{n,s}\, \langle n,s|$

$W_{\text{hop}} = \sum_{n,s} |n{+}1,s\rangle\langle n,s|\, e^{\frac{\mathcal{E}_{n,s}}{2}} + |n,s\rangle\langle n{+}1,s|\, e^{-\frac{\mathcal{E}_{n,s}}{2}}$ $\qquad \gamma_{n,s} = 1 + e^{\mathcal{E}_{n,s}/2} + e^{-\mathcal{E}_{n-1,s}/2}$

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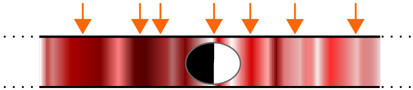

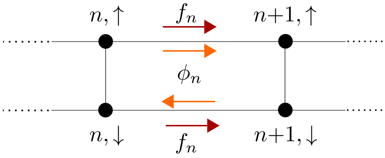

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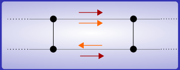
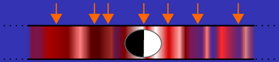


Clean system



Drift = 0, increasing propulsion ($\bar{\phi}$)

- No propulsion ($\bar{\phi} = 0$):
  $\lambda_{k,+} = 2 - 2\cos(k) \qquad k = 2\pi n/N$
  $\lambda_{k,-} = 4 - 2\cos(k)$

Adding propulsion ($\bar{\phi}$)

- Bloch (two bands): $|k, s\rangle$
  $\boldsymbol{W}^{(k)} = b\boldsymbol{\sigma}_x - ia\boldsymbol{\sigma}_z + c\boldsymbol{1}$
- PT symmetry breaking $a(k, \bar{\phi}) > b$
- Spectrum is complex for $\bar{\phi} > \phi_c$
  ($\phi_c \approx 0.96$)
- Eigenstates become polarized
  $|k, \pm\rangle = \sum_n e^{ikn} \left(|n, \uparrow\rangle \pm e^{\pm i\varphi} |n, \downarrow\rangle\right)$

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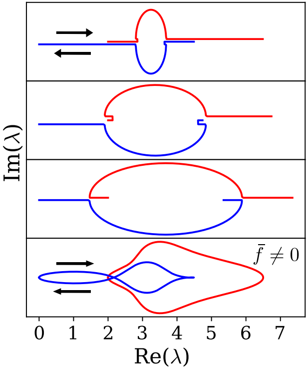
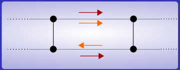


$a = \left[2\sinh\left(\frac{\bar{\phi}}{2}\right)\right]\sin(k) \qquad c = \left[2\cosh\left(\frac{\bar{\phi}}{2}\right)\right]\cos(k) - \left[1 + 2\cosh\left(\frac{\bar{\phi}}{2}\right)\right]$

$\lambda_{k,\pm} = -[c \pm \sqrt{b^2 - a^2}\,] \qquad b = 1$



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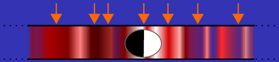

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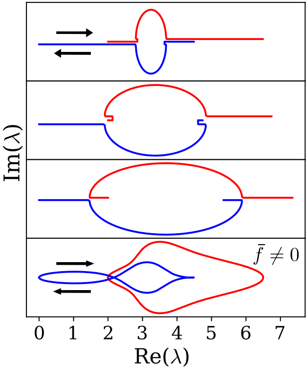

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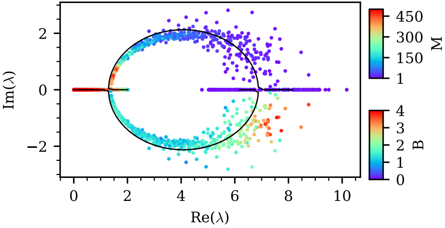
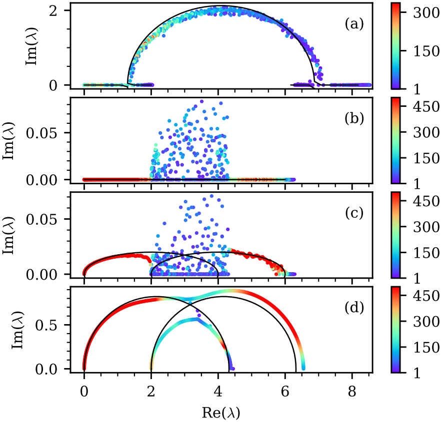
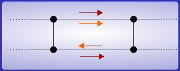
Adding disorder - Gallery

- Spectrum: $\boldsymbol{W}\boldsymbol{\psi} = -\lambda\boldsymbol{\psi}$
- Four parameters
  Propulsion: $\phi_n = \bar{\phi} + [-\sigma_\phi, \sigma_\phi]$
  Drift: $f_n = \bar{f} + [-\sigma_f, \sigma_f]$





Participation number: $M = \left[\sum_{n,s} P_{n,s}^2\right]^{-1}$



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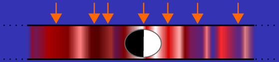
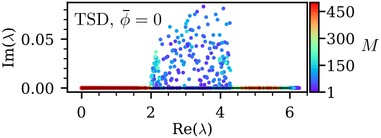
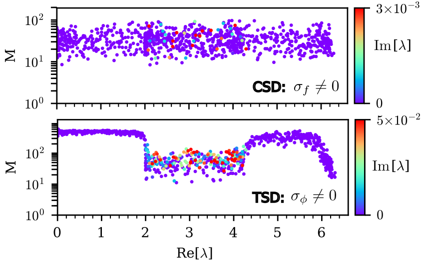


Adding disorder $\Rightarrow$ Localization ($\sim M^{-1}$)

- CSD (random f_n) $\Rightarrow$ Spectrum is real, localization is uniform (one channel)





- TSD (ϕ_n is random) $\Rightarrow$ Spectrum is complex, no finite threshold for ϕ_n
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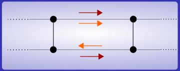

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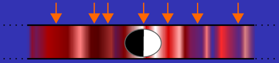

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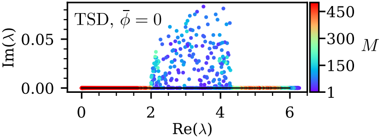

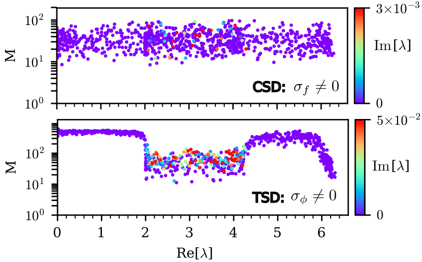

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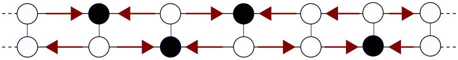
Floor level

Large TSD: The eigenvalues λ stretch along the real axis.

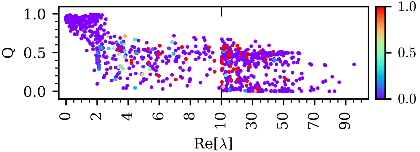
- 25% of the eigenstates stay within the limits $0 < \mathrm{Re}[\lambda] < 2$.
- These eigenvalues remain real.

A floor-level band is formed: symmetric "virtual transitions" occur between the floor sites.



Probability of $|n, s\rangle$ to be in the floor-band:

$$p(\mathcal{E}_{n,s} = \rightarrow) \times p(\mathcal{E}_{n+1,s} = \leftarrow) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$



Q - Probability to be in the floor-band

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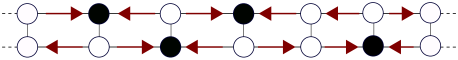

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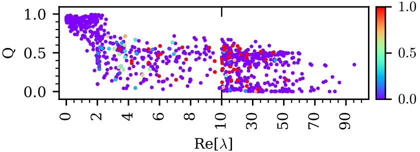

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Discussion

- Relaxation modes of a stochastic network can be either over-damped or under-damped depending on whether their λ-s are real or complex.
- Without disorder ($\phi_n = \bar{\phi}$), under-damped relaxation require $\bar{\phi} > \phi_c$
- Random ϕ_n - No finite threshold for under-damped relaxation
- Random ϕ_n very different than random f_n (complexity, localization)
- Complexity and de-localization do not come together (contrary to Hatano-Nelson)

[1] DS, Dganit Meidan and Doron Cohen (Phys. Rev. E 98, 012107)