

Quantum irreversibility of quasistatic protocols

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Abstract

In quantum mechanics, a driving process is expected to be reversible in the quasistatic limit, aka adiabatic theorem. This statement stands in opposition to classical mechanics, where mixed chaotic dynamics implies irreversibility. A paradigm for demonstrating the signatures of chaos in quantum irreversibility, is a sweep process whose objective is to transfer condensed bosons from a source orbital. Such protocol is dominated by an interplay of adiabatic-shuttling and chaos-assisted depletion processes. The latter is implied by interaction-terms that spoil the Bogolyubov integrability of the Hamiltonian. As the sweep rate is lowered, a crossover to a regime that is dominated by quantum fluctuations is encountered, featuring a breakdown of quantum-to-classical correspondence. The major aspects of this picture are not captured by the common two-orbital approximation, which implies failure of the familiar manybody Landau-Zener paradigm.

Introduction

In Classical Mechanics, contrary to a prevailing misconception, the quasi-static limit is in general *not* adiabatic. This observation implies that protocols become irreversible, even if their control parameters are varied very very slowly. Adiabaticity and reversibility in the quasistatic limit are guaranteed only if the phase-space of the system does not undergo structural changes. Accordingly, one distinguishes between integrable-dynamics version of adiabaticity [1] where action integrals serve as adiabatic invariants, and chaotic-dynamics version of adiabaticity [2–8] where the phase-space volume is the adiabatic invariant. Generic systems feature *mixed phase-space* dynamics and therefore do not obey the standard adiabatic theorems. The simplest demonstration for such irreversibility is the *separatrix crossing* scenario that has been discussed extensively in the mathematical literature [9–19]. But generic systems have more than a single degree-of-freedom, and therefore chaos becomes a central theme in the analysis [20–24].

In this paper we would like to explore how the above picture is reflected or modified *in the quantum framework*. The most suitable arena for such studies concern the dynamics of condensed bosons. In order to avoid an abstract discussion, let us consider a specific generic scenario. Let us assume that initially the bosons are condensed in a source orbital. A sweep protocol is designed to transfer them to a different orbital. *Naively*, one is inclined to speculate that this would be merely a many-

body version of the Landau-Zener (LZ) adiabatic passage problem. The classical limit, aka nonlinear LZ problem, has been studied extensively [25, 26]. It features a *diabatic ejection* stage (Fig.1, left panel) that is related to a swallow-tail structure in its bifurcation diagram. The full quantum version has been addressed as well [27]. Irreversibility has not been discussed there, but it is expected due to the separatrix crossing, per the conditions of the Kruskal-Neishtadt-Henrard theorem [9–19].

We claim that in general the manybody LZ problem cannot serve as a paradigm for depletion. Typically the dynamics involves more than two orbitals, meaning that we are dealing with more than one degree of freedom. Consequently the role of *chaos* cannot be ignored [22–24]. Using different phrasing, we say that the inapplicability of the LZ paradigm is related to the failure of the two-orbital approximation (TOA). Once additional orbitals are taken into account, the integrability of the Hamiltonian is spoiled. Consequently, the depletion stage involves competing mechanisms which we call *adiabatic shuttling* and *chaos-assisted depletion* (Fig.1, middle and right panles).

Our interest is to address the *irreversibility* theme, and to contrast *quantum* against *semiclassical* dynamics. In our semantics the term ‘semiclassical’ replaces the term ‘classical’ whenever the quantum state is represented in phase-space by a cloud of points, that are propagated using classical equations of motion. This is also known as the ‘truncated-Wigner-approximation’, and goes much beyond the single-trajectory dynamics of Mean Field theory. Nevertheless, semiclassical approximation, in this restricted sense, is not capable of taking into account neither tunneling [28–31] nor interference of separated trajectories.

In quantum mechanics, contrary to the semiclassical picture, the quasi-static limit of a closed finite system is always adiabatic, and therefore reversible. This is because the energies are quantized, and therefore the system follows the (gaped) ground state for slow enough driving. However, this quantum adiabaticity has no experimental significance once we deal with a mesoscopic system. In the example that we discuss in this work, the condensate is a flow-state of a superfluid ring. As the control parameter is varied, the flow-state becomes *metastable*. But the tunnel coupling to the new ground state is exponentially small in the number of particles [28], and therefore can be ignored. Hence the system fails to follow the ground state. This is in fact the essence of superfluidity. The question remains, what is the fate of the flow-state as the control parameter is further varied.

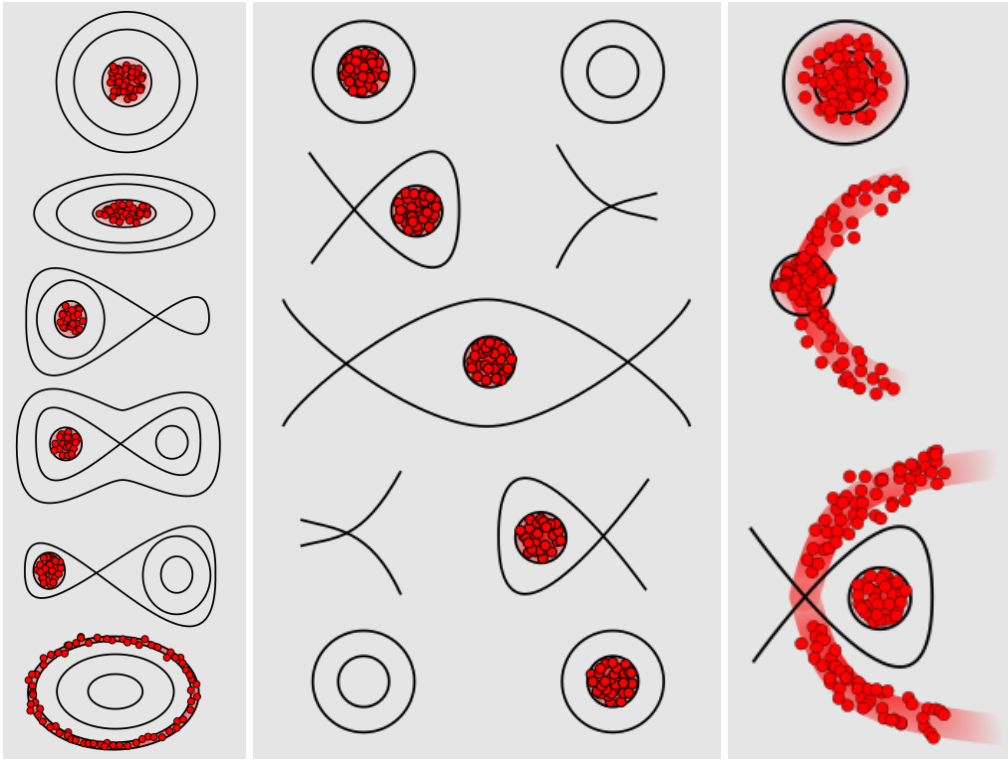


FIG. 1. **Schematics of phase-space evolution.** Each panel provides a sequence of phase-space snapshots. In the left and middle panels the curves are $\mathcal{H} = \text{const}$ contours of a one-degree-of-freedom system. The evolving cloud is red. Initially the cloud is located in the minimum of the energy landscape. The left panel displays an *adiabatic shuttling* process. As a control parameter is varied a second local minimum appears due to a saddle-node bifurcation (3rd snapshot), and the cloud becomes metastable (5th snapshot). In a quantum perspective the evolution is *diabatic*, meaning that quantum tunneling does not have the time to take place. The process ends with *diabatic ejection* (last snapshot). If the sweep is reversed (not shown), the cloud can *split* into the two minima of the 5th snapshot (assuming that both basins are expanding). The middle panel displays a *relay-shuttling* process. It consists of pitchfork bifurcation; swap of separatrices; and inverse pitchfork bifurcation. The right panel displays the effect of a *chaos-assisted depletion* mechanism that competes with the pitchfork bifurcation of the relay-shuttling process. Strictly speaking we display in this panel a Poincare section of a two-degree-of-freedom system. Due to spoiled integrability, there is a chaotic strip along which spreading is allowed. The outer part of the cloud starts to spread away before the central fixed-point becomes unstable.

What is the mechanism of depletion? Do we have the same irreversibility as in the semiclassical analysis?

The question that we pose is not merely related to the foundations of physics (irreversibility, quantum vs classical). It is also of practical importance for the design of protocols whose objective is to manipulate many-body states of cold atoms, aka atomtronics [32]. Specifically, we consider bosons that are described by the Bose-Hubbard Hamiltonian (BHH). This model is of major interest both theoretically and experimentally [33–36]. There is a particular interest in lattice-ring circuits that can serve as a SQUID or as a useful Qubit device [37–40]. The hope is to achieve coherent operation for BHH configurations that involve a few orbitals. This is the natural extension of studies that concern two orbitals, aka Bosonic Josephson Junction. The most promising configuration is naturally the 3-site trimer [41–59]. For the analysis of such circuits one has to confront the handling of an underlying mixed phase space [57, 58, 60].

We are inspired by hysteresis experiments, as done for double well geometry [61], and by protocols that have been realized experimentally for bosons in a ring (or SQUID) geometry [32, 62–65]. The related theoretical studies adopt the TOA, and highlight the appearance of swallow-tail bifurcations [66–70]. As opposed to that, our interest below is to push the discussion of irreversibility into the realm of high-dimensional dynamics, addressing the fingerprints of chaos and mixed phase-space in the quantum-mechanical reality.

The failure of the TOA is anticipated by observing that the Bogolyubov pairing interaction requires 3 orbitals, and by the further observation that there are additional terms in the Hamiltonian that spoils the integrability of the Bogolyubov approximation. The classical analysis of the forward sweep process follows our previous publication [24]. In the present paper we further illuminate that the integrable mechanism that is implied by the Bogolyubov approximation is a variant of adia-

batic shuttling that we call *relay shuttling* (Fig. 1). In the quasistatic limit this mechanism is overwhelmed by chaos-assisted depletion. We explore the *quantum* scenario, and append an *inverse-sweep* of the control parameter, in order to study the *irreversibility* due to the interplay of the various mechanisms involved.

Outline.— We present the Bose-Hubbard Hamiltonian that describes a superfluid ring, and display some results of simulations that probe irreversibility. Then we illuminate our findings by performing step-by-step analysis: We clarify the failure of the TOA; we provide predictions that are based on the Bogolyubov approximation; and then, going beyond that, we discuss the implications of chaos. This is followed by a discussion, where we highlight the manifestation of universal quantum fluctuations (UQF), and the breakdown of quantum-to-classical correspondence (QCC).

Results

The suggested experiment.— Consider a ring with condensed bosons. The optical potential that holds the bosons is possibly painted as in [64]. The ring has several weak links (as in SQUID geometry), or it can be an L -site lattice ring (as assumed below). Initially the ring is at rest, and the condensed bosons have zero momentum. In a *quench* protocol the ring starts abruptly to rotate. Superfluidity means that the rotation velocity Ω should be larger than a critical value Ω_c in order to induce current. The appearance of a non-zero current (depletion of the zero momentum orbital) can be verified using a standard time-of-flight measurement procedure. We would like to consider a *sweep* protocol, such that the rotation velocity is increased gradually (quasi-statically) from zero to a finite value that is larger than Ω_c . Then we ask whether this sweep process is reversible. Accordingly, we decrease gradually Ω back to zero. Our main message, from the perspective of an experiment, is that the quasi-static protocol features novel *quantum irreversibility*. A secondary message is that the value of Ω_c is affected by the sweep rate, and provides an indication for the underlying depletion mechanism.

The model.— Consider N bosons in an L -site ring, described by the Bose-Hubbard Hamiltonian (BHH) with hopping frequency K and on-site interaction U . The sweep control-parameter is the Sagnac phase Φ , which is proportional to the rotation velocity Ω of the device. This phase can be regarded as the Aharonov-Bohm flux that is associated with a Coriolis field in the rotating frame. The Hamiltonian is

$$\mathcal{H} = \sum_{j=0}^{L-1} \left[\epsilon_j a_j^\dagger a_j + \frac{U}{2} (a_j^\dagger a_j^\dagger a_j a_j) - \frac{K}{2} \left(e^{i\frac{\Phi}{L}} a_{j+1}^\dagger a_j + e^{-i\frac{\Phi}{L}} a_j^\dagger a_{j+1} \right) \right] \quad (1)$$

where $\epsilon_j = -\epsilon \cos(2\pi j/L)$ is included, as in [69]. It sig-

nifies an external gravitation potential that may arise due to an optional tilt of the ring. Unless stated otherwise we assume $\epsilon=0$. The momentum orbitals are labelled $k = (2\pi/L) \times \text{integer}$. In this basis the Hamiltonian takes the form

$$\mathcal{H} = \sum_{k=0}^{L-1} \mathcal{E}_k b_k^\dagger b_k - \frac{\epsilon}{2} \sum_{k,\pm} b_{k\pm 1}^\dagger b_k + \frac{U}{2L} \sum_{k_1, k_2, k_3, k_4}' b_{k_1}^\dagger b_{k_2}^\dagger b_{k_3} b_{k_4} \quad (2)$$

where the prime in the k summation implies that conservation of total momentum is required. The presence of the control parameter Φ is implicit via

$$\mathcal{E}_k = -K \cos \left(k - \frac{\Phi}{L} \right) \quad (3)$$

We start with a non rotating ring ($\Phi=0$). Initially the bosons are condensed in the zero momentum orbital ($k_0=0$). Keeping only the 3 lowest orbitals, labeled as (k_0, k_+, k_-) , it is convenient to describe their subsequent occupation using the depletion coordinate n , and the imbalance coordinate M , that are defined as follows:

$$n = \sum_{k \neq 0} n_k = n_+ + n_- \quad (4)$$

$$M = n_+ - n_- \quad (5)$$

Simulations.— In order to motivate the subsequent analysis we first present some results of numerical simulation for an $L=3$ ring, aka trimer. Initially all the particles are condensed in $k=0$, meaning that the initial value of the depletion coordinate is $n=0$. The protocol consist of 3 stages: a forward sweep of Φ from $\Phi=0$ up to $\Phi=2.5\pi$, an optional waiting period, and a backward sweep to $\Phi=0$. Note that once Φ exceeds $\Phi_{\text{mts}} = \pi$ (to be indicated by black vertical line in the time axis of our figures) the condensate becomes metastable. But its depletion happens only in a later stage, as discussed below.

We display in Fig. 2 the variation of (E, n) as a function of time using both quantum and semiclassical simulations. The variation of n is color coded. In the semiclassical simulations we propagate an ensemble of trajectories, starting with a cloud that mimics the initial condensate. In the quantum simulations we propagate the evolving manybody state $\Psi(t)$, and calculate the probabilities

$$p_\nu(t) = |\langle E_\nu | \Psi(t) \rangle|^2 \quad (6)$$

The energy levels $E_\nu(\Phi(t))$ are plotted as a function of time: gray color indicates levels whose weight is vanishingly small (less than 3.5%); and the other levels whose p_ν is non-negligible are color-coded by $\langle n \rangle_\nu = \langle E_\nu | n | E_\nu \rangle$.

One observes that for "slow" sweep the spreading in E is worse, indicating that irreversibility is enhanced. For the semiclassical simulation we show in Fig. 2 (3rd row)

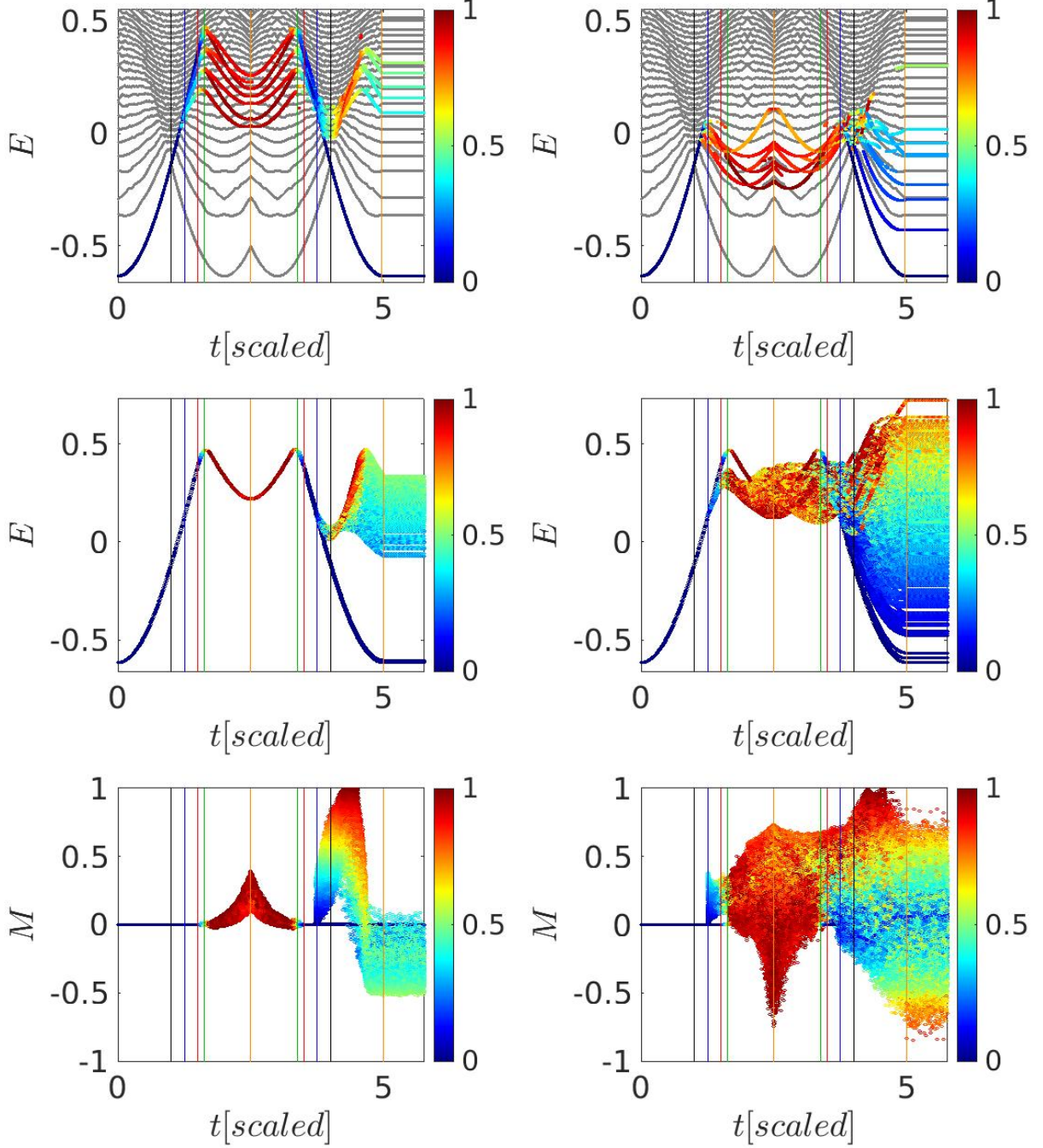


FIG. 2. **Simulations that test irreversibility.** The control parameter is swept from $\Phi=0$ to $\Phi=2.5\pi$ and back to $\Phi=0$. The horizontal axis is the scaled time $(1/\pi)|\dot{\Phi}|t$. The vertical lines are the thresholds $\Phi_{\text{mts}}=1\pi$ (black), $\Phi_{\text{stb}}=1.26\pi$ (blue), $\Phi_{\text{dyn}}=1.5\pi$ (red), $\Phi_{\text{swp}}=1.62\pi$ (green). The vertical orange lines indicate where the sweep is reversed and then stopped. Note that the quantum simulations may include an additional waiting period at $\Phi=2.5\pi$. The initial state is the ground-state condensate. The upper panels are quantum simulation for $N=30$ particles, with $K=1$ and $NU=2.3$. The energy levels $E_\nu(\Phi(t))$ are plotted. Gray color indicates levels whose weight p_ν is vanishingly small (for presentation purpose this set is diluted by factor 10). The participation levels whose p_ν is non-negligible are color-coded by $\langle n \rangle_\nu$. The left panel is for $\dot{\Phi}=5\pi \cdot 10^{-4}$, and the right panel is for $\dot{\Phi}=3.33\pi \cdot 10^{-7}$. The 2nd and 3rd rows display semiclassical simulations for the same system. We plot E and M for an ensemble of trajectories, starting with a cloud that mimics the initial condensate. The value of the n -coordinate is color-coded. The left panel is for an optimal sweep rate $\dot{\Phi}=5\pi \cdot 10^{-4}$, while the right is for a very slow sweep $\dot{\Phi}=5\pi \cdot 10^{-5}$. In this figure, and in all subsequent figures, the units are normalized ($n := n/N$, $M := M/N$, $E := E/N$).

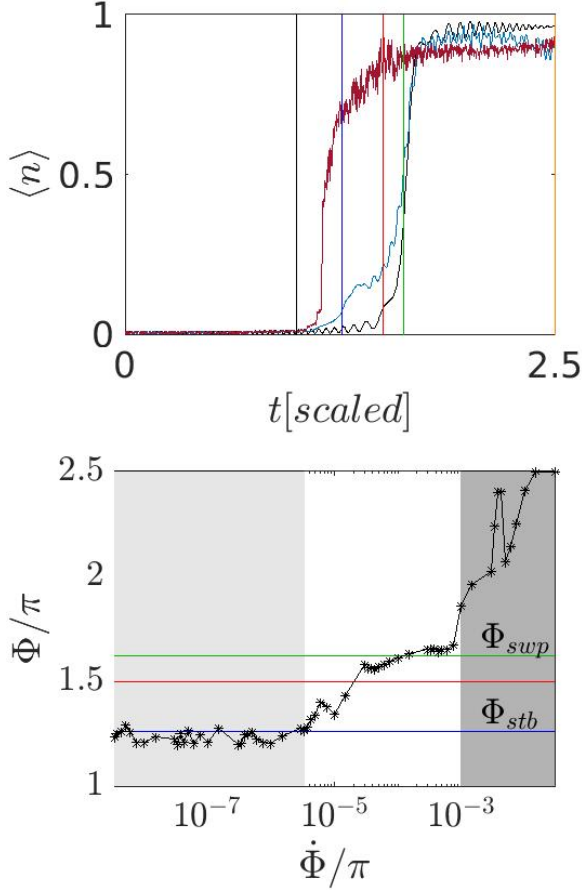


FIG. 3. **Depletion vs sweep-rate.** The upper panel displays the depletion $\langle n \rangle$ versus time for a quantum simulation with sweep rate $\dot{\Phi} = 5\pi \cdot 10^{-4}$ (blue), and with very slow rate $\dot{\Phi} = 3.33\pi \cdot 10^{-7}$ (red). The former is compared with simulation (black) that is generated by the Bogolyubov-approximated Hamiltonian. The vertical lines and the parameters are as in Fig. 2. From such plots we determine the time t_d at which the depletion happens. The dependence of $\Phi(t_d)$ on the sweep rate $\dot{\Phi}$ is displayed in the lower panel. The dark gray background indicates non-quasistatic regime where the depletion time lags and becomes numerically ill-defined. In the quasistatic regime the observed dependence on $\dot{\Phi}$ indicates the crossover from chaos-assisted depletion (light gray background) to adiabatic shuttling. Namely, the depletion shifts from Φ_{stb} to Φ_{swp} .

how this spreading is expressed in M . The optional Fig. S2 of the SM shows how the spreading looks like in occupation space, using (n, M) coordinates.

In Fig. 3 we plot the depletion $\langle n \rangle$ versus time. In the quasistatic regime the time of the depletion t_d is determined by inspection of the sharp rise in $\langle n \rangle$. We indicate by dark gray background color the range of $\dot{\Phi}$ where t_d becomes ill-defined, reflecting a lag with respect to the parametric variation of Φ . In the quasistatic regime we observe that $\Phi(t_d)$ is shifted as $\dot{\Phi}$ is increased. Later we interpret this shift as an indication for a crossover from

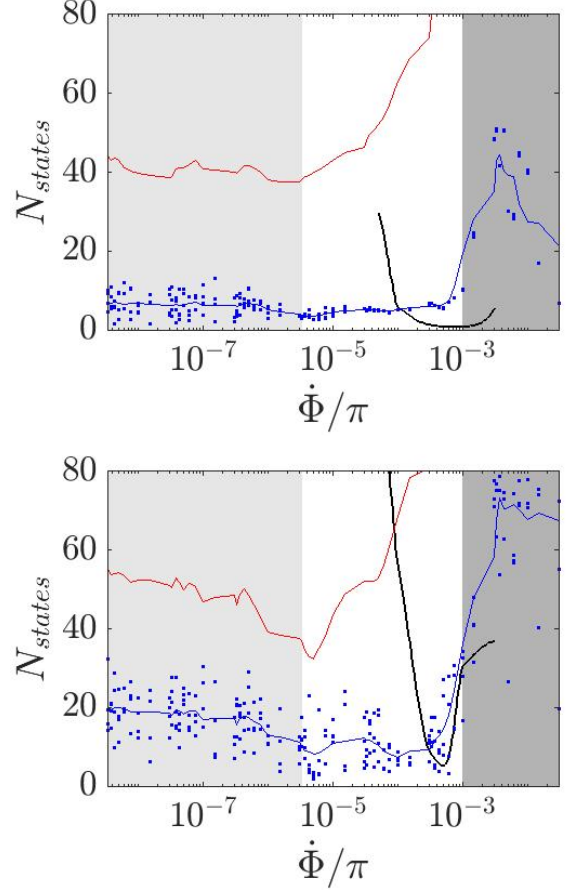


FIG. 4. **Irreversibility vs sweep-rate.** Irreversibility is indicated by the growth of the the number N_{states} of energy levels that participate in the evolution. We show the dependence of N_{states} (blue dots) on the sweep rate $\dot{\Phi}$ before the reversed sweep (upper panel) and at the end of the reversed sweep (lower panel), for misc values of the waiting time. The erratic dependence on the waiting is illustrated in Fig. S6 of the SM. The blue lines provides the average value of N_{states} , and the red lines provided the average value $\sum \nu p_\nu$. The black lines are based on the semiclassical simulations. The gray background is the same as in Fig. 3.

chaos-assisted depletion to adiabatic shuttling.

In order to quantify the *adiabaticity* in the quantum simulations, we characterize the spreading in energy by estimating the number of participating energy levels

$$N_{\text{states}}(t) = \left[\sum_{\nu} |p_{\nu}(t)|^2 \right]^{-1} \quad (7)$$

Illustrations for the variation of $N_{\text{states}}(t)$ are provided in the Methods. It should be noted that this function is expected to be monotonic increasing only for a strictly quasistatic process, which is not the case here (because we have mixed phase space and bifurcations along the way). Nevertheless, the final spreading can be used as a measure for the *irreversibility* of the sweep protocol. Its

dependence on the rate $\dot{\Phi}$ is displayed in Fig. 4.

We see that in the quasistatic regime slowness is bad for adiabaticity. This is very pronounced in the semiclassical simulation, and has modest reflection in the quantum evolution. On the average, irreversibility is suppressed quantum-mechanically compared with the semiclassical expectation. But more interestingly, the dependence of N_{states} on $\dot{\Phi}$ becomes erratic, indicating a crossover to a regime of chaos-assisted-depletion. This crossover is further reflected in the timing of the depletion, as we already saw in Fig. 3.

Two orbital approximation.— As we sweep the parameter Φ , orbitals k_0 and k_+ cross each other. It is therefore natural to adopt TOA as in [69]. This naturally leads to an effective 2 sites (dimer) problem as in [27], that can be regarded as second-quantized version of the well known nonlinear LZ problem [25, 26].

With TOA, the 3rd term in Eq. (2) does not generate transitions between orbitals. Therefore we need a tilt $\epsilon \neq 0$ in order to get non-trivial dynamics. Indeed this was the approach in [69]. But clearly for a BHH ring we should have non-trivial dynamics even without a tilt. So clearly TOA is an over-simplification. Nevertheless one may wonder whether with $\epsilon \neq 0$ there is a regime such that TOA makes sense. We address this secondary question in the Methods section.

Bogolyubov approximation.— The Bogolyubov approximation keeps in Eq. (2) transitions of *pairs* from the k_0 condensate to the $k_{\pm}$ orbitals. The textbook version further makes the substitution $b_0 \mapsto \sqrt{N}$, but we avoid below this over-simplification. Either way, it is clear that the Bogolyubov approximation implies that in the absence of tilt ($\epsilon=0$) the occupation imbalance (M) is a constant of motion. Consequently, for the $L=3$ trimer (or for any ring if we keep the 3 lowest orbitals k_0, k_+, k_-) the BHH becomes formally identical to a generalized dimer Hamiltonian, that differs from the standard TOA dimer.

We present the derivation of the effective $\mathcal{H}_{\text{dimer}}$ in the Methods, and further discuss it below. The same $\mathcal{H}_{\text{dimer}}$ can be exploited to simulated the TOA dynamics using appropriate set of effective parameters, and to simulate the Bogolyubov dynamics using a different set of effective parameters. The dynamics that is generated in the two cases is illustrated in Fig. 5. One observes that the TOA dynamics (with tilt) features *adiabatic ejection*. As opposed to that, the Bogolyubov-approximated dynamics features what we call *relay shuttling*. The snapshots of the evolution that are provided in Fig. 5 correspond to the scenarios that have been caricatured in Fig. 1.

Coming back to Fig. 3 we observe that the t_d of the Bogolyubov (black) line agree with that of the blue line, but not with that of the red line. This implies that in the latter case (very slow sweep) the depletion mechanism is *not* a relay-shuttling process.

The generalized dimer problem.— Both the TOA (with tilt) and the Bogolyubov approximation (with

or without a tilt) lead to an effective dimer problem. The technical derivation is provided in the Methods. The dimer Hamiltonian can be written using generators of spin-rotations. Namely, S_z is defined as half the occupation difference in the site representation, while $S_x = (n_0 - n_1)/2 = (N/2) - n$ is half the occupation difference in the momentum orbital representation. Thus, S_x is merely a shifted version of the depletion coordinate. What we call generalized dimer Hamiltonian contains two distinct interaction terms:

$$\mathcal{H}_{\text{dimer}} = -\mathcal{E}S_x - \epsilon S_z + U_{\parallel}S_z^2 + U_{\perp}S_x^2 \quad (8)$$

In the Methods section we show that the TOA reduces to this form with

$$U_{\parallel} = 0, \quad U_{\perp} = -\frac{1}{L}U, \quad [\text{TOA}] \quad (9)$$

In contrast, the Bogolyubov approximations features, due to the pairing interaction,

$$U_{\parallel} = \frac{2}{L}U, \quad U_{\perp} = \frac{1}{4L}U, \quad [\text{Bogolyubov}] \quad (10)$$

The detuning parameter $\mathcal{E}$ reflects the excitation energy of the condensate. For the TOA it is $\mathcal{E} = \mathcal{E}_+ - \mathcal{E}_0$, while for Bogolyubov it is

$$\mathcal{E}(\Phi) = \frac{1}{2}(\mathcal{E}_+ + \mathcal{E}_-) - \mathcal{E}_0 + \frac{NU}{4L} \quad (11)$$

As Φ is increased, $\mathcal{E}$ decreases, and at Φ_{swp} it swaps sign, namely $\mathcal{E}(\Phi_{\text{swp}}) = 0$. The swap location is indicated by green vertical line in the time axis of our figures.

We further show in Methods that the bifurcation scenario depends on the relative magnitudes of the U -s. The parameters $U_{\perp}$ and $U_{\parallel}$ have the same sign (the latter is zero for TOA). Accordingly, phase space contours on the Bloch spheres are ellipses (or parabolas) in the (S_x, S_z) coordinates. If we vary the control parameter $\mathcal{E}(\Phi)$, there are two different bifurcations scenarios depending which interaction is larger. The two scenarios are compared in Fig. 5 and Fig. 6, and further discussed below.

Consider the TOA, for which we have $|U_{\perp}| > |U_{\parallel}|$. For large $\mathcal{E}$ the lowest energy is in the East pole, which supports condensation in orbital $\#0$. As $\mathcal{E}$ is decreased, a bifurcation appears at the West hemisphere, with separatrix that move to the East. This leads eventually to a diabatic ejection of the condensed cloud. We show in the Methods that the pertinent bifurcations happens at

$$\mathcal{E}_c = \left[(|U_{\parallel} - U_{\perp}|N)^{2/3} - \epsilon^{2/3} \right]^{3/2} \quad (12)$$

Consider the Bogolyubov approximation, for which we have $|U_{\perp}| < |U_{\parallel}|$. Here two bifurcations take place: The first bifurcation appears at the West hemisphere, and is formally the same as that of Eq. (12). The same expression for $\mathcal{E}_c$ applies. However, this bifurcation has no significance, as implied by Fig. 5. It is followed by a second

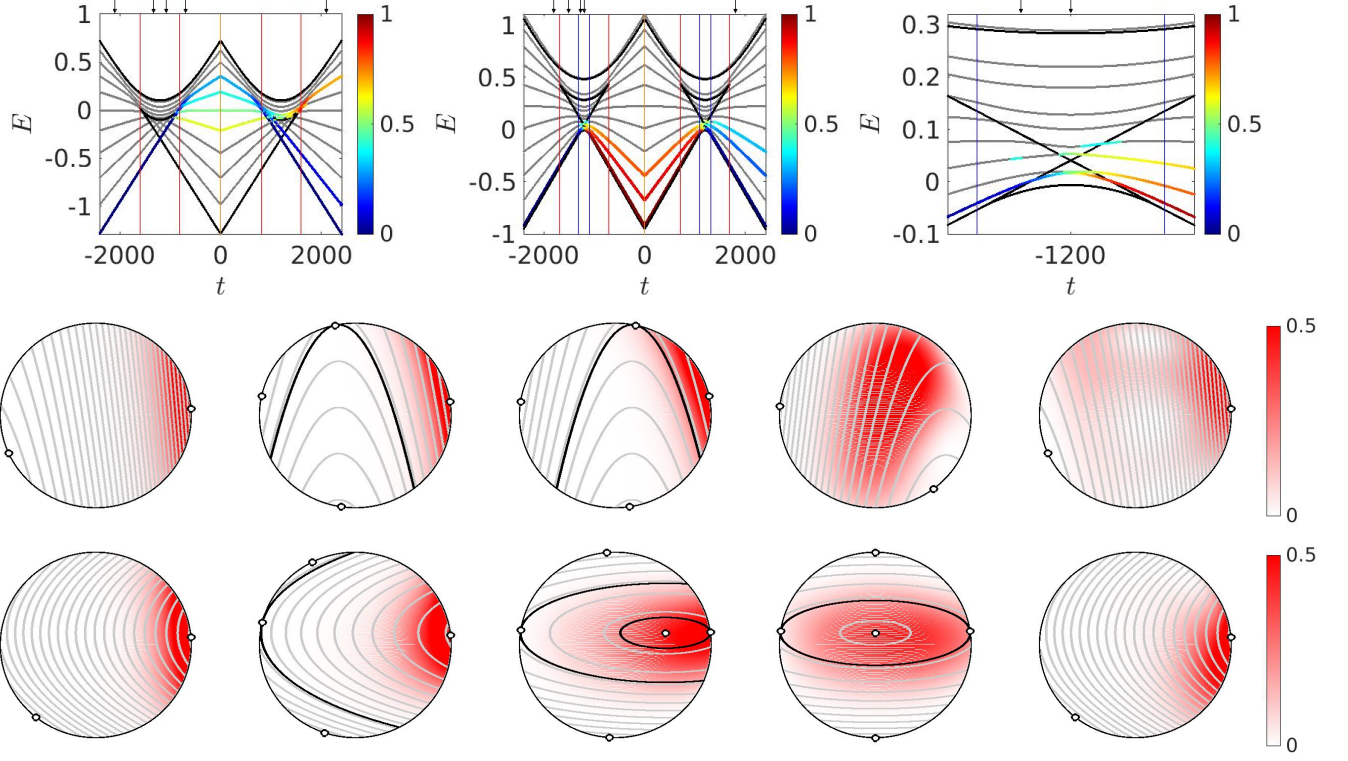


FIG. 5. **Quantum simulations for the dimer.** We consider $N=10$ particles whose dynamics is generated by the Hamiltonian Eq.(8). The upper panels are for diabatic ejection scenario (left), and relay shuttling (middle), and zoom of the latter (right). The units of time are such that $K=1$, and the bias is $\epsilon=0.2$. The interaction parameters are given respectively by Eq.(9) with $NU=3.45$ and by Eq.(10) with $NU=2.3$, with $L=3$. The sweep is from $\mathcal{E}=2$ to $\mathcal{E}=-2$ and back to $\mathcal{E}=2$, with rate $\dot{\mathcal{E}} = 1/600$. Energy levels E_ν are plotted versus time. Levels whose p_ν is vanishingly small are in gray. The other levels are color-coded by $\langle n \rangle_\nu$. The energies of the minima, maxima and separatrices are indicated by black lines. Bifurcation points are indicated by vertical lines. Snapshots of the evolution are taken at times that are indicated by small black arrows, and placed at the 2nd row (diabatic ejection) and at the 3rd row (relay shuttling). At each snapshot we plot the Husimi representation of the quantum state, using (S_x, S_z) phase-space coordinates. We overplot energy contours of the Hamiltonian, and indicate in black the extremal points and the separatrices.

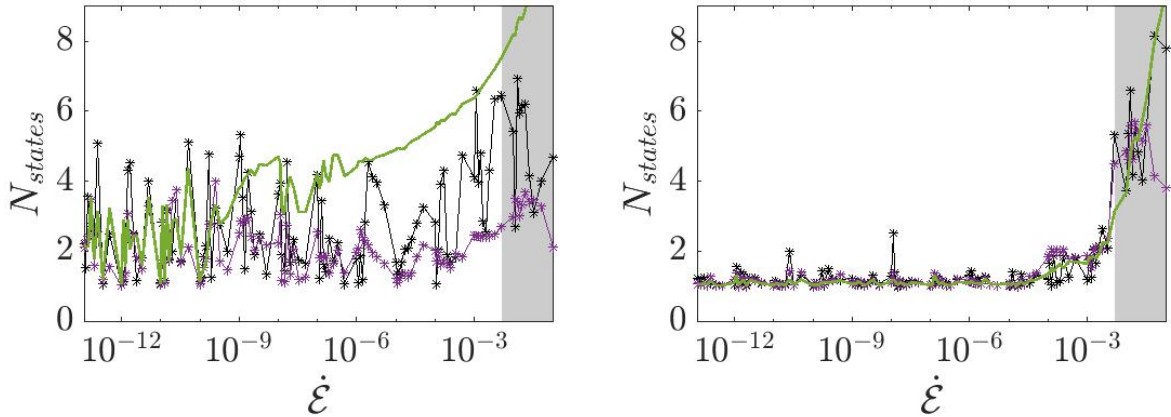


FIG. 6. **Irreversibility vs sweep-rate for the dimer.** For simulations as in Fig.5, we plot N_{states} versus the sweep rate $\dot{\mathcal{E}}$ at the end of the forward sweep (purple), and at the end of the reversed sweep (black). The left panel is for the diabatic ejection scenario, and the right panel is for the relay shuttling scenario. Additionally we plot (in green) the average level index at the end of the forward sweep. The non-quasistatic region (gray) is determined by inspection of N_{orbitals} plot, see SM.

bifurcation of the East pole that for zero bias is determined by the condition $\mathcal{E}(\Phi) = \mathcal{E}_{\text{dyn}}$, where $\mathcal{E}_{\text{dyn}} = NU_{\perp}$. For non-zero bias we derive in [Methods](#) the more general expression

$$\mathcal{E}_{\text{dyn}} = \left[1 - \left(\frac{\epsilon}{U_{\parallel} N} \right)^2 \right]^{1/2} NU_{\perp} \quad (13)$$

This bifurcation signifies the loss of dynamical-stability of the condensate (elliptic fixed-point becomes hyperbolic), and therefore the above condition can be used to determine Φ_{dyn} . Due to the bifurcation a new minimum is born, and a relay-shuttling process is initiated. Subsequently, at Φ_{swp} , there is a swap of separatrices, and consequently, hereafter, the minimum that had bifurcated from the East belongs to the basin of the West. The net effect is relay-shuttling from East to West that ends when $\mathcal{E}(\Phi) = -\mathcal{E}_{\text{dyn}}$. This scenario is illustrated in [Fig.5](#).

Chaos.— Once we go beyond the Bogolyubov approximation, the imbalance M is no longer a constant of motion. Using action angle variables $(n, M, \text{and their conjugates})$ it is possible to express the 3-orbital Hamiltonian as the sum of integrable Bogolyubov term $\mathcal{H}^{(0)}(n, \varphi; M)$ that conserves M , and additional terms $\mathcal{H}^{(\pm)}$ that spoil the integrability. See [\[24\]](#) and the [SM](#) for explicit expressions. The $\mathcal{H}^{(\pm)}$ terms allow slow depletion of the cloud by drifting away from $M=0$. The phase-space landscape is illustrated in [Fig.7](#). It is the inspiration for the caricature in the right panel of [Fig.1](#). The small island that we see in the Poincare section resides above the shuttled cloud. The latter evolves adiabatically to lower energy, and can be located in a Poincare section at a slightly lower energy (not displayed). The chaotic region allows an optional depletion process that we further discuss in the next paragraph.

A necessary condition for chaos-assisted depletion is to have a potential floor that goes down from $n=M=0$ in the $M \neq 0$ direction. This is the Landau criterion for instability of the superflow. Namely, $n=0$ becomes a saddle rather than a local minimum in the energy landscape. For illustration of the crossover, see the additional phase-space landscape plots in [Fig.S1](#) of the [SM](#). The Landau-instability is encountered once we cross Φ_{stb} , which is indicated by the blue vertical line in the time axis of our figures. Bogolyubov analysis [\[24\]](#) provides the explicit expression

$$\Phi_{\text{stb}} = 3 \arccos \left(\frac{1}{6} \left(\sqrt{u^2 + 9} - u \right) \right) \quad (14)$$

where $u = NU/K$ is the dimensionless interaction strength. But we have to remember that only later, at Φ_{dyn} , the $n=0$ location becomes dynamically unstable, as in [Fig.7](#). This means that for $\Phi_{\text{stp}} < \Phi < \Phi_{\text{dyn}}$ only the outer piece of the cloud can drift away from $M=0$ via the chaotic region. The implied branching is clearly

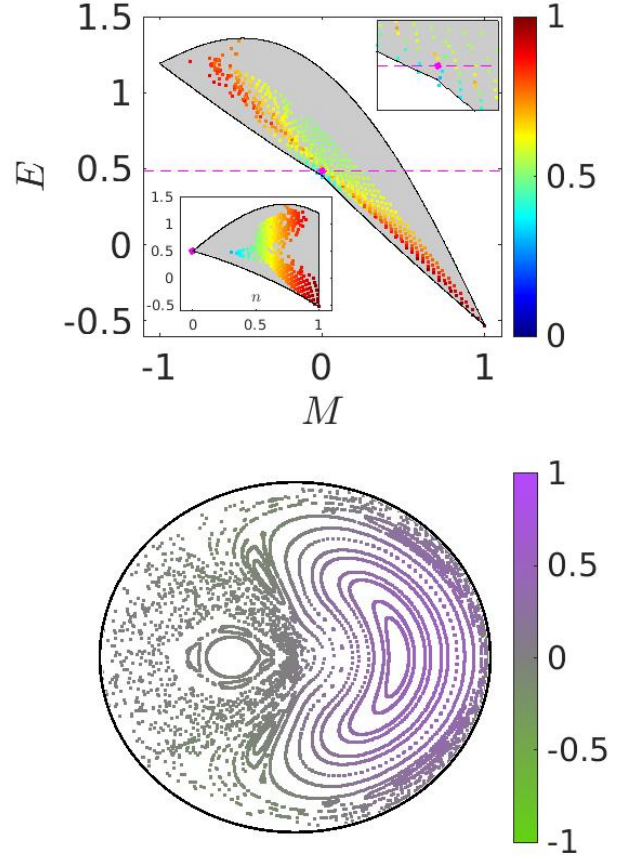


FIG. 7. Mixed chaotic phase space. The upper panel displays the energy landscape of $\mathcal{H}_{\text{trimer}}$ for $u=2.3$ and $\Phi = 1.6\pi$. Each point represents an eigenstate that is positioned according to its E_{ν} and $\langle M \rangle_{\nu}$, and color-coded according to its $\langle n \rangle_{\nu}$. Looking on the classical Hamiltonian $\mathcal{H}(n, \varphi; M, \phi)$, for each M we find the floor (minimum) and the ceiling (maximum) of the energy, and get the Black solid lines that bound the spectrum from below and from above. The $n=M=0$ central fixed-point is indicated by a pink dot. Its vicinity is zoomed in the upper inset, and its energy is indicated by a dashed line. For this energy a Poincare section of phase space is displayed in the lower panel, where each section-point of a trajectory is color-coded by its M , and displayed using its (n, φ) as polar coordinates. We have here $\Phi > \Phi_{\text{dyn}}$, for which the central point is an unstable saddle immersed in chaos. Therefore it cannot support an eigenstate. This observation is better delivered by the lower inset of the upper panel, where the same spectrum is plotted with n serving as horizontal axis.

demonstrated in [Fig.2](#) and optionally in [Fig.S2](#) of the [SM](#).

The splitting of cloud, into an $M=0$ shuttling branch and $M \neq 0$ chaotic spreading, is responsible for the crossover to chaos-assisted depletion. The latter is a very slow process, and therefore becomes noticeable only for very slow sweep rate. It is clearly distinct from shuttling, because it starts earlier, at Φ_{stb} , unlike the shuttling that starts at Φ_{dyn} .

In the reversed sweep we see once again this branching effect. In fact it is more conspicuous on the way back: the cloud stretches further in the M direction, which becomes possible because the ceiling of the potential is going up, hence not blocking further expansion. An optional way to illustrate this branching is provided by Fig.S2 of the SM.

Mechanisms for irreversibility.— In linear response theory (Kubo formalism), irreversibility is related to accumulated deviation from adiabaticity. It is controlled by the ratio between the sweep rate and the natural frequency of the driven system. This picture assumes that the cloud follows an evolving adiabatic-manifold in phase-space. In the quasistatic limit, linear response theory implies reversibility. But this picture breaks down if during the sweep a violent event takes place. In the non-linear LZ problem the local minimum is diminished at a particular moment of the sweep process due to an inverse saddle-node bifurcation, see Fig.1, consequently the cloud is ejected and stretched along the fading separatrix. This is what we call *adiabatic ejection*. On the way back the cloud can split between two regions as implied by the Kruskal-Neishtadt-Henrard theorem [9–19]. This type of dynamics is reflected in the quantum dynamics, see Fig.5 for demonstration.

In the problem under consideration, *adiabatic ejection* is an artefact of the TOA. Instead we find that the Bogolyubov approximation predicts *relay-shuttling*. A gentle type of irreversibility can arise when the shuttling process starts or ends (pitchfork bifurcations). See Fig.S4 of the SM for demonstration. A quantitative comparison of the irreversibility that is associated with the two mechanisms is provided in Fig.6.

As we already discussed, for very slow sweep a different depletion mechanism takes over, that goes beyond Bogolyubov, namely, chaos-assisted depletion. This mechanism gives rise to “free expansion” of the cloud in phase space (M is not constant of motion). Furthermore, once the sweep is reversed the cloud undergoes a conspicuous branching process, as discussed previously for Fig.2 and Fig.S2 of the SM. Thus, irreversibility is extremely enhanced in the semiclassical simulations. Quantitatively this has a modest manifestation in the quantum mechanical case. On the other hand, we observe a novel regime of quantum irreversibility that exhibits “quantum chaos” characteristics and breakdown of QCC that we further discuss below.

Universal Quantum Fluctuations.— Classical evolution of expectation values reflect ergodization. Namely, fluctuations are completely smoothed away if we wait enough time. As opposed to that, quantum fluctuations persist and are not smoothed away. This means that quantum mechanically the quasistatic limit does not exist. At any moment the state of the system cannot be regarded as stationary. In Fig.S6 of the SM we demonstrate the dependence of N_{states} on the waiting time T .

The same fluctuations are reflected if we plot N_{states} versus Φ . We re-emphasize that such fluctuations are absent in the semiclassical simulations. (Therefore we set $T=0$ in the semiclassical simulations of Fig.2.)

QCC and its breakdown.— We already pointed out that the semiclassical dynamics is reflected in the quantum simulations, see Fig.2. The term “reflected” does not imply “correspondence”. We would like to explain the observed breakdown of QCC for slow sweep.

For an extremely slow sweep (that cannot be realized in practice), the quantum dynamics would follow the ground state. This can be regarded as a *quantum detour* of the classical non-adiabatic arena that was looming ahead. For realistic sweep rate the dynamics follows *adiabatically* the metastable minimum. But still the probability can leak to levels that are crossed along the way. This early leakage becomes more probable as the forbidden-area shrinks (low energetic barrier), and definitely once it is replaced by dynamical barriers of the Kolmogorov-Arnold-Moser (KAM) type [29, 30].

The lifetime τ of the condensate can be extracted from the local density of states (LDOS) of the Hamiltonian, see Methods. The interesting range, as explained above, is $\Phi_{\text{stb}} < \Phi < \Phi_{\text{dyn}}$. In this range the classical cloud has a piece that is trapped on a dynamically stable island, and therefore cannot decay. But quantum mechanically the cloud can tunnel through the KAM barriers, and therefore has a finite lifetime $\tau(\Phi)$.

We are now equipped to estimate the border between the various Φ regimes. The quantum adiabatic regime is determined by the standard condition $|d\mathcal{H}/dt| < \kappa^2$, where κ is the tunnel coupling, that determines the level splitting. As discussed earlier this condition is never satisfied in practice due to the smallness of κ . Using $\alpha \equiv |d\mathcal{H}/d\Phi| \sim K$, we can re-write the adiabatic condition as follows,

$$\tau(\Phi) < \frac{\Delta\Phi}{\dot{\Phi}} \quad (15)$$

where $\Delta\Phi = \kappa/\alpha$, is the parametric width of the avoided crossing, and $\tau \sim 1/\kappa$ is the time to make a Rabi transition. We can extend this reasoning to the Fermi-Golden-Rule (FGR) regime where κ becomes larger than the effective levels spacing Δ_0 . The latter refers to the participating levels of the LDOS. There we expect $\tau = 1/\gamma$, with $\gamma = 2\pi\kappa^2/\Delta_0$. The condition for having an escape before Φ_{dyn} is obtained from Eq.(15), and implies a crossover at $\dot{\Phi} \sim 10^{-4}\pi$, in rough agreement with Fig.3.

Discussion

Considering a closed classical Hamiltonian driven system, such as a particle in a box with moving wall (aka the piston paradigm), the common claim in Statistical Mechanics textbooks is that quasistatic processes are adiabatic, with vanishing dissipation, and hence reversible.

This statement is indeed established for *integrable* [1] and for fully *chaotic* systems [2–8]. But generic systems are neither integrable nor completely chaotic. Rather they have *mixed phase space*. For such system the quasistatic limit is not adiabatic [20–24], and therefore we expect irreversibility. This irreversibility can be regarded as the higher-dimensional version of separatrix crossing [9–19], where the so-called Kruskal-Neishtadt-Henrard theorem is followed.

In the present work we wanted not just to expand the analysis of classical irreversibility, but also to explore the quantized version. We asked whether the distinct mechanisms of classical irreversibility are reflected in the quantum mechanical arena, and how this reconciles with the observation that quantum dynamics, unlike classical dynamics, is always reversible in the strict quasistatic (adiabatic) limit. Our main observations are as follows: **(1)** The TOA, and the associate LZ picture, do not provide a proper framework for the analysis of the depletion process. We need at least 3 orbitals in order to capture the essential features of the dynamics. This means that we are dealing here with a “quantum chaos” problem. **(2)** The Bogolyubov approximation, unlike the naive TOA, implies gentle type of irreversibility that is related to relay shuttling, and not to diabatic ejection. **(3)** Beyond the Bogolyubov approximation we have chaos-assisted mechanism that competes with the relay shuttling process. This mechanism becomes dominant in the deeper quasistatic regime. **(4)** Accordingly, with regard to the sweep rate, one has to distinguish between non-quasistatic regime; relay-shuttling regime; chaos-assisted regime; and quantum adiabatic regime. For a manybody condensate, the latter is not accessible in practice. **(5)** Quantum features dominate the quantum adiabatic regime and the chaos-assisted regime. The most prominent effect can be described as a version of universal quantum fluctuations. **(6)** In the same regime, breakdown of QCC is conspicuous. It is related to leakage of probability along the diabatic transitions. Such leakage does not exist in the semiclassical simulations.

On the practical side one observes that the optimization of a protocol is related to the crossovers between the various regimes. Sweeping a control parameter ‘too fast’ takes us out of the quasistatic regime, while ‘too slow’ is affected by chaos. UQF possibly can be exploited for fine-tuning, whose purpose is to minimized chaos-related irreversibility. In analogy with the claim that diagonalization of the Hamiltonian can provide in “one shot” phase-space tomography [71], also here we can say that relatively cheap quantum simulations, can provide information on the classical dynamics for a cloud of trajectories.

Methods

Participating orbitals.— The one-particle reduced probability matrix that is associated with a manybody state is $\rho_{k',k} = (1/N) \langle \mathbf{b}_k^\dagger \mathbf{b}_{k'} \rangle$. We define

$$N_{\text{orbitals}} = \left[\text{Tr}(\rho^2) \right]^{-1} \quad (16)$$

This is a measure for fragmentation. For a manybody coherent state $N_{\text{orbitals}}=1$, meaning that all the particles are condensed in a single orbital. Semiclassically, such state can be pictured as a localized Gaussian-like distribution in phase-space. It is important to realize that the at the end of a relay shuttling process we get $N_{\text{orbitals}}=1$ in the reduced dimer representation, but $N_{\text{orbitals}}=2$ in the proper trimer representation, reflecting a Twin Fock state (half of the particles in each orbital). At the swap we have $N_{\text{orbitals}}=3$. The **SM** provides plots of $N_{\text{orbitals}}(t)$ and $N_{\text{states}}(t)$ for the protocols that are discussed in the main text.

BHH interaction term for a dimer.— The interaction term of Eq.(2) for an $L=2$ dimer, disregarding a constant is

$$\frac{U_o}{2} n_+ n_0 + \frac{U_{\parallel}}{4} \left(b_+^\dagger b_+^\dagger b_0 b_0 + \text{h.c.} \right) \quad (17)$$

with $U_o=U_{\parallel}=U$. Making the substitution $b_j \mapsto \sqrt{n_j} e^{i\varphi_j}$, and $n_0 = N-n$, and $n_+ = n$, we get

$$\frac{U_o}{2} (N-n)n + \frac{U_{\parallel}}{2} (N-n)n \cos(2\varphi) \quad (18)$$

BHH interaction term for a ring.— Consider an $L > 2$ ring. Let us exclude from the interaction term of Eq.(2) scattering events that involve 4 different orbitals. Dropping a constant, we are left with

$$\frac{U}{L} \left[\sum_{\langle k, k' \rangle} n_k n_{k'} + \sum_{k=0, \pm} \left(b_{k+1}^\dagger b_{k-1}^\dagger b_k b_k + \text{h.c.} \right) \right] \quad (19)$$

where the summation $\langle k, k' \rangle$ is over pairs, excluding double counting. The Bogolyubov approximation is obtained if we keep in the second term only the $k=0$ transitions. Then we get

$$\frac{U}{L} \left[n_0 n_+ + n_0 n_- + n_+ n_- + n_0 \sqrt{n_+ n_-} \cdot 2 \cos(2\varphi) \right] \quad (20)$$

Using n and M as coordinates this expression takes the form

$$\begin{aligned} & \frac{U}{L} \left[(N-n)n + \frac{1}{4}(n^2 - M^2) \right] \\ & + \frac{U}{L} (N-n) \sqrt{n^2 - M^2} \cos(2\varphi) \end{aligned} \quad (21)$$

Setting $M=0$, the above terms are formally the same as that of the dimer, provided we allow different coefficients

$U_o = (3/(2L))U$ and $U_{||} = (2/L)U$, and include the term $NU/(4L)$ in Eq.(11). On the other hand, if we keep in Eq.(20) only the n_0 and the n_+ , we get the TOA where $U_o = (2/L)U$ and $U_{||} = 0$.

The effective dimer Hamiltonian.— One can write the interaction term Eq. (17) using generators of spin-rotations. We define $S_x = (n_+ - n_0)/2$, while $n_+ + n_0 = N$, and use the identity

$$S_z^2 - S_y^2 = \frac{1}{2} (b_+^\dagger b_+^\dagger b_0 b_0 + \text{h.c.}) \quad (22)$$

In order to get rid of S_y^2 we exploit that $S_x^2 + S_y^2 + S_z^2$ is a constant of motion. Thus the final expression can be written as in Eq.(8), with $U_{\perp} = (U_{||} - U_o)/2$.

The Husimi representation.— For a dimer, the coherent states are related as follow to the Fock states $|n\rangle$,

$$|\theta, \varphi\rangle = \sum_{n=0}^N \sqrt{\binom{N}{n}} \left[\cos \frac{\theta}{2} \right]^{N-n} \left[\sin \frac{\theta}{2} \right]^n e^{in\varphi} |n\rangle \quad (23)$$

The Husimi function is:

$$Q(\theta, \varphi) = |\langle \theta, \varphi | \psi \rangle|^2 \quad (24)$$

If $n = (N/2) - S_z$ were the occupation-coordinate in the position (site) basis, then $\theta=0$ would be located at North pole. But we have defined $n = (N/2) - S_x$ as the occupations-coordinate in the momentum (orbital) basis. Therefore our $n=0$ is located at the East pole, which is re-defined as the origin for θ . Accordingly $S_x = (N/2) \cos(\theta)$. We plot images of the Husimi function using (S_x, S_z) coordinates.

Bifurcations.— The (S_x, S_z) contour lines of the Hamiltonian Eq.(8) are ellipses that are chopped by the circle $S_x^2 + S_z^2 = (N/2)^2$. If the circle is ignored, the minimum is at

$$(S_x, S_z) = \left(\frac{\mathcal{E}}{2U_{\perp}}, \frac{\epsilon}{2U_{||}} \right) \quad (25)$$

In the relay shuttling scenario, as $\mathcal{E}$ is varied, a bifurcation takes place at the East pole once this minimum enters into the circle. This happens at Eq.(13). In the adiabatic ejection scenario the relevant bifurcation happens on the bounding circle: before the bifurcation we have on the circle one minimum and one maximum; after the bifurcation a secondary minimum and an associated saddle point appears. In order to find the bifurcation we define the function

$$h(\theta) = H \left(S_x := \frac{N}{2} \cos(\theta), S_z := \frac{N}{2} \sin(\theta) \right) \quad (26)$$

Then we write the equations $h'(\theta) = 0$ and $h''(\theta) = 0$, for the first and the and second derivatives, as

required at the bifurcation point. The combined equations $\sin \theta h''(\theta) - \cos \theta h'(\theta) = 0$ and $\cos \theta h''(\theta) + \sin \theta h'(\theta) = 0$ are solved to get Eq.(12).

TOA vs Bogolyubov.— One wonders whether the discussion of “Nucleation in finite topological systems during continuous metastable quantum phase transitions” [69] is flawed. In order to answer this question we have to appreciate the physical significance of the continuum limit $L \rightarrow \infty$ that was considered there. It is physically clear that “rotation” of a flat clean ring (that has neither tilt nor lattice potential) is an empty notion: nothing changes in the Hamiltonian. Furthermore, in this limit, chaos is not an issue (the $L \rightarrow \infty$ is integrable). The physics that we discuss becomes relevant as L becomes finite, and irreversibility is most pronounced for $L=3$.

Still one may insist to adopt TOA for a finite L ring. How the results would be in comparison with the correct picture? Looking in Fig.3 of [69] we see that the interest there is in simple adiabatic shuttling along the upper level, during which no bifurcation occurs. In this energy range there is no major difference between the TOA and the Bogolyubov versions, as we see from looking on the higher levels in Fig.5. But for the scenario that we consider, starting at $n=0$, the TOA completely fails. Demonstration of this colossal failure is provide in Fig.S3 that is placed in the SM.

Quantum stability of the condensate.— For a frozen value of Φ we perform simulations whose purpose is to monitor the stability of the quantum condensate. The interest is in the regime $\Phi_{\text{stb}} < \Phi < \Phi_{\text{dyn}}$. In this regime the classical cloud has a piece that is trapped on a dynamically stable island, and therefore cannot decay. But quantum mechanically the cloud can tunnel through the KAM barriers, and therefore has a finite lifetime τ . The survival probability $P(t) = |\langle \Psi(0) | \Psi(t) \rangle|^2$ of the condensate has been found for representative values of Φ . From that τ has been extracted. In the range of interest, for our choice of parameters, $\tau \sim 90$. The survival amplitude is related to the LDOS via a Fourier transform, and therefore one can say that we employ here an LDOS based determination of τ .

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Quantum irreversibility of quasistatic protocols (Supplementary Material)

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This supplementary contains a somewhat expanded version of the Methods: the Hamiltonian in momentum representation; the TOA and the Bogolyubov approximation; and the relation to the generalized dimer Hamiltonian. We also provide a few snapshots of phase-space, to clarify how it changes during the sweep process; and additional (optional) plots that illustrate the dynamics that has been discussed in the main text.

===== [1] The BHH - standard representation

The dimer Hamiltonian is

$$\mathcal{H}_{\text{dimer}} = \sum_{j=0,1} \left[\epsilon_j a_j^\dagger a_j + \frac{U}{2} a_j^\dagger a_j^\dagger a_j a_j \right] - \frac{K}{2} (a_1^\dagger a_0 + a_0^\dagger a_1) \quad (\text{S-1})$$

where K is hopping amplitude, U is the on-site interaction, and $a_j^\dagger$ and a_j are the creation and annihilation operators, and ϵ_j is the on-site potential. The L site ring Hamiltonian is

$$\mathcal{H}_{\text{ring}} = \sum_{j=0}^{L-1} \left[\epsilon_j a_j^\dagger a_j + \frac{U}{2} (a_j^\dagger a_j^\dagger a_j a_j) - \frac{K}{2} (e^{i\frac{\Phi}{L}} a_{j+1}^\dagger a_j + e^{-i\frac{\Phi}{L}} a_j^\dagger a_{j+1}) \right] \quad (\text{S-2})$$

where Φ is a Coriolis gauge-field due to rotation. If the ring is tilted relative to the gravitational field of Earth, the on site potential is

$$\epsilon_j = -\epsilon \cos\left(\frac{2\pi}{L} x_j\right), \quad \text{where } x_j = j \quad (\text{S-3})$$

Note that the Fourier transformed potential is

$$\sum_j \epsilon_j \exp\left(i\frac{2\pi}{L} qx_j\right) = -\frac{\epsilon}{2} L \delta_{q,\pm 1} \quad (\text{S-4})$$

===== [2] The BHH - momentum representation

In momentum representation the Hamiltonian of the dimer takes the form

$$\mathcal{H}_{\text{dimer}} = \sum_{k=0,1} \mathcal{E}_k n_k - \frac{\epsilon}{2} (b_0^\dagger b_1 + b_1^\dagger b_0) + \frac{U_o}{4} ((N-1)N + 2n_0 n_1) + \frac{U_{||}}{4} (b_1^\dagger b_1^\dagger b_0 b_0 + \text{h.c.}) \quad (\text{S-5})$$

where $U_o = U_{||} = U$. For an L site ring we get

$$\mathcal{H}_{\text{ring}} = \sum_{k=0}^{L-1} \mathcal{E}_k b_k^\dagger b_k - \frac{\epsilon}{2} \sum_{k,\pm} b_{k\pm 1}^\dagger b_k + \frac{U}{2L} \sum_{k_1, k_2, k_3, k_4}' b_{k_1}^\dagger b_{k_2}^\dagger b_{k_3} b_{k_4} \quad (\text{S-6})$$

The momentum index k can be defined mod(L) such that $k := (2\pi/L)k$ is the quasi momentum in standard units. For a trimer this index takes the values $k = 0, \pm 1$ or shortly $k = 0, \pm$. The prime in the k summation implies that conservation of total momentum is required. The interaction term can be arranged as follows:

$$\frac{1}{2} \sum_k' b_{k_1}^\dagger b_{k_2}^\dagger b_{k_3} b_{k_4} = \frac{(N-1)N}{2} + \sum_{\langle k, k' \rangle} n_k n_{k'} + \text{pairing} + \text{scattering} \quad (\text{S-7})$$

The second term reflects the cost of fragmentation. The summation is over pairs (without double counting). Additionally there are scattering events that involve 4 different orbitals, and pairing events that involve 3 orbitals (two k particles split into $k \pm q$ orbitals, and vice versa). In the special case $L=3$, the 4-orbitals scattering events are absent, and we write

$$\mathcal{H}_{\text{trimer}} = \sum_{k=0,\pm} \mathcal{E}_k b_k^\dagger b_k - \frac{\epsilon}{2} \sum_{\langle k, k' \rangle} b_k^\dagger b_{k'} + \frac{U}{3} \left[\frac{(N-1)N}{2} + \sum_{\langle k, k' \rangle} n_k n_{k'} + \sum_{k=0,\pm} (b_{k+1}^\dagger b_{k-1}^\dagger b_k b_k + \text{h.c.}) \right] \quad (\text{S-8})$$

The Bogolyubov approximation is obtained if we eliminate the $k \neq 0$ terms of the pairing events.

===== [3] The dimer Hamiltonian

We define the depletion coordinate as $n_1 = n$ such that the occupation of the source orbital is $n_0 = N - n$. Then the dimer Hamiltonian can be written in action-angle variables as follows:

$$\mathcal{H}_{\text{dimer}} = E_0 + \mathcal{E}n - \epsilon\sqrt{(N-n)n}\cos(\varphi) + \frac{U_o}{2}(N-n)n + \frac{U_{\parallel}}{2}(N-n)n\cos(2\varphi) \quad (\text{S-9})$$

where $E_0 = N\mathcal{E}_0 + (U/4)(N-1)N$ is a constant that can be dropped, and $\mathcal{E} = K$ is the detuning of the two orbitals. Note that $U_o = U_{\parallel} = U$.

An optional representation is based on the definition of spin-rotation generators, such that S_z is half the occupation difference in the site representation, while $S_x = (n_0 - n_1)/2 = (N/2) - n$ is half the occupation difference in the momentum orbital representation. Thus S_x is merely a shifted version of the depletion coordinate. In this language, dropping a constant, the dimer Hamiltonian takes the form $\mathcal{H}_{\text{dimer}} = -\mathcal{E}S_x - \epsilon S_z + US_z^2$. Later we describe the depletion process for an L site ring using an effective dimer Hamiltonian that has the following generalized form:

$$\mathcal{H}_{\text{dimer}} = \text{const} - \mathcal{E}S_x - \epsilon S_z - \frac{U_o}{2}S_x^2 + \frac{U_{\parallel}}{2}S_{\Delta}^2 \quad (\text{S-10})$$

$$= \text{const} - \mathcal{E}S_x - \epsilon S_z + U_{\parallel}S_z^2 + U_{\perp}S_x^2 \quad (\text{S-11})$$

The first version is the generalized dimer Hamiltonian in the momentum (orbital) representation, where the pairing interaction term is:

$$S_{\Delta}^2 \equiv S_z^2 - S_y^2 = \frac{1}{2} \left(b_1^{\dagger} b_1^{\dagger} b_0 b_0 + \text{h.c.} \right) \quad (\text{S-12})$$

The second version is obtained by exploiting that $S_x^2 + S_y^2 + S_z^2$ is a constant of motion, with the identification $U_{\perp} = (U_{\parallel} - U_o)/2$. For the ordinary dimer $U_{\perp} = 0$ and $U_o = U_{\parallel} = U$, with $U > 0$ without loss of generality. For naive TOA we have $U_{\parallel} = 0$, and $U_{\perp} = -U_o/2$ is negative. For the Bogolyubov approximation we have $0 < U_{\perp} < U_{\parallel}$.

===== [4] The naive TOA

As far as U is concerned, naive TOA for any ring ($L > 2$) gives no hopping. Later we focus of the trimer ($L = 3$) for numerical demonstrations. Keeping only the two lowest orbitals, we define $\mathcal{E} = \mathcal{E}_1 - \mathcal{E}_0$, and use the notations $n_1 = n$ and $n_0 = N - n$. We get

$$\mathcal{H}_{\text{ring}} \approx \mathcal{E}n - \frac{\epsilon}{2}(b_1^{\dagger}b_0 + b_0^{\dagger}b_1) + \frac{U}{L}(N-n)n \quad (\text{S-13})$$

This is formally like the dimer Hamiltonian $\mathcal{H}_{\text{dimer}}$ with

$$\mathcal{E} = \mathcal{E}_1 - \mathcal{E}_0 \quad (\text{S-14})$$

$$U_o = \frac{2U}{L} \quad (\text{S-15})$$

$$U_{\parallel} = 0 \quad (\text{S-16})$$

We can compare it to the approximation that [69] is using for an continuous ring of length $2\pi R \equiv La$. to get this limit the lattice constant a should be taken to zero, keeping La constant. In this limit $K = (ma^2)^{-1}$ is related to the mass of the particle. The gauge field is $\Phi = (\pi R^2) \times 2m\Omega$, where Ω is the rotation frequency. The single particle energies are

$$\mathcal{E}_k = \frac{1}{2mR^2} (k - m\Omega R^2)^2 \quad (\text{S-17})$$

Hence, up to a constant, $\mathcal{E}$ is identified as the rotation frequency:

$$\mathcal{E} = \mathcal{E}_1 - \mathcal{E}_0 = \frac{1}{2mR^2} - \Omega \quad (\text{S-18})$$

[5] The Bogolyubov approximation

We go to action angle variables, with focus on the condensation orbital. We define the depletion coordinate n and the imbalance coordinates M . Then we write the Hamiltonian such that $\mathcal{H}^{(0)}$ is the Bogolyubov approximation,

$$\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(\epsilon)} + \mathcal{H}^{(+)} + \mathcal{H}^{(-)} \quad (\text{S-19})$$

Dropping a constant the interaction term in $\mathcal{H}^{(0)}$ is

$$= \frac{U}{L} [n_0(n_+ + n_-) + n_+n_- + n_0\sqrt{n_+n_-} \cdot 2\cos(2\varphi)] \quad (\text{S-20})$$

$$= \frac{U}{L} \left[(N-n)n + \frac{1}{4}(n^2 - M^2) + (N-n)\sqrt{n^2 - M^2} \cos(2\varphi) \right] \quad (\text{S-21})$$

Thus we get

$$\mathcal{H}^{(0)} = E_0 - \frac{U}{12}M^2 + \mathcal{E}_\perp M + \mathcal{E}_n + \frac{U_o}{2} (N-n)n + \frac{U_\parallel}{2} (N-n)\sqrt{n^2 - M^2} \cos(2\varphi) \quad (\text{S-22})$$

with

$$U_o = \frac{3}{2L}U \quad (\text{S-23})$$

$$U_\parallel = \frac{2}{L}U \quad (\text{S-24})$$

and

$$\mathcal{E} = \frac{1}{2}(\mathcal{E}_+ + \mathcal{E}_-) - \mathcal{E}_0 + \frac{NU}{4L} \quad (\text{S-25})$$

$$\mathcal{E}_\perp = \frac{1}{2}(\mathcal{E}_+ - \mathcal{E}_-) \quad (\text{S-26})$$

For $M = 0$ the $\mathcal{H}^{(0)}$ of Eq.(S-22) is formally like Eq.(S-9) of the generalized dimer. For illustration we write the explicit expression for the additional (non Bogolyubov) terms for the $L=3$ trimer:

$$\mathcal{H}^{(\epsilon)} = \epsilon \left[\frac{1}{2}\sqrt{n^2 - M^2} \cos(2\phi) + \frac{1}{\sqrt{2}}\sqrt{(N-n)(n \pm M)} \cos(\phi \pm \varphi) \right] \quad (\text{S-27})$$

$$\mathcal{H}^{(\pm)} = \frac{U}{3\sqrt{2}\sqrt{(N-n)(n \pm M)}}(n \mp M) \cos(3\phi \mp \varphi) \quad (\text{S-28})$$

For $L > 3$ there are additional non-Bogolyubov terms for scattering events that involve 4 different orbitals.

[6] The variation of phase space

The following figure is a modified version of Fig.S1 of [24]. Its purpose is to clarify how phase-space changes as Φ is varied. Snapshots are taken after Φ_{mts} , after Φ_{stb} , and after Φ_{dyn} . It shows how the $n=0$ fixed-point changes from metastable minimum to elliptic fixed-point and then becomes unstable. We also have an indication for the emerging shuttling island. The small island that we see in panel (c) is in fact a section of torus that resides above the captured cloud. The latter evolves adiabatically to lower energy, and therefore its identification requires a Poincare section at a slightly lower energy.

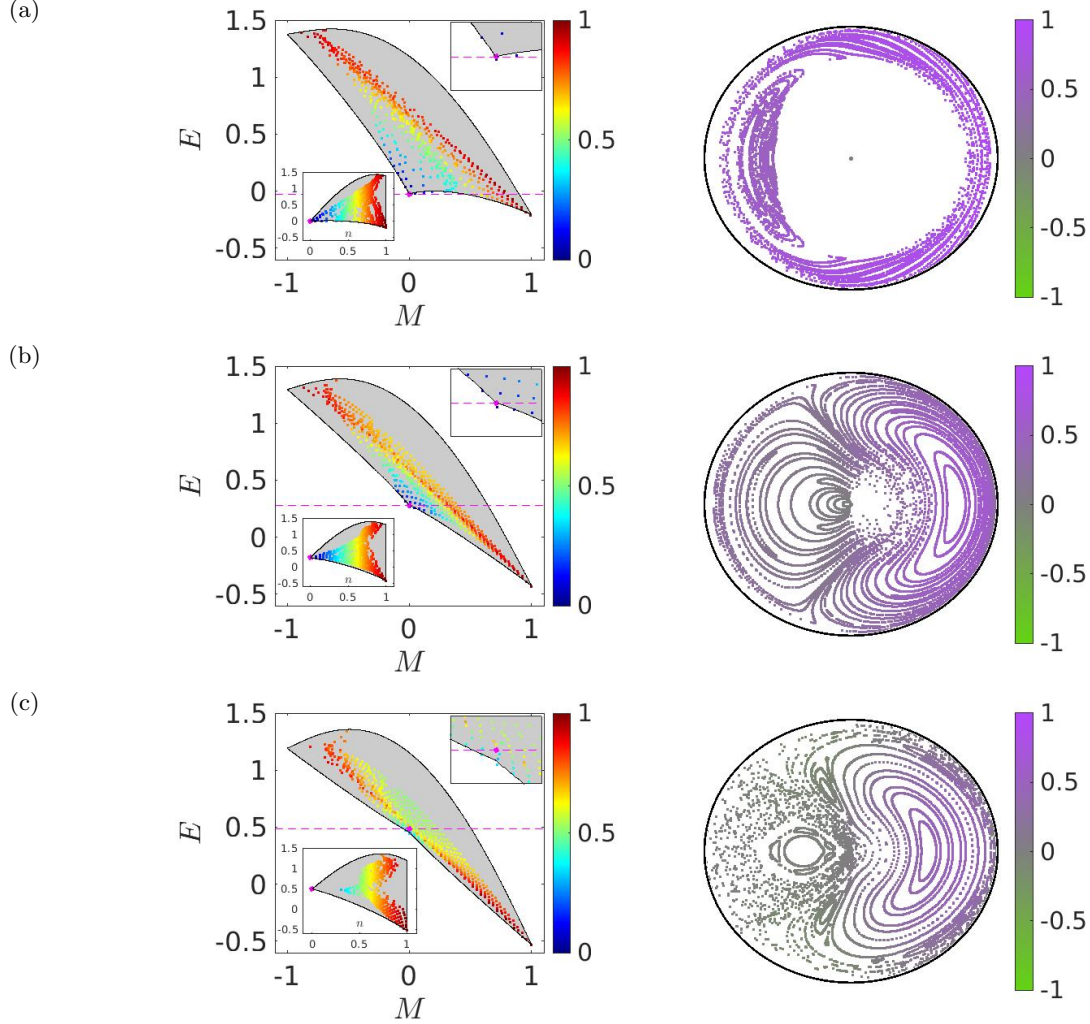


FIG. S1. **Mixed chaotic phase space.** The energy landscape of $\mathcal{H}_{\text{trimer}}$ for $u=2.3$. Panels (a)-(c) are for $\Phi = 1.1\pi, 1.4\pi, 1.6\pi$. Panel (c) is the same as Fig.7 of the main text. It is contrasted with panel (a) that illustrates energetic stability, and panel (b) that illustrates dynamical stability. For further details see the caption of Fig.7.

[7] The branching of the cloud

In the following figure we show how the evolution of the cloud of Fig.2 look like in occupation space, using n, M coordinates. This figure provides an optional view of the branching: one piece of the cloud drifts away from $M=0$ starting at Φ_{stb} , and another piece is shuttles along $M=0$ starting at Φ_{dyn} . The branching is visible only for very slow sweep. In the forward sweep the drift stops after a short duration because the ceiling of the potential is going down, hence blocking further expansion. But in the reversed sweep the ceiling of the potential is going up, and therefore the branching becomes conspicuous.

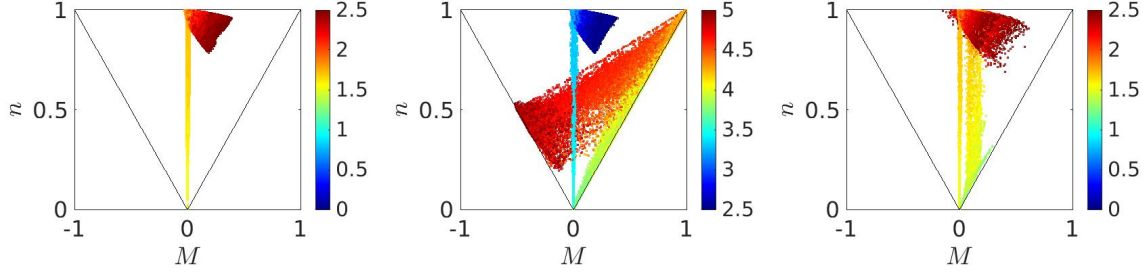


FIG. S2. **Evolution of the cloud in occupation space.** Optional plots for the semiclassical simulations of Fig.2. The left and the middle panels are for the forward and for the reversed sweep, with the optimal sweep rate $\dot{\Phi}=5\pi \cdot 10^{-4}$. The right panel is for the very slow forward sweep with $\dot{\Phi}=5\pi \cdot 10^{-5}$.

[8] Simulations with a biased ring

The following figure provides additional panels for Fig.2. We compare the dynamics that is generated by $\mathcal{H}$ with the dynamics that is generated using TOA. Namely, in the TOA Hamiltonian we keep just two momentum orbitals. Without bias the TOA Hamiltonian is identical with the $U=0$ Hamiltonian, and therefore its failure is trivial (not displayed). We therefore add bias $\epsilon \neq 0$ as in [69]. We see that the TOA completely fails to reproduce the dynamics.

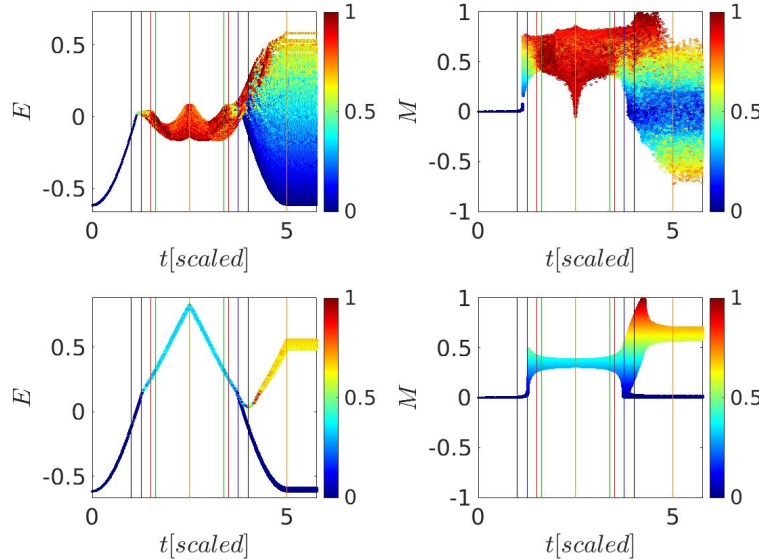


FIG. S3. **Simulations with a biased ring.** These are additional panels for Fig.2 of the main text. The parameters are the same as for the left panels there ($\dot{\Phi}=5\pi \cdot 10^{-4}$), with added bias $\epsilon=0.1$. The upper panels are generated with the full Hamiltonian, while the lower panels use TOA.

===== [9] Depletion and spreading as a function of time

The following figures provide examples for the temporal variation of $\langle n \rangle$ and N_{states} and N_{orbitals} . Fig.S4 is for the dimer simulations, while Fig.S5 and Fig.S5 are for the trimer. Fig.S4 demonstrates that relay shuttling is rather reversible. As opposed to that, in diabatic ejection we have splitting in the reversed sweep, which is reflected in N_{states} and N_{orbitals} , and also spoils $\langle n \rangle$. In Fig.S5 we include a black line that is generated by the Bogolyubov-approximated Hamiltonian $\mathcal{H}^{(0)}$. This approximation is formally equivalent to the relay-shuttling scenario of Fig.S4. Note that its t_d agree with the blue line, but not with the red line (very slow sweep), reflecting that different depletion scenarios are involved.

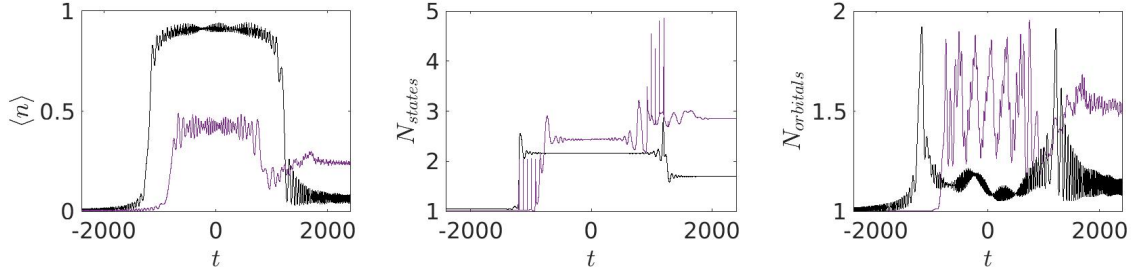


FIG. S4. **Depletion vs time for the dimer.** The depletion $\langle n \rangle$, and N_{states} and N_{orbitals} , are plotted as a function of time for diabatic ejection scenario (purple) and for relay shuttling (black). Simulations parameters are as in Fig.5. There is no relation between the two scenarios: they are combined in one plot for presentation purpose. The only meaningful comparison concerns the question whether the reversed sweep is capable of restoring the initial state.

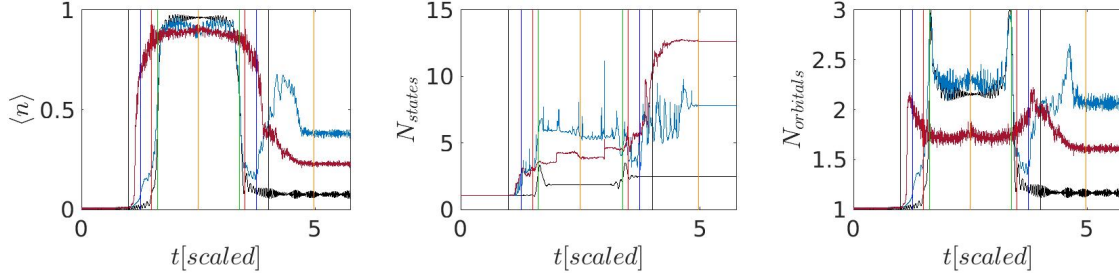


FIG. S5. **Depletion vs time for the trimer.** The depletion $\langle n \rangle$, and N_{states} and N_{orbitals} , are plotted as a function of time for $\dot{\Phi} = 5\pi \cdot 10^{-4}$ (blue), and for very slow rate $\dot{\Phi} = 3.33\pi \cdot 10^{-7}$ (red). The black line is generated with the Bogolyubov-approximated Hamiltonian $\mathcal{H}^{(0)}$ for $\dot{\Phi} = 5\pi \cdot 10^{-4}$. The other model parameters and the vertical lines are as in Fig.2.

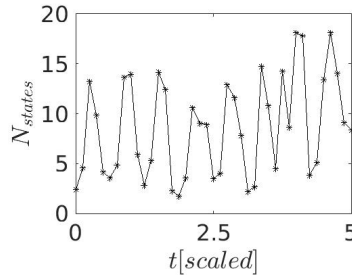


FIG. S6. **Irreversibility vs Waiting time.** This is an additional panel for Fig.4. It illustrates the erratic dependence of N_{states} on the waiting time for $\dot{\Phi} = 5\pi \cdot 10^{-6}$. In the main-text figure a few values of N_{states} are sampled for each $\dot{\Phi}$.