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Energy absorption by 'sparse' systems: beyond linear response theory

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Abstract
The analysis of the response to driving in the case of weakly chaotic or weakly interacting systems should go beyond linear response theory. Due to the 'sparsity' of the perturbation matrix, a resistor-network picture of transitions between energy levels is essential. The Kubo formula is modified, replacing the 'algebraic' average over the squared matrix elements by a 'resistor-network' average. Consequently, the response becomes semi-linear rather than linear. Some novel results have been obtained in the context of two prototype problems: the heating rate of particles in billiards with vibrating walls; and the Ohmic Joule conductance of mesoscopic rings driven by electromotive force. The results obtained are contrasted with the 'Wall formula' and the 'Drude formula'.

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(Some figures may appear in color only in the online journal)

1. Introduction

This paper is about driven systems, like those illustrated in figure 1, whose dynamics is generated by a Hamiltonian that is represented by a matrix that has the generic structure

$$\mathcal{H}_{\text{total}} = \text{diag}\{E_n\} - f(t)\{V_{nm}\}. \tag{1}$$

Here E_n are the ordered energy levels of the unperturbed system. Their density $\varrho(\text{states/energy})$ is assumed to be roughly uniform. The system is driven by a low-frequency stationary driving source $f(t)$. The elements of the perturbation matrix are V_{nm}. The induced transitions have rates that are proportional to $|V_{nm}|^2$. We define

$$\boldsymbol{X} = \{|V_{nm}|^2\}. \tag{2}$$

Our interest is in Hamiltonians that have a 'sparse' perturbation matrix. This means that the majority of elements in $\boldsymbol{X}$ are small. To be more precise, we assume that these elements have a log-wide distribution with a *median* that is much smaller compared with the *average*. An example of such a matrix is given in figure 2, and a typical histogram of the elements is presented in figure 3.

The question we ask is simple: given $\boldsymbol{X}$, what is the calculation that should be performed in order to obtain the energy absorption rate (EAR). It makes sense that the result should be proportional to some weighted average $\langle\!\langle \boldsymbol{X} \rangle\!\rangle$ over the matrix elements. Indeed, within the framework of linear response theory (LRT) the Kubo formula is doing just that—a weighted *algebraic* average.

One should realize that the use of the Kubo formula for EAR calculation can be justified only in the very weak driving limit, provided that there is a background 'bath' that maintains quasi-equilibrium at any moment. However, if the driving is not very weak, compared with the relaxation, then one should be worried: in order for the system to heat up, it is essential to have *connected* sequences of transitions; else the system is 'stuck'. There is an obvious analogy here with a resistor-network calculation: due to the sparsity the energy absorption somewhat resembles a percolation process.

The bottom line of the above considerations is that the *algebraic* average of Kubo $\langle\!\langle \boldsymbol{X} \rangle\!\rangle_{\text{a}}$ should be replaced by a resistor-network average $\langle\!\langle \boldsymbol{X} \rangle\!\rangle_{\text{s}}$, whose value is much smaller if the matrix is 'sparse'. This should be regarded as an *anomaly* in the theory of response: it is an effect that arises upon quantization. Namely, one can characterize the sparsity of $\boldsymbol{X}$ by a parameter $0 < s < 1$ that is absent in the classical context, but has a dramatic effect in the quantum analysis.

A few words are in order regarding the literature. We go here beyond the conventional random matrix theory (RMT)

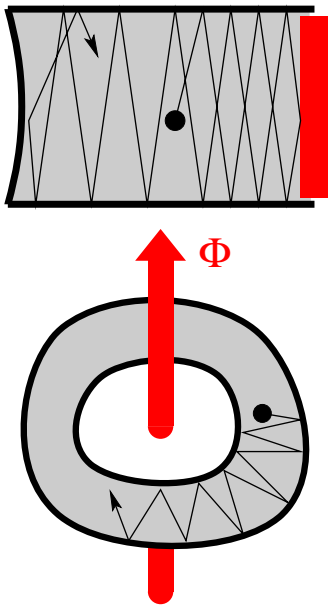

Figure 1. Model systems: a billiard with a moving wall (upper panel) and a ring with a time-dependent magnetic flux (lower panel). The deviation of the billiard from integrability is quantified by a parameter u. It is due to a deformation of the boundary (as in the figure) or due to a deformation of the potential floor (not shown). In the lower panel the ring is regarded as a rectangular-like billiard with periodic boundary conditions in one of its coordinates. In the numerics the ring has been modeled as a tight binding array of dimensions $L \times \mathcal{M}$. In the latter case the non-integrability was due to on-site disorder W.

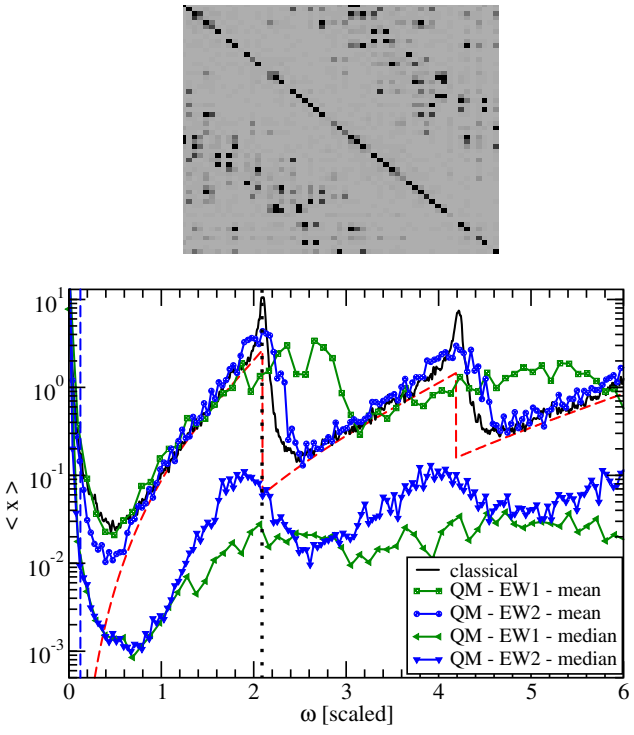

Figure 2. The upper panel is an image of the matrix $\boldsymbol{X}$ for a billiard very similar in shape to that of figure 1. This matrix is 'sparse'. This can be deduced either by inspection or by looking in the lower panel where the *average* value $\langle x \rangle$ of the elements (upper curves) is calculated as a function of the distance ω from the diagonal, and contrasted with the much smaller *median* values (lower curves). For further details see [39, 40].

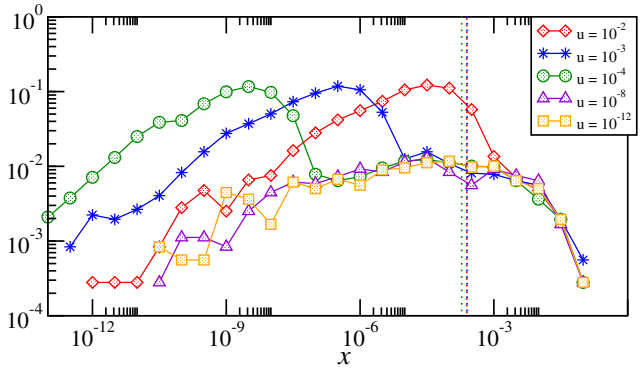

Figure 3. Histogram of the in-band elements x of the matrix $\boldsymbol{X}$ for a rectangular billiard that has deformed potential floor. Different symbols refer to different values of the deformation parameter u. The vertical line is the *average* value of the elements, while the *median* is roughly at the locations of the peaks. The former, unlike the latter, is not sensitive to the degree of deformation. For details see [38].

perspective of [1, 2], because we are dealing with 'sparse' matrices [3–6], possibly banded [7–12]. Our view of LRT follows that of [13–20]. In the context of billiards, LRT implies the 'Wall formula' [21–26], while in the context of mesoscopic conductance LRT implies the 'Drude formula'[1]. In both cases one should take into account *corrections* that are related to correlations and level statistics. The quest for *anomalies* that cannot be explained by introducing corrections within the framework of LRT, but go beyond LRT, has some history [18, 19, 28–32]. The line of study regarding the anomaly that arises due to 'sparsity' is documented in [33–40], see the acknowledgment. The resistor-network analysis introduced below is inspired by [41–46], but generalizes its scope in a somewhat revolutionary way.

The paper is organized as follows. First we introduce in detail the model systems of figure 1. Then we outline the formalism of the EAR calculation, which is based on a simple Fermi golden rule (FGR) picture. Finally, we present the results we obtained on the dependence of the absorption coefficient or conductance on the sparsity, where the latter is controlled by the degree of deformation (figure 1) or by the disorder in the system. Two appendices give extra details of the resistor-network calculation and what we call the 'resistor-network average'.

2. The model systems

Assume that we have N non-interacting particles in a 'box'. For presentation purpose, assume a rectangular-like two-dimensional (2D) billiard-shaped box as in figure 1(a), or optionally, imposing periodic boundary conditions in one direction, a ring-shaped box as in figure 1(b). The box is slightly deformed: either its walls are slightly curved or optionally the potential floor is not flat, e.g. due to some scatterer or disorder. The system is driven by a low-frequency stationary source. In figure 1(a), the driving is induced by moving a 'piston', while in figure 1(b) it is induced by varying a magnetic flux through the ring. The Hamiltonian matrix in

[1] For a review and further references, see the paper by Kamenev and Gefen [27].

the unperturbed energy basis takes the form of equation (1). In the case of the driven ring, we have the identifications

$$f(t) \;\mapsto\; \Phi(t), \tag{3}$$

$$V_{nm} \;\mapsto\; (e/L)\, v_{nm}, \tag{4}$$

where $\Phi(t)$ is the magnetic flux, and v_{nm} are the matrix elements of the velocity operator. Note that by the Faraday law $-\dot{\Phi}$ is the electromotive force (EMF).

The driving induces transitions between energy levels. We assume a stationary driving source, and define its power spectrum as

$$\tilde{S}(\omega) = \mathrm{FT}\langle \dot{f}(t)\dot{f}(0)\rangle. \tag{5}$$

Note that as far as EAR is concerned, it is a non-zero $\dot{f}$ that makes the Hamiltonian time dependent, corresponding to the EMF in the case of the ring. We assume low-frequency driving. Accordingly, we write

$$\tilde{S}(\omega) = 2\pi\,\overline{|\dot{f}|^2}\;\delta_{\mathrm{c}}(\omega), \tag{6}$$

where the prefactor is the rms value that characterizes the driving intensity, and $\delta_{\mathrm{c}}(\omega)$ is a broadened delta function whose line shape reflects the spectral content of the driving.

3. The energy absorption rate

The driving induces transitions between energy levels, which implies diffusion in the energy space. This diffusion is characterized by a coefficient D (energy^2 per time) for which we would like to have a formula. Assuming that D is known, the EAR is given by the following expression:

$$\mathrm{EAR} = \mathrm{density} \times D. \tag{7}$$

This is a straightforward generalization of the Einstein-type relations discussed in [18] and in greater detail in [24]. We can call it a diffusion–dissipation relation. What we label in equation (7) as 'density' stands for the number of particles (N) per energy, meaning N/T in the case of a Boltzmann occupation at temperature T or ϱ in the case of a low-temperature Fermi occupation. In the latter case the role of temperature is overtaken by the Fermi energy, namely $\varrho \sim N/E_F$.

What we would like to have is a theory that allows the calculation of D. The formulae that we would like to advertise is

$$D = \pi\varrho\;\langle\!\langle |V_{nm}|^2\rangle\!\rangle\;\overline{\dot{f}^2}. \tag{8}$$

Depending on the interpretation of $\langle\!\langle |V_{nm}|^2\rangle\!\rangle$, this is the Kubo formula of LRT or its resistor-network variation. In the latter case we refer to the results as the outcome of semi-linear response theory (SLRT). The latter term indicates that $\langle\!\langle |V_{nm}|^2\rangle\!\rangle_{\mathrm{s}}$ unlike $\langle\!\langle |V_{nm}|^2\rangle\!\rangle_{\mathrm{a}}$ is a semi-linear rather than linear operation. The derivations of both the LRT and the SLRT variations of equation (8) are outlined in the next sections.

In the case of an EMF-driven ring, it is convenient to rewrite the EAR formulae equation (7) as

$$\mathrm{EAR} = G\,\overline{\dot{f}^2}, \tag{9}$$

where the so-called mesoscopic conductance is given by the expression

$$G = \pi\varrho^2\,\left(\frac{e}{L}\right)^2\,\langle\!\langle |v_{mn}|^2\rangle\!\rangle. \tag{10}$$

One can regard G as a mesoscopic version of the absorption coefficient, while equation (9) can be regarded as the mesoscopic version of the Joule law.

4. The Fermi golden rule picture and the Kubo formula

The Hamiltonian in the standard basis is equation (1). We can transform it into the adiabatic basis:

$$\tilde{\mathcal{H}} = \mathrm{diag}\{E_n\} - \dot{f}(t)\left\{\frac{\mathrm{i}V_{nm}}{E_n - E_m}\right\}. \tag{11}$$

The FGR transition rate from E_m to E_n due to the low-frequency noisy driving is

$$w_{nm} = \left|\frac{V_{nm}}{E_n - E_m}\right|^2\;\tilde{S}(E_n - E_m). \tag{12}$$

The FGR transitions lead to diffusion in energy space. Assuming that there is a background relaxation process that maintains at any moment quasi-equilibrium with occupation probabilities p_n that are the same as those in the absence of driving, we obtain for the driving-induced diffusion

$$D = \sum_n p_n\left[\frac{1}{2}\sum_m (E_m - E_n)^2\; w_{mn}\right], \tag{13}$$

leading to the Kubo formula

$$D = \int_0^\infty \tilde{C}(\omega)\tilde{S}(\omega)\frac{\mathrm{d}\omega}{2\pi} \tag{14}$$

with the spectral function

$$\begin{aligned}\tilde{C}(\omega) &= \mathrm{FT}\langle V(t)V(0)\rangle \\ &= \sum_n p_n \sum_m |V_{nm}|^2\; 2\pi\delta(\omega - (E_m - E_n)).\end{aligned} \tag{15}$$

Note that this spectral function reflects the band profile of $\boldsymbol{X}$, as defined in equation (2), and illustrated in figure 2.

It is important to realize that the Kubo formula equation (14) is a linear functional of $\tilde{S}(\omega)$. The dependence on the matrix elements of $\boldsymbol{X}$ is linear too:

$$D = \left[\pi\sum_{n,m} p_n |V_{mn}|^2\;\delta_{\mathrm{c}}(E_m - E_n)\right]\;\overline{\dot{f}^2}. \tag{16}$$

This can be formally written as equation (8) with an implied definition of the weighted algebraic average $\langle\!\langle |V_{nm}|^2\rangle\!\rangle_{\mathrm{a}}$.

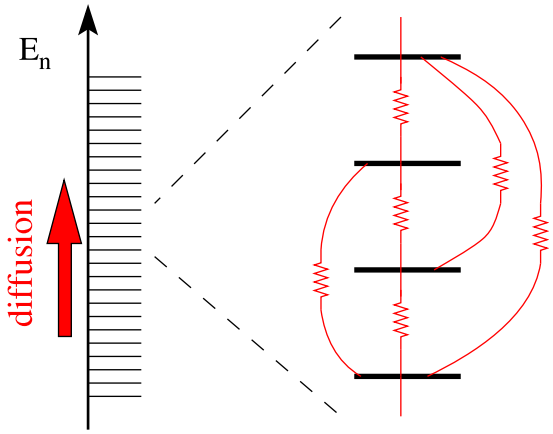

Figure 4. The driving induces transitions between levels E_n of a closed system, leading to diffusion in energy space and hence an associated heating. The diffusion coefficient D can be calculated using a resistor-network analogy. Connected sequences of transitions are essential in order to have a non-vanishing result, as in the theory of percolation.

For completeness we note that in the context of conductance calculation the popular textbook version of the Kubo formula is

$$G = \pi \left(\frac{e}{L}\right)^2 \sum_{n,m} |v_{mn}|^2 \, \delta_T(E_n - E_F) \, \delta_c(E_m - E_n). \quad (17)$$

This expression is implied by equation (16), with averaging over the levels in the vicinity of the Fermi energy. Namely, $p_n = \varrho^{-1}\delta_T(E_n - E_F)$, where the width of $\delta_T()$ is determined by the temperature.

5. The calculation of the diffusion coefficient

We would like to consider circumstances in which the driving-induced transitions are faster compared with the background relaxation. In such circumstances the occupation probabilities p_n are no longer as in equilibrium. For the purpose of analysis we simply neglect the bath, and describe the dynamics by a rate equation

$$\frac{\mathrm{d}p_n}{\mathrm{d}t} = -\sum_m w_{nm}(p_n - p_m), \quad (18)$$

where the rates w_{nm} are determined by equation (12). The matrix $\boldsymbol{w} = \{w_{nm}\}$ can be regarded as a quasi-1D network; see figure 4. Optionally, one may interpret the rate equation as a probabilistic description of a random walk process, where w_{nm} is the probability to hop from m to n per unit time. The local spreading is described by

$$\mathrm{Var}(n) = \sum_n [w_{n,n_0} t]\,(n - n_0)^2 \quad (19)$$

$$\equiv 2\tilde{D}_{\mathrm{local}} t. \quad (20)$$

It follows that the course-grained spreading should be described by a diffusion equation

$$\frac{\partial p_n}{\partial t} = \tilde{D}\frac{\partial^2}{\partial n^2} p_n, \quad (21)$$

where n is regarded as a continuous variable. The diffusion equation is formally a continuity equation

$$\frac{\partial p_n}{\partial t} = -\frac{\partial}{\partial n} I_n, \quad (22)$$

where the current is given by Fick's law:

$$I_n = -\tilde{D}\frac{\partial}{\partial n} p_n. \quad (23)$$

If we have a sample of length N, then

$$I = -\frac{\tilde{D}}{N} \times [p_N - p_0]. \quad (24)$$

This shows that $\tilde{D}/N$ is formally like the inverse resistance of the chain: it is the ratio between the current and the 'potential difference'. We therefore can use standard recipes of electrical engineering in order to calculate its value. For example, if we have only near-neighbor transitions, then 'adding connectors in series' implies

$$\frac{\tilde{D}}{N} = \left[\sum_{n=1}^{N} \frac{1}{w_{n,n-1}}\right]^{-1}. \quad (25)$$

In general, we use the notation $\tilde{D} = [[\boldsymbol{w}]]$, where double brackets stand for the inverse resistivity calculation, as discussed in appendix A.

In the above analysis, we have assumed unit distance between sites. If the mean level spacing is ϱ^{-1}, the expression for the diffusion coefficient should be re-scaled as follows:

$$D = \varrho^{-2}[[\boldsymbol{w}]]. \quad (26)$$

Recall that the w_{nm} are given by equation (12); it follows that

$$D = \varrho^{-2}\left[\left[\left|\frac{V_{nm}}{E_n - E_m}\right|^2 2\pi|\overline{\dot{f}}|^2 \delta_c(E_n - E_m)\right]\right], \quad (27)$$

leading to equation (10), with an implied definition of the resistor-network average $\langle\langle |V_{nm}|^2 \rangle\rangle_s$. The definition and calculation of the latter are further discussed in appendix B.

6. The wall formula and beyond

The roughest estimate for the diffusion that is induced by a vibrating wall is known as the 'Wall formula' [21–26]. Its original derivation is based on a simple kinetic picture: it is based on the assumption that collisions with the vibrating wall are not correlated. This leads in the 2D case [24, 40] to the result

$$D_0 = \frac{4}{3\pi}\frac{m^2 v_E^3}{L_x}\overline{\dot{f}^2}, \quad (28)$$

where m is the mass of the particles, and L_x is the linear dimension of the box as illustrated in figure 1. The result assumes a microcanonical preparation at energy E, and we have defined $v_E = [2E/m]^{1/2}$. If we have a Boltzmann occupation, the expression should be averaged accordingly.

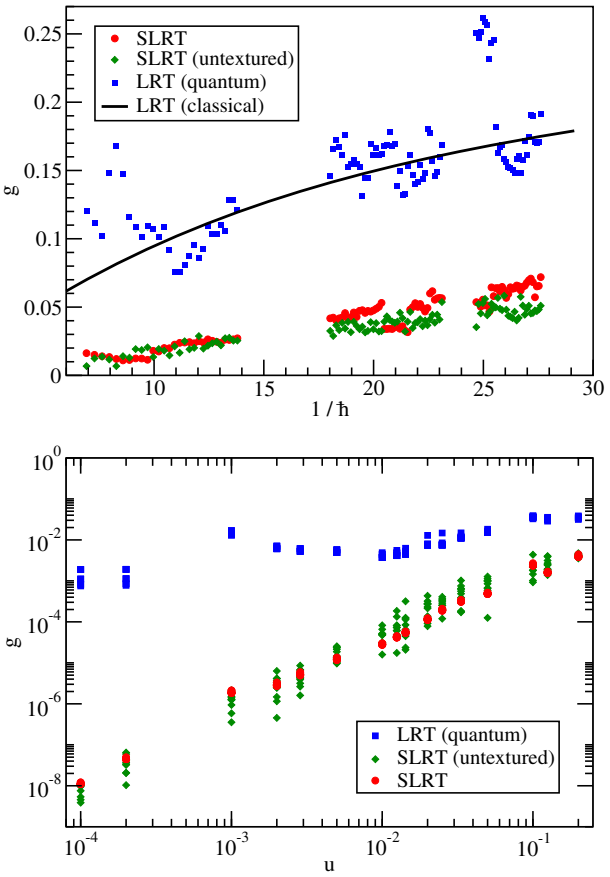

Figure 5. The scaled absorption coefficient g_c (LRT) and $g = g_s g_c$ (SLRT) versus the dimensionless $1/\hbar$ (upper panel) and versus the dimensionless deformation parameter u (lower panel). The input for this analysis is the matrix $\boldsymbol{X}$ of figure 2. The calculation of each point has been carried out on a 100×100 sub-matrix of $\boldsymbol{X}$ centered around the $\hbar$ implied energy E. The 'untextured' data points are calculated for artificial random matrices with the same band profile and sparsity. The complementary lower panel is oriented to show the small u dependence within an energy window that corresponds to $1/\hbar \sim 9$. For further details see [39, 40].

If we have low-temperature Fermi occupation, what counts in equation (7) is the value at $E = E_F$.

Within the framework of LRT the same result is obtained from the Kubo formula equation (14), provided that $\tilde{C}(\omega)$ is flat, i.e. provided there are no correlations between collisions. In practice, there are correlations leading to $D = g_c D_0$ with g_c that can be either smaller or larger than unity depending on the geometry. In the quantum calculation g_c is slightly affected by the level spacing statistics.

Within the framework of SLRT one has to calculate the resistor-network average of the $\boldsymbol{X}$ matrix. If this matrix is sparse, the result becomes very suppressed, leading to $D = g_s g_c D_0$, where

$$g_s \equiv \frac{\langle\langle |V_{nm}|^2 \rangle\rangle_s}{\langle\langle |V_{nm}|^2 \rangle\rangle_a}. \tag{29}$$

We emphasize that g_s reflects an anomaly: it depends on the sparsity s of the matrix, a parameter that has no meaning in the classical context. For more details, including the RMT analysis of this dependence, see [37].

Some numerical results for g_c (LRT) and $g = g_s g_c$ (SLRT) are presented in figure 5. The calculation is performed

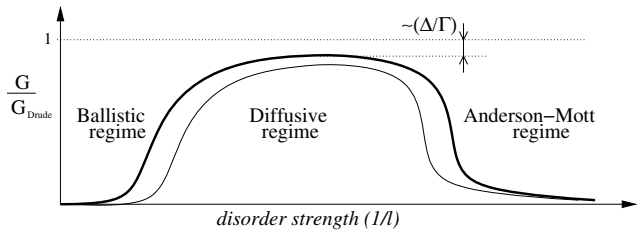

Figure 6. Schematic illustration that sketches the dependence of the dc mesoscopic conductance on the strength of the disorder. It should be regarded as a caricature of figure 7. The level width $\Gamma = \hbar\omega_c$ affects the so-called weak localization correction in the diffusive regime. In the other two regimes of either weak or strong disorder, the perturbation matrix becomes 'sparse' and consequently G is suppressed compared with Drude.

with the $\tilde{C}(\omega)$ of figure 2. The $\hbar$ dependence of the LRT 'classical' result is due to the lower cutoff of the $d\omega$ integral in the Kubo formula equation (14), which is sensitive to the mean level spacing. If $\tilde{C}(\omega)$ were 'flat' the result would not be very sensitive to $\hbar$. As expected, the SLRT calculation gives a much smaller result. The effect becomes more conspicuous for smaller deformations, for which the sparsity is 'stronger'.

7. The Drude formula and beyond

The EAR by particles driven by an oscillating electric field, due to an induced EMF, is a very similar problem as that of particles driven by an oscillating wall. Here the simple result that is based on kinetic considerations is known as the 'Drude formula'. As in the case of the 'Wall formula', it is assumed that scattering events are uncorrelated, leading to the estimate

$$D_0 = \left(\frac{e}{L}\right)^2 v_E \ell \ \overline{\dot{\Phi}^2}, \tag{30}$$

where e is the charge of the electron, L is the length of the ring and ℓ is the mean free path between collisions. Note that $\mathcal{D} = v_E \ell$ is the spatial diffusion coefficient, while $(e\dot{\Phi}/L)$ is the energy that is gained per circulation. The implied result for the conductance can be written as

$$G_0 = \frac{e^2}{2\pi\hbar} \mathcal{M} \frac{\ell}{L}, \tag{31}$$

where $\mathcal{M} = \mathsf{m} v_E L_y/\pi$ is the number of open modes. This way of writing the Drude formula is very illuminating because it reflects the Ohm law, and the units are the same as in the Landauer formula. For clarity we have restored the $\hbar$ in this formula.

Within the framework of LRT, the same result is obtained from the Kubo formula equation (14), provided that $\tilde{C}(\omega)$ of equation (15) is a Lorentzian that reflects an exponential decay of the velocity–velocity correlation function. In practice, there are extra correlations leading to $D = g_c D_0$ with g_c that can be either smaller or larger than unity depending on the geometry. In the quantum calculation g_c is slightly affected by the level spacing statistics, and the correction is of order $(\varrho\omega_c)^{-1}$. This is sometimes regarded as a variation of weak localization corrections [27].

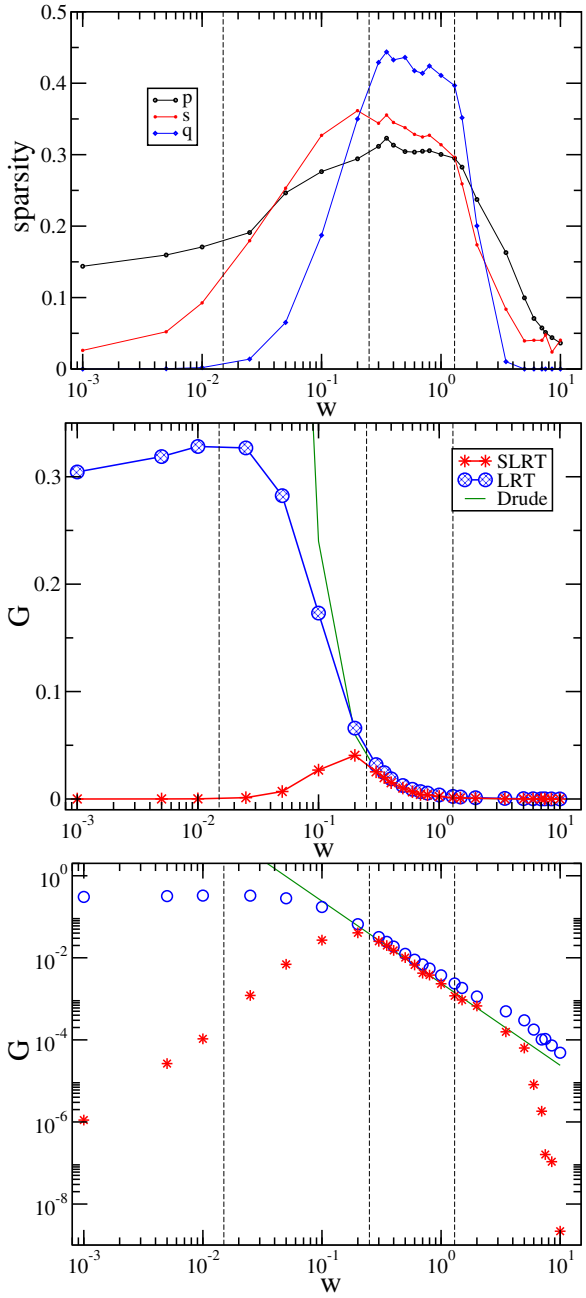

Figure 7. Various measures of sparsity (top panel) and the mesoscopic conductance (middle and bottom panels) as a function of the disorder W in the ring. The vertical lines separate the clean, ballistic, diffusive and localization regimes. Note that the scaled conductance in arbitrary units is equal to $\langle\langle|v_{mn}|^2\rangle\rangle$. The Drude, LRT and SLRT results are displayed in both the normal and the log scale. We see that in the ballistic regime the SLRT conductance gets worse as the disorder becomes weaker, in opposition to the Drude expectation. For further details see [36, 37].

Within the framework of SLRT one has to calculate the resistor-network average of the matrix $\{|v_{nm}|^2\}$. Here one should distinguish between different regimes, depending on the strength W of the disorder. In the Born approximation the mean free path is $\ell \propto W^2$, while the localization length is $\ell_\infty \approx \mathcal{M}\ell$. The diffusive regime, where there is no issue of sparsity, requires an intermediate strength of disorder, such that $\ell \ll L \ll \ell_\infty$. For either stronger or weaker disorder, the matrix $\{|v_{nm}|^2\}$ becomes 'sparse'. This is because the eigenstates become non-ergodic: either they are localized in real space (for strong disorder) or in mode space (for weak disorder). Note also that for very weak disorder ('clean' ring) each eigenstate occupies a single mode (up to small correction). For a detailed analysis that supports the above picture see [37].

The expectation with regard to the dependence of G/G_0 on the strength of the disorder is sketched in figure 6. Some numerical results on both the LRT and the SLRT conductance are presented in figure 7. The calculation is performed for a tight-binding model. The SLRT result in the Anderson localization regime is completely analogous to the reasoning of variable range hopping [41–45], as explained in [46]. It should be appreciated that in our approach all regimes are treated on an equal footing.

In the ballistic regime, contrary to the Drude expectation, the conductance becomes worse as the disorder is reduced. This looks strange, but can easily be rationalized if we think about the extreme case of no disorder: in the absence of scattering the particle stays all the time in the same mode; hence an irreversible diffusive spreading in energy is impossible.

8. Conclusions

The random matrix approach of Wigner (∼1955) is based on the observation that in generic circumstances the perturbation can be represented by a random matrix whose elements are taken from a Gaussian distribution. In this paper our interest is in a restricted class of 'sparse' systems for which this observation does not hold. In such *weak* quantum chaos circumstances, the elements are characterized by a log-wide distribution. Consequently, the response, and in particular the energy absorption, are similar to a percolation process, and their analysis requires a novel resistor-network approach.

Besides the quantitative issue, the experimental fingerprint of the resistor-network calculation is the implied semi-linearity of the response. In the SLRT regime, i.e. if the driving is the predominant mechanism for transitions between levels, one expects the combined effect of two independent sources to be super-linear, namely

$$D[\tilde{S}_a(\omega) + \tilde{S}_b(\omega)] > D[\tilde{S}_a(\omega)] + D[\tilde{S}_b(\omega)] \qquad (32)$$

but still semi-linear $D[\lambda\tilde{S}(\omega)] = \lambda\, D[\tilde{S}(\omega)]$. We have provided in this paper two prototype examples where an SLRT anomaly can arise: heating of particles that are trapped in billiards with vibrating walls; and Joule heating of charged carriers that are driven by an induced EMF.

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Appendix A. The resistor-network calculation

In this appendix, we explain how the inverse resistivity $G = [[\boldsymbol{G}]]$ of a 1D resistor network $\boldsymbol{G} \equiv \{G_{nm}\}$ is calculated. We use the language of electrical engineering for this purpose. In general, this relation is semi-liner rather than linear, namely $[[\lambda \boldsymbol{G}]] = \lambda[[\boldsymbol{G}]]$, but $[[\boldsymbol{A}+\boldsymbol{B}]] \neq [[\boldsymbol{A}]]+[[\boldsymbol{B}]]$.

There are a few cases when an analytical expression is available. If only near-neighbor nodes are connected, allowing $G_{n,n+1} = g_n$ to be different from each other, then 'addition in series' implies that the inverse resistivity calculated for a chain of length N is

$$G = \left[\frac{1}{N}\sum_{n=1}^{N}\frac{1}{g_n}\right]^{-1}. \tag{A.1}$$

If $G_{nm} = g_{n-m}$ is a function of the distance between the nodes n and m, then it is a nice exercise to prove that 'addition in parallel' implies

$$G = \sum_{r=1}^{\infty} r^2 g_r. \tag{A.2}$$

In general, an analytical formula for G is not available, and we have to apply a numerical procedure. For this purpose we imagine that each node n is connected to a current source I_n. The Kirchhoff equations for the voltages are

$$\sum_m G_{mn}(V_n - V_m) \;=\; I_n. \tag{A.3}$$

This set of equation can be written in a matrix form,

$$\tilde{\boldsymbol{G}}\boldsymbol{V} \;=\; \boldsymbol{I}, \tag{A.4}$$

where the so-called discrete Laplacian matrix of the network is defined as

$$\tilde{G}_{nm} = \left[\sum_{n'} G_{n'n}\right]\delta_{n,m} - G_{nm}. \tag{A.5}$$

This matrix has an eigenvalue zero which is associated with a uniform voltage eigenvector. Therefore, it has a pseudo-inverse rather than an inverse, and the Kirchhoff equation has a solution if and only if $\sum_n I_n = 0$. In order to find the resistance between nodes $n_{\rm in} = 0$ and $n_{\rm our} = N$, we set $I_0 = 1$ and $I_N = -1$ and $I_n = 0$ otherwise, and solve for V_0 and V_N. The inverse resistivity is $G = [(V_0 - V_N)/N]^{-1}$.

Appendix B. The resistor-network average

We use the notation $\langle\!\langle \boldsymbol{X} \rangle\!\rangle$ in order to indicate the weighted *average* value of its elements. First we would like to define the standard *algebraic* average. It is essential to introduce a *weight* function that defines the band of interest. In the physical context this function reflects the spectral content of the driving source. Namely, we define $F(r)$ as the normalized version of $\tilde{S}(\omega)$, such that $\sum_r F(r) = 1$, where $r = n - m$ is the energy difference $\omega = E_n - E_m$ in integer units. The bandwidth in these dimensionless units ($b_{\rm c} = \varrho\omega_{\rm c}$) is assumed to be quantum mechanically large ($b_{\rm c} \gg 1$). The algebraic average is defined in the standard way:

$$\langle\!\langle \boldsymbol{X} \rangle\!\rangle_{\rm a} = \frac{1}{N}\sum_{n,m} F(n-m)\; X_{nm}, \tag{B.1}$$

where N is the size of the matrix, which is assumed to be very large. The algebraic average is a *linear* operation, meaning that

$$\langle\!\langle \lambda \boldsymbol{X} \rangle\!\rangle = \lambda \langle\!\langle \boldsymbol{X} \rangle\!\rangle, \tag{B.2}$$

$$\langle\!\langle \boldsymbol{X} + \boldsymbol{Y} \rangle\!\rangle = \langle\!\langle \boldsymbol{X} \rangle\!\rangle + \langle\!\langle \boldsymbol{Y} \rangle\!\rangle. \tag{B.3}$$

There are different types of 'averages' in the literature, such as the *harmonic* average and the *geometric* average and we can include the *median* in the same list. All these 'averages' are *semi-linear* operations because only the $\langle\!\langle \lambda \boldsymbol{X} \rangle\!\rangle = \lambda \langle\!\langle \boldsymbol{X} \rangle\!\rangle$ property is satisfied for them. Irrespective of the semi-linearity issue *any* type of average should satisfy the following requirement: if all the elements are equal to the same number, then also the average should equal the same number.

In this paper, we highlight a new type of average that we call a *resistor-network* average:

$$\langle\!\langle \boldsymbol{X} \rangle\!\rangle_{\rm s} \;\equiv\; \left[\left[2F(n-m)\,\frac{X_{nm}}{(n-m)^2}\right]\right]. \tag{B.4}$$

Writing the above expression as $[[\tilde{w}_{nm}]]$, one should realize that the $\tilde{w}_{nm}$ can be regarded as FGR transition rates. Using equation (A.2) it is not difficult to show that if all the elements X_{nm} are the same number, then also their resistor-network average is the same number. In general,

$$\langle\!\langle \boldsymbol{X} \rangle\!\rangle_{\rm s} < \langle\!\langle \boldsymbol{X} \rangle\!\rangle_{\rm a}. \tag{B.5}$$

Typically, the resistor-network average is bounded from below by the median. In order to get a realistic estimate in the case of a 'sparse' matrix, one can use a generalized variable range hopping scheme that we have developed in [37].

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