Triangular Bose-Hubbard trimer as a minimal model for a superfluid circuit

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The Model

A rotating 3 site system with N bosons.

\[ \mathcal{H} = \sum_{j=1}^{M} \left[ \frac{U}{2} a_j^{\dagger} a_j^{\dagger} a_j a_j - \frac{K}{2} \left( e^{i(\Phi/M)} a_{j+1} a_j + e^{-i(\Phi/M)} a_j a_{j+1} \right) \right] , \quad M=3 \]

\[ N = \sum_{j=1}^{M} n_j \quad , \quad n_j = a_j^{\dagger} a_j \]

like M ”coupled oscillators”.

Dimensionless parameters (Φ, u):

\[ u = \frac{NU}{K} \]

\[ \Phi = \frac{M^2}{2\pi} \left( \frac{m}{m_{\text{eff}}} \right) \frac{\Omega}{K} \]

Upon quantization we have \( \hbar = 1/N \).
For a fixed particle number $N = n_1 + n_2 + n_3$ the BH trimer can be regarded as a 2 degrees of freedom system with coordinates: $(n_1 - n_2, n_3)$.

$$|\psi(r)|^2 = |\langle r | E_\alpha \rangle|^2, \quad r = \frac{1}{N}(n_1 - n_2, n_3)$$

For each eigenstate $|E_\alpha\rangle$ we calculate:

one-body reduced probability matrix:

$$\rho_{ij} = \langle a_j^\dagger a_i \rangle_\alpha$$

$$S_\alpha \equiv \text{trace}(\rho^2), \quad 1/S \in [1, 3]$$

$1/S = \#$ of participating orbitals.

$1/S = 1$ means a coherent state.

$1/S = 3$ is a maximum fragmentation.

Bond averaged current:

$$\mathcal{I}_\alpha \equiv -\left\langle \frac{\partial \mathcal{H}}{\partial \Phi} \right\rangle_\alpha$$
The energy spectrum of the trimer ($N = 42$)

$(\Phi, u) = (0.2\pi, 0.2)$

($0.2\pi, 2.5$)

($0.6\pi, 1470$) $1/S$

- Vortex States $m = 0, \pm 1$

$$I_m = \frac{N}{M} K \sin \left( \frac{1}{M} (2\pi m - \Phi) \right)$$

- Mott Transition
- Self-Trapping (Bright Solitons)
- Metastable Vortex state

Vortex state = Condensation in momentum orbital.
Self-trapped state = Condensation in site orbital.
Self Trapping

\[
\begin{align*}
u & = \frac{6 - 9 \cos \left( \frac{\pi + 2\Phi}{3} \right) - 3 \cos \left( \frac{\pi - 4\Phi}{3} \right)}{6 \cos \left( \frac{\pi - \Phi}{3} \right) - 2 \cos(\Phi)} \\
I_m & = \frac{N}{M} K \sin \left( \frac{1}{M} (2\pi m - \Phi) \right) \\
m & = -1
\end{align*}
\]

- Self-trapping can occur for arbitrarily small interaction.
Quasi-Stability

\[ u = \frac{6 - 9 \cos \left( \frac{\pi - 2\Phi}{3} \right) - 3 \cos \left( \frac{\pi + 4\Phi}{3} \right)}{6 \cos \left( \frac{\pi + \Phi}{3} \right) - 2 \cos (\Phi)} \]

- The quantum metastability regime extends beyond the classically expected.
- Quantum scarring related effect(?).

\[ \frac{1}{S} \]
\[ I \]
The energy landscape

\[ \mathcal{H} = \sum_{j=1}^{M} \left[ \frac{U}{2} n_j^2 - K \sqrt{n_{j+1} n_j} \cos \left( (\varphi_{j+1} - \varphi_j) - \frac{\Phi}{M} \right) \right] , ~ a_j = \sqrt{n_j} e^{i\varphi_j} \]

\[ \mathcal{H} = \frac{U}{2} (n_1^2 + n_2^2 + n_3^2) - K (\sqrt{n_2 n_3} \cos(q_1) + \sqrt{n_3 n_1} \cos(q_2) + \sqrt{n_1 n_2} \cos(\Phi + q_1 + q_2)) \]

\[ V(r) = \min_{\varphi} [\mathcal{H}(r, \varphi)] \quad \text{or} \quad \max_{\varphi} [\mathcal{H}(r, \varphi)] \]

\[ E_m = V_m(r_0) = \frac{1}{6} N^2 U - NK \cos \left( \frac{2\pi m - \Phi}{3} \right), \quad m = 0, \pm 1 \]

\[ I_m = \frac{N}{M} K \sin \left( \frac{1}{M} (2\pi m - \Phi) \right) \]
The energy landscape (cont.)

\[ V''_\pm(r_0) = \left. \frac{d^2 H(q_\pm(n), n)}{dn^2} \right|_{N/3} = 6U - 9\frac{K}{N} \left[ \frac{2 - 3 \cos \left( \frac{\pi \pm 2\Phi}{3} \right) - \cos \left( \frac{\pi \mp 4\Phi}{3} \right)}{3 \cos \left( \frac{\pi \mp \Phi}{3} \right) - \cos (\Phi)} \right] \]

Some eigenstates
Concluding Remarks

- The essence of superfluidity is the possibility to witness metastable vortex states.
- In the standard classical stability analysis one finds that vortex states whose rotation velocity is less than a critical velocity are metastable (“Landau criterion”).
- For a non-rotating 3-site model the same type of classical analysis implies that there are no metastable vortex states [1,2].
- We have explored the full regime diagram. In the presence of rotation we find $(\Omega, u)$ regimes where metastable vortex states exist.

- In the quantum analysis we find that the metastable vortex state is quasi-stable in a much wider regime, even for a small rotation frequency, contrary to the classical expectation.

[1] PC with J. Anglin, T. Leggett, P. Ghosh