Quantal Brownian motion—dephasing and dissipation

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Abstract. We analyse quantal Brownian motion in d dimensions using the unified model for diffusion localization and dissipation, and Feynman–Vernon formalism. At high temperatures the propagator possesses a Markovian property and we can write down an equivalent master equation. Unlike the case of the Zwanzig–Caldeira–Leggett model, genuine quantum mechanical effects manifest themselves due to the disordered nature of the environment. Using Wigner's picture of the dynamics we distinguish between two different mechanisms for destruction of coherence: scattering perturbative mechanism and smearing non-perturbative mechanism. The analysis of dephasing is extended to the low-temperature regime by using a semiclassical strategy. Various results are derived for ballistic, chaotic, diffusive, both ergodic and non-ergodic motion. We also analyse loss of coherence at the limit of zero temperature and clarify the limitations of the semiclassical approach. The condition for having coherent effect due to scattering by low-frequency fluctuations is also pointed out. It is interesting that the dephasing rate can be either larger or smaller than the dissipation rate, depending on the physical circumstances.

1. Introduction

Classical Brownian motion is described by the Langevin equation

$$m\ddot{x} + \eta\dot{x} = \mathcal{F} \tag{1}$$

where x is the position of the particle, η is the friction coefficient, and $\mathcal{F} = -\nabla\mathcal{U}(x, t)$ is a stochastic force. This equation is meaningful only in a statistical sense. The time evolution of a phase-space distribution $\rho(x, p)$ is obtained by solving (1) for various realizations of the stochastic potential, and then averaging over all these realizations. The stochastic potential is zero on the average and its correlations are

$$\langle\mathcal{U}(x'', t'')\mathcal{U}(x', t')\rangle = \phi(t'' - t') \cdot w(x'' - x'). \tag{2}$$

Typically these correlations are characterized by a 'short' temporal scale τ_c and a 'microscopic' spatial scale ℓ. Usually, it is further assumed that higher moments are determined by Gaussian statistics. The Langevin description can be derived by considering a general Hamiltonian of the form

$$\mathcal{H} = \mathcal{H}_0(x, p) + \mathcal{H}_{\text{env}}(x, Q_\alpha, P_\alpha) \tag{3}$$

where x and p are the canonical variables that correspond to the distinguished degree of freedom, $\mathcal{H}_0 = p^2/2m$ is the free-motion Hamiltonian, and (Q_α, P_α) are environmental degrees of freedom. The dynamical variable x may represent the position of a large particle. In the one-dimensional version of Brownian motion it may represent the position of a piston. The actual conditions for having the reduced Langevin description turn out to be quite

weak [1]. The environment should consist of at least three degrees of freedom with *fast* chaotic dynamics. *Fast* implies that the classical motion is characterized by a continuous spectrum with high-frequency cut-off, such that the motion of the environment can be treated adiabatically with respect to the *slow* motion of the particle. It is essential to assume that the following condition is fulfilled

$$\text{generic Brownian motion} \qquad \Leftrightarrow \qquad \frac{v}{\ell} \ll \frac{1}{\tau_c} \tag{4}$$

where τ_c and ℓ characterize the correlations of the stochastic potential which is experienced by the particle. Equation (4) is the condition for using the white noise approximation (WNA). Thus, the noise is characterized by its intensity

$$\nu \equiv \int_{-\infty}^{\infty} \phi(\tau)\, \mathrm{d}\tau \cdot |w''(0)| \tag{5}$$

and its temporal correlations are adequately described by the formal expression $\phi(\tau) = \nu\delta(\tau)$. Without loss of generality we assume from now on the normalization $w''(0) = -1$. The correlations of the stochastic force satisfy

$$\langle \mathcal{F}(t)\mathcal{F}(t') \rangle_{\text{at } x} = \phi(t - t'). \tag{6}$$

In the general case, (6) is less informative than (2). However, in the case of a classical particle that experiences white noise, the additional information is not required at all! It should be emphasized that this observation does not hold if (4) is not satisfied. If τ_c is larger than ℓ/v, then the particle will perform a stochastic motion that depends crucially on the 'topography' of the stochastic potential.

The classical analysis further reveals [1] that associated with the stochastic potential there is also a dissipation effect. If the environment is characterized by either microcanonical or canonical temperature T, then the friction parameter will be

$$\eta = \frac{\nu}{2k_B T} \qquad \text{fluctuations} \rightsquigarrow \text{dissipation.} \tag{7}$$

Thus, any generic environment, in the sense specified above, leads to the universal ohmic behaviour. The motion of the particle is determined by the interplay between the friction and the noise. The friction leads to damping of the particle's velocity, while the noise pumps energy back into its motion. Eventually we have diffusion with coefficient $D_\eta = \nu/\eta^2$. If (4) is not satisfied, then we will have diffusion even in the absence of dissipation. The latter 'non-dissipative diffusion' is characterized by the coefficient $D_0 \sim \ell v$, where v is the velocity of the particle. This latter type of diffusive behaviour should not be confused with the generic 'dissipative diffusion' that characterizes Brownian motion.

One wonders whether there is a well-defined quantized version of the above Langevin equation. Is it possible, quantum mechanically, to characterize a universal dynamical behaviour that corresponds to classical Brownian motion? Are stronger physical conditions required in order to guarantee generic behaviour? In this paper we have no intention to fully answer all these questions. Rather, as in previous publications [2], we follow the Caldeira–Leggett strategy [3] and consider the motion of a particle under the influence of an effective (non-chaotic) bath that is composed of infinitely many oscillators. The proper model that corresponds (classically) to the Langevin equation (1) with (2) is the 'diffusion, localization and dissipation' (DLD) model that has been introduced in [2]. These three effects (diffusion, localization and dissipation) result naturally from the quantum mechanical solution of the DLD model.

Quantum mechanically, it is convenient to use the Wigner function $\rho(R, P)$ in order to represent the reduced-probability density matrix. We can use the Feynman–Vernon (FV)

formalism [4] in order to find an expression for the propagator $\mathcal{K}(R, P|R_0, P_0)$. This propagator is uniquely determined once the friction (η), the temperature of the bath (T) and the spatial correlations $w(r)$ are specified. The dependence of the propagator on the bath temperature is via the appearance of a quantum mechanical version of the noise kernel $\phi(\tau)$. The latter is related to the friction coefficient via a universal fluctuation–dissipation relation. At high temperatures the quantum mechanical $\phi(\tau)$ coincides with its classical expression. At the limit of zero temperature $\phi(\tau)$ does not vanish, rather it develops a large negative tail. In the latter case, $\phi(\omega)$, the power spectrum of the noise, reflects the zero point fluctuations (ZPF) of the environmental modes.

The Zwanzig–Caldeira–Leggett (ZCL) model [3] constitutes a special formal limit of the DLD model. It is obtained by taking the limit $\ell \to \infty$. Both the ZCL model and the DLD model will be generalized in this paper to allow the analysis of Brownian motion in d dimensions. A modified version of the the DLD model incorporates the effect of long-range (power-law) spatial correlations. On the other hand, we are not discussing the DLD model in its full generality (as in [2]), but rather restricting ourselves to the particular circumstances that correspond in the classical limit to generic Brownian motion. Thus, we are considering the *ohmic* DLD model, and further assume that (4) is satisfied.

In the case of the ZCL model, the quantal propagator $\mathcal{K}(R, P|R_0, P_0)$ is a Gaussian stochastic kernel. Consequently the dynamics can be obtained by solving the Langevin equation (1) with (6). At low temperatures the quantum mechanical version of $\phi(\tau)$ should be used. At high temperatures $\phi(\tau)$ becomes classical-like and consequently the ZCL propagator coincides with its classical limit. These observations do not hold in the case of the DLD model [2]. Furthermore, the distinction between the quantal DLD propagator and its classical limit persists even in the limit of high temperatures.

It is important to define what is the meaning of 'high temperatures'. As in the classical case, a relatively simple description of the dynamics is obtained if it possesses a Markovian property. With a Markovian property it is possible to obtain the long-time evolution by composing short-time evolution steps. It is also possible then to write down a corresponding master equation for the Wigner function. The FV path-integral expression for the propagator contains the quantum mechanical version of $\phi(\tau)$ rather than the classical one. In order to have a Markovian property one should argue that it is possible to use the WNA, meaning to replace $\phi(\tau)$ by the effective classical-like delta function. Still, in the case of the DLD model, the result of the path integration is not classical-like: the high-temperature Markovian limit is not the same as the classical limit.

It turns out that the quantum mechanical condition for using the WNA, thus having a Markovian property, is more restrictive than (4). For ballistic-like motion we shall see that the actual condition is, in most circumstances,

$$\text{high temperatures} \qquad \Leftrightarrow \qquad \frac{v}{\ell} \ll \frac{k_B T}{\hbar}. \tag{8}$$

A recent derivation of the high-temperature ohmic DLD model which is based on a synthetic random matrix theory (RMT) Hamiltonian has been reported in [5]. The existence of such a derivation is most encouraging since it further supports the idea of having a universal description of quantal Brownian motion, at least in the Markovian limit. It has been speculated by the authors of the latter reference that a future extension of their formalism will lead to an agreement with the general result of [2]. The high-temperature ohmic DLD model has been discussed in [2] and has been further analysed in [5]. The physical picture of high-temperature dynamics will be further illuminated in this paper.

Loss of coherence, or dephasing, is a central issue in the analysis of quantal Brownian motion. Wigner's picture of the dynamics leads to the distinction between two different

mechanisms for loss of coherence. In the case of the ZCL model one should consider the '*spreading mechanism*'. This classical-like non-perturbative mechanism is very effective in smearing away the interference pattern. In the case of the DLD model, coherence is much better maintained, and one should consider the perturbative '*scattering mechanism*' for dephasing.

At the limit of zero temperature, the spreading mechanism is still effective in suppressing interference. One can use the Langevin formalism in order to analyse the smearing of the interference pattern. It is important, however, to take into account the negative temporal correlations of the effective noise [6]. On the other hand, the analysis of low-temperature dephasing in the case of the 'scattering mechanism' is quite a subtle issue, and constitutes a main concern of this paper. The Langevin formalism is no longer applicable, and the lack of a Markovian property enforces a semiclassical approach. This has the added advantage that it is possible to go beyond the analysis of a simple ballistic-like Brownian motion and to analyse other types of transport. We shall distinguish between ballistic, diffusive and chaotic motions through cavities. We shall derive various results and contributions to the dephasing rate in the various temperature regimes and depending on the physical circumstances. Some of these results coincide with similar computations that are related to electrons in metal [7].

An important question is whether the scattering mechanism is still effective in suppressing interference at the limit of zero temperature [12, 13]. It turns out that our semiclassical approach, in spite of its other advantages, has a limited range of validity. It can be trusted if the kinetic energy of the particle is sufficiently large. For ballistic-like motion large energy means

$$\text{large energy} \qquad \Leftrightarrow \qquad \frac{v}{\ell} \ll \frac{E}{\hbar} \qquad\qquad (9)$$

in analogy with (8). The latter condition can be cast into the more suggestive form $\lambda_B \ll \ell$, where λ_B is the de Broglie wavelength of the particle. The large-energy condition is obviously satisfied in the case of the ZCL model ($\ell \to \infty$). The large-energy condition may not be satisfied in the case of the DLD model, and consequently the semiclassical result should be modified. In particular, in case of a low-temperature thermal motion, the contribution of the ZPF to the dephasing rate should be excluded.

2. The ohmic DLD model

We consider a bath that consists of infinitely many oscillators whose canonical coordinates are (Q_α, P_α). The bath Hamiltonian is

$$\mathcal{H}_{\text{bath}} = \sum_\alpha \left(\frac{P_\alpha^2}{2m_\alpha} + \frac{1}{2} m \omega_\alpha^2 Q_\alpha^2 \right). \qquad\qquad (10)$$

In the case of the DLD model the interaction of the particle with the oscillators is described by

$$\mathcal{H}_{\text{int}} = -\sum_\alpha c_\alpha Q_\alpha u(x - x_\alpha) \qquad\qquad (11)$$

where x_α is the location of the α oscillator, $u(x - x_\alpha)$ describes the interaction between the particle and the α oscillator, and c_α are coupling constants. It is assumed that the function $u(r)$ depends only on $|r|$. The range of the interaction will be denoted by ℓ. The oscillators

are distributed uniformly all over space. Locally, the distribution of their frequencies is ohmic. Thus,

$$\frac{\pi}{2} \sum_\alpha \frac{c_\alpha^2}{m_\alpha \omega_\alpha} \delta(\omega - \omega_\alpha) \delta(\boldsymbol{x} - \boldsymbol{x}_\alpha) = \eta \omega \qquad \text{for } \omega < 1/\tau_c. \tag{12}$$

It is useful to define the spatial autocorrelation function

$$w(\boldsymbol{r}) = \int_{-\infty}^{\infty} u(\boldsymbol{r} - \boldsymbol{x}') u(\boldsymbol{x}') \, \mathrm{d}\boldsymbol{x}'. \tag{13}$$

Without loss of generality $u(\boldsymbol{r})$ is normalized such that $w''(0) = -1$. For example, we may consider a Gaussian $u(\boldsymbol{r})$ for which

$$w(\boldsymbol{r}) = \ell^2 \exp\left(-\frac{1}{2}\left(\frac{\boldsymbol{r}}{\ell}\right)^2\right). \tag{14}$$

The d-dimensional Fourier transform (FT) of $w(\boldsymbol{r})$ will be denoted by $\tilde{w}(\boldsymbol{k})$. The mode density (after angular integration) is $g(k) = (C_d/(2\pi)^d)k^{d-1}\tilde{w}(k)$. In general, we shall assume that

$$g(k) = C\ell^{2+\sigma}k^{\sigma-1} \qquad \text{for } k < 1/\ell \tag{15}$$

where ℓ characterizes the spatial scale of the correlations, and C is a dimensionless constant. In the case of the short-range Gaussian-type correlations (14), the parameter σ is simply equal to the dimensionality d. For long-range power-law interaction it may be less than d, possibly negative. In a moment we shall argue that in order to have a well-defined model we must have $|w''(0)| < \infty$. Therefore only $-2 < \sigma$ is meaningful. The regime $-2 < \sigma \leqslant 0$ is well defined but it requires special treatment since $w(0)$ diverges.

The formal limit $\ell \to \infty$ corresponds to the ZCL model. The ZCL model describes the interaction of a particle with environmental modes whose wavelength is much larger than the distances explored by the particle. Consequently, the interaction with the α-field mode is approximated by a linear function. The interaction term in the Hamiltonian that *defines* the ZCL model is of the following form (here generalized to d dimensions):

$$\mathcal{H}_{\text{int}} = \sqrt{d} \sum_\alpha c_\alpha Q_\alpha \hat{n}_\alpha \cdot \boldsymbol{x}. \tag{16}$$

The unit vectors $\hat{n}_\alpha$ are assumed to be distributed isotropically. The distribution of the oscillators with respect to their frequencies is assumed to be ohmic, as in (12) with the $\delta(\boldsymbol{x} - \boldsymbol{x}_\alpha)$ omitted.

The classical analysis of the d-dimensional DLD model constitutes a trivial generalization of the one-dimensional DLD model that has been considered in [2]. Equation (3.10) in the latter reference should be replaced by the following expression for the friction:

$$\mathcal{F}_{\text{friction}}(\boldsymbol{v}) = \int_{-\infty}^{\infty} \alpha(\tau) \nabla w(\boldsymbol{v}\tau) \, \mathrm{d}\tau \tag{17}$$

where $\boldsymbol{v}$ is the velocity of the particle. For an ohmic bath $\alpha(\tau)$ can be replaced by $-\eta\delta'(\tau)$. One obtains then $\mathcal{F}_{\text{friction}} = -\eta\boldsymbol{v}$. Obviously, in order to obtain a finite result it is crucial to have $|w''(0)| < \infty$. Recall that by convention we use the normalization $w''(0) = -1$. Thus, the generalized version (15) of the DLD model is well defined provided $-2 < \sigma$. The reduced motion of the particle will be described by the Langevin equation (1), where the noise satisfies (2), with $\phi(\tau) = 2\eta k_B T \delta(\tau)$. For the realization of a classical trajectory, the global functional form of $w(\boldsymbol{r})$ is of no importance. Only the force correlations (6) are important. The latter correlations are well defined as long as $|w''(0)| < \infty$, and by our

convention they are equal $\phi(\tau)$. The parameter ℓ is insignificant in the classical analysis of the ohmic DLD model, and therefore the classical description of Brownian motion is the same as in the case of the ZCL model.

Quantum mechanically, the reduced dynamics of the system may be described by the propagator $\mathcal{K}(R, P|R_0, P_0)$ of the probability density matrix. For the sake of comparison with the classical limit one uses the Wigner function $\rho(R, P)$ in order to represent the latter. Using FV formalism [4], an exact path-integral expression for this propagator is obtained. The FV expression is a double sum $\iint \mathcal{D}x'\mathcal{D}x''$ over the path variables $x'(\tau)$ and $x''(\tau)$. It is convenient to use new path variables $R = (x'+x'')/2$ and $r = (x''-x')$, and to transform the $\iint \mathcal{D}R\mathcal{D}r$ integral into the form [2]

$$\mathcal{K}(R, P|R_0, P_0) = \int_{R_0, P_0}^{R, P} \mathcal{D}R\mathcal{K}[R] \tag{18}$$

where $\mathcal{K}[R]$ is a real functional defined by the expression

$$\mathcal{K}[R] = \int \mathcal{D}r\, e^{i\frac{1}{\hbar}(S_{\text{free}}+S_F)} e^{-\frac{1}{\hbar^2}S_N}. \tag{19}$$

The $\mathcal{D}r$ integration is unrestricted at the endpoints, and the free action functional is $S_{\text{free}}[\boldsymbol{R}, \boldsymbol{r}] = -m\int_0^t \mathrm{d}\tau\, \ddot{R}r$. The action functional $S_F[\boldsymbol{R}, \boldsymbol{r}]$ corresponds to the friction, and the functional $S_N[\boldsymbol{R}, \boldsymbol{r}]$ corresponds to the noise. The explicit expression for the friction action functional, assuming an ohmic bath, is

$$S_F[\boldsymbol{R}, \boldsymbol{r}] = \eta \int_0^t \mathrm{d}\tau\, \nabla w(\boldsymbol{r}) \cdot \dot{\boldsymbol{R}}. \tag{20}$$

The general expression for the noise functional is

$$S_N[\boldsymbol{x}', \boldsymbol{x}''] = \tfrac{1}{2} \int_0^t \int_0^t \mathrm{d}t_1\, \mathrm{d}t_2\, \phi(t_2 - t_1)[w(\boldsymbol{x}_2'' - \boldsymbol{x}_1'') + w(\boldsymbol{x}_2' - \boldsymbol{x}_1') - 2w(\boldsymbol{x}_2'' - \boldsymbol{x}_1')] \tag{21}$$

where $\boldsymbol{x}_i$ is a short notation for $\boldsymbol{x}(t_i)$. The power spectrum of the noise $\phi(\omega)$ is the FT of the noise kernel $\phi(\tau)$. For an ohmic bath

$$\phi(\omega) = \eta\omega\hbar \coth\left(\frac{\hbar\omega}{2k_BT}\right) \qquad \text{for } \omega < 1/\tau_c. \tag{22}$$

The ZCL version for S_F and S_N is obtained by taking the limit $\ell \to \infty$ which is equivalent to the formal substitution $w(\boldsymbol{r}) = -r^2/2$.

The power spectrum (22) of the quantum mechanical noise is the same as that of a classical white noise in the restricted frequency regime $\omega < k_BT/\hbar$, where $\phi(\omega) = 2\eta k_BT$. If the temperature is high enough, such that $1/\tau_c < k_BT/\hbar$, then the power spectrum is essentially classical. At lower temperatures the power spectrum contains a non-trivial frequency zone $k_BT/\hbar < \omega < 1/\tau_c$, where one may use the approximation $\phi(\tau) = \hbar\eta\omega$. This component of the power spectrum does not vanish in the limit of zero temperature. It corresponds to the ZPF of the environmental modes. The temperature becomes effectively zero if $k_BT/\hbar < 1/t$. At intermediate temperatures, namely $1/t < k_BT/\hbar < 1/\tau_c$, it is convenient to write the power spectrum as a sum $\phi(\omega) = \phi_0(\omega) + \phi_T(\omega)$, where $\phi_0(\omega)$ corresponds to the ZPF and $\phi_T(\omega)$ is the excess thermal noise:

$$\phi_T(\omega) = 2\eta \frac{\hbar|\omega|}{\exp(\frac{\hbar|\omega|}{k_BT}) - 1} \qquad \text{for } \omega < 1/\tau_c. \tag{23}$$

The propagator (18) possesses a Markovian property if it is legitimate to make the WNA. The actual condition for having effectively white noise will be discussed in a later section. At low temperatures it is essential to consider the non-trivial nature the quantal noise in order to obtain the proper quantum mechanical dynamical behaviour [8].

3. Propagation in a high-temperature environment

In the absence of noise and friction, the free-motion propagator of the Wigner function is the same as in the classical limit. Namely,

$$\mathcal{K}(R, P|R_0, P_0) = \mathcal{K}_{\text{free}}^{\text{(cl)}} = 2\pi \delta(P - P_0) \, \delta\left((R - R_0) - \frac{P}{m}t\right). \tag{24}$$

For simplicity, but without loss of generality, we shall consider whenever possible one-dimensional motion. In the case of the ZCL model, both S_F and S_N are quadratic forms, and therefore the propagator is a Gaussian kernel. It follows that the dynamics of the Wigner function is purely stochastic, and it can be described by Langevin equation (1) with (2) and (22). In this sense, there are no genuine quantum mechanical effects that are associated with Brownian motion, as long as the ZCL model is used for its description. The situation is quite different in the case of the DLD model. Here the propagator is non-Gaussian, and, in general, it may have non-stochastic features.

In this section we shall restrict ourselves to the high-temperature regime, where the WNA $\phi(\tau) = \nu\delta(\tau)$ can be applied. Whenever the WNA applies, the ZCL propagator becomes identical with the classical propagator for damped motion, namely

$$\mathcal{K}(R, P|R_0, P_0) = \mathcal{K}_{\text{damped}}^{\text{(cl)}} = \text{Gaussian}. \tag{25}$$

In contrast, in the case of the DLD model, the high-temperature limit of the propagator *does not coincide* with its classical limit. An explicit calculation, as in [2], gives the following expression:

$$\mathcal{K} = W_{\hbar/\ell} \star \mathcal{K}_{\text{damped}}^{\text{(cl)}} + e^{-\frac{2\eta k_B T \ell^2}{\hbar^2}t}(1 - W_{\hbar/\ell}\star)\mathcal{K}_{\text{free}}^{\text{(cl)}}. \tag{26}$$

This expression is also valid for the generalized DLD model provided $0 < \sigma$. The precise value of the parameter ℓ is defined here via $w(0) \equiv \ell^2$. Above, $W(R - R', P - P')$ is a smooth Gaussian-like kernel that has unit normalization. Its spread in phase space is characterized by the momentum scale $\hbar/\ell$, and by an associated spatial scale. The symbol $\star$ stands for convolution. Thus, the classical propagator is smeared on a phase-space scale that correspond to $\Delta p = \hbar/\ell$ and there is an additional unscattered component that decays exponentially and eventually disappears. The structure of the propagator is illustrated in figure 1. The significance of this structure will be discussed in the next section.

To write down an explicit expression for the propagator is not very useful. Rather, it is more illuminating to follow the standard procedure [3] and to write an equivalent master equation. This is possible since at high temperatures the path-integral expression possesses a Markovian property. Namely, since both S_F and S_N are local functionals of the paths, it is possible to regard the finite-time propagation as the convolution of (infinitesimal) short-time kernels. The derivation of the master equation is explained in appendix B. The final result is
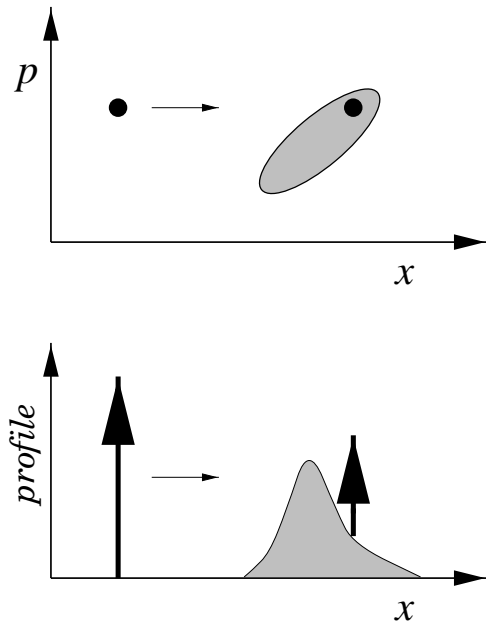
$$\frac{\partial \rho}{\partial t} = \left[-\partial_R \frac{1}{m}P + \partial_P \frac{\eta}{m}G_F \star P + \nu G_N\star\right]\rho. \tag{27}$$

The friction kernel is defined as follows

$$G_F \equiv \mathcal{FT}\left(\frac{w'(r)}{r}\right) = \frac{1}{\hbar/\ell}\hat{G}_F\left(\frac{P - P'}{\hbar/\ell}\right) \tag{28}$$

and the noise kernel is

$$G_N \equiv \frac{1}{\hbar^2}\mathcal{FT}(w(r) - w(0)) = \left(\frac{\ell}{\hbar}\right)^2\left[\frac{1}{\hbar/\ell}\hat{G}_N\left(\frac{P - P'}{\hbar/\ell}\right) - \delta(P - P')\right]. \tag{29}$$

Figure 1. Upper plot: phase-space illustration for the structure of the DLD propagator. Lower plot: the projected phase-space density.

Both $\hat{G}_F$ and $\hat{G}_N$ are smooth Gaussian-like scaling functions, properly normalized to unity.

If the Wigner function does not possess fine details on the momentum scale $\hbar/\ell$, then the convolution with G_F can be replaced by multiplication with 1, and the convolution with G_N can be replaced by $\partial^2/\partial P^2$. These replacements are formally legitimate both in the classical limit $\hbar \to 0$, and in the ZCL limit $\ell \to \infty$. One obtains then the classical Fokker–Planck equation

$$\frac{\partial \rho}{\partial t} = \left[-\partial_R \frac{1}{m} P + \partial_P \frac{\eta}{m} P + \nu \frac{\partial^2}{\partial P^2} \right] \rho. \tag{30}$$

Inspection of (24) yields the same observation. Namely, that if the propagator acts on a smooth Wigner function (no features on momentum scale $\hbar/\ell$), then the second term becomes vanishingly small, while the first term becomes effectively classical.

4. Dephasing within the Wigner picture

The Wigner function may have some modulation on a fine scale due to an interference effect. The standard textbook example of a two-slit experiment will be analysed below, where the interference pattern has the momentum scale $\hbar/d$ (see figure 2). d is the distance between the slits. In this section we shall explain the possible mechanisms that lead to the disappearance of such an interference pattern. In view of the semiclassical point of view of the next section, we use the term 'dephasing' in order to refer to this disappearance effect. In this section, as in the previous one, we are still limiting ourselves to the high-temperature regime.

In the case of the ZCL model, the propagator is the same as the classical one, and therefore we may adopt a simple Langevin picture in order to analyse the dephasing process. Alternatively, we may regard the dephasing process as arising from Gaussian smearing of the interference pattern by the propagator. In the case of the DLD model we should distinguish between two possible mechanisms for dephasing:
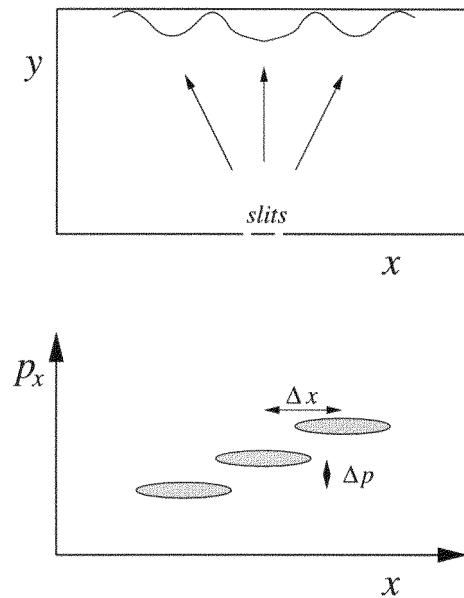
Figure 2. Upper plot: the geometry of a two-slit experiment. The propagation of the wavepacket is in the y direction, and the interference pattern is resolved on the screen. Lower plot: phase-space picture of the dynamics. The Wigner function of the emerging wavepacket is projected onto the (x, p_x) plane.

- scattering (perturbative) mechanism;
- spreading (non-perturbative) mechanism.

Actually, it is better to regard them as mechanisms to maintain coherence. The first mechanism to maintain coherence is simply not to be scattered by the environment. The second mechanism to maintain coherence is not to be smeared by the propagator. The first mechanism is absent in the case of the ZCL model. We shall see shortly that the first mechanism is much more effective in maintaining coherence. The notions perturbative and non-perturbative are used in order to suggest a connection with an earlier work [6]. The smearing effect can be regarded as arising from the accumulation of many small-momentum scattering events. A conventional perturbative approach cannot be applied in order to analyse such a dephasing process, even in the limit of very faint noise.

Having in mind the specific example of a two-slit experiment, we should distinguish between two cases. If the environment is characterized by a spatial correlation scale ℓ that is much larger than d, then we can 'forget' about the DLD model and just use the ZCL model. The relevant mechanism for destruction of coherence in this case is the spreading mechanism (see the detailed analysis in the next paragraph). If the environment characterized by a spatial correlation scale $\ell \ll d$ then a totally different picture emerges. Now, the Wigner function contains a modulation on a momentum scale much finer than $\hbar/\ell$. This modulation is not affected by the friction, but its intensity decays exponentially in time. This is based on inspection of either the propagator (24), or the equivalent master equation (27). In the latter case note that the convolution with G_N can be replaced by multiplication with $-(\ell/\hbar)^2$. The decay rate is

$$\frac{1}{\tau_\varphi} = \frac{2\eta k_B T \ell^2}{\hbar^2} \qquad \text{WNA for } 0 < \sigma. \qquad (31)$$

This is a universal result for the dephasing rate due to the 'scattering mechanism', since it does not depend on details of the quantum mechanical state involved. However, the validity of the result is restricted to the high-temperature regime, where WNA can be applied.

For completeness we turn back to the detailed analysis of dephasing due to spreading.

This mechanism leads to non-universal results, since the calculation of the dephasing time depends on actual details of the interference pattern. For concreteness we consider the simplest case of a two-slit interference experiment. The wavepacket that emerges from the two slits is assumed to be a superposition $\psi(x) = \sum_{\pm} \varphi(|x|) \exp(ik|x \mp d/2|)$, where $\hbar k$ is the momentum of the incident particle, and $\varphi(|x|)$ is a radial envelope function. The corresponding Wigner function is

$$\rho(x, p) \approx (1 + \cos(\vec{d} \cdot p/4)) \times \rho_{\text{one-slit}}(x, p) = \sum_{n=-\infty}^{\infty} \rho_n(x, p). \tag{32}$$

The one-slit Wigner function is multiplied by $\cos^2(\frac{1}{2}d \cdot p_x)$, and therefore it is natural to regard the two-slit Wigner function as composed of partial wavepackets, each characterized by a different transverse momentum. By definition the partial wavepacket ρ_n equals ρ for $|p_x - 2\pi n/d| < \pi/d$ and equals zero otherwise. Each partial wavepacket represents the possibility that the particle, being scattered by the slits, had acquired a transverse momentum which is an integer multiple of $\Delta p_x = 2\pi\hbar/d$. Note that the corresponding angular separation is $\Delta p_x/(\hbar k) = \lambda_B/d$, as expected, and that the associated spatial separation is $\Delta x = (\Delta p/m) \cdot t$ where $t = y/(\hbar k/m)$ is the time up to the screen. It is important to distinguish between the 'preparation' zone $y < d$ which is excluded from our considerations, and the far-field (Franhoufer) zone $d^2/\lambda_B \ll y$ where $\hbar \ll \Delta x \Delta p_x$ and consequently the individual partial wavepackets can be resolved. Due to the noise the interference pattern is smeared on a momentum scale $\delta p_x = \sqrt{\nu t}$, and on a corresponding spatial scale $\delta x = \sqrt{\nu t} \cdot t$. The interference pattern disappears once $\delta p_x \sim \Delta p_x$ or equivalently $\delta x \sim \Delta x$. This leads to the following expression for the dephasing time:

$$\frac{1}{\tau_\varphi} = \frac{\eta k_B T}{\hbar^2} d^2 \qquad \text{WNA for the ZCL model (two-slit).} \tag{33}$$

Comparing with (31) we observe that indeed the smearing mechanism is very effective in suppressing the interference pattern. The result is also non-universal since it depends on details of the interference pattern. In some other circumstances, where the Wigner function is characterized by a different type of interference pattern (notably the case of a 'vertical' interference pattern), the dephasing time may be proportional to some fractional power of the noise intensity [6]. In general, in the full analysis of the smearing process, one should take into account the effect of friction. In the above example this effect has been neglected.

The analysis of dephasing in the case of the ZCL model is easily extended to the low-temperature regime. Langevin formalism is still applicable, provided the appropriate $\phi(\tau)$ is being used. The smearing of the interference pattern can be analysed as in the previous paragraph. At the limit of zero temperature one should take into account the negative correlations of the noise [6]. For the ballistic-like motion that has been considered above, the smearing scale δp^2 is proportional, at zero temperature, to $\ln(t)$ rather than to t, leading to an anomalous (non-exponential) dephasing factor [6, 2].

5. Dephasing—the semiclassical point of view

There is a very different way to analyse dephasing, based on the semiclassical point of view. The advantage of the semiclassical approach is that it allows one to extend the study of dephasing to the low-temperature regime as well as to circumstances in which the simple ballistic-like Brownian motion scenario is not applicable. Specific examples that will be discussed later are the transmission via either chaotic or diffusive cavity. From now on we regard the two-slit experiment as a special case of transport problem (see figure 3). The
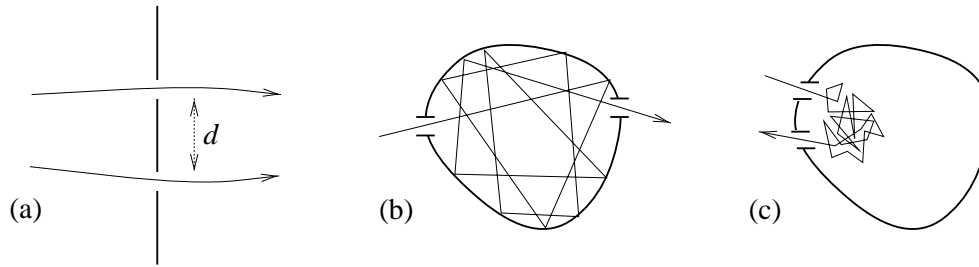
Figure 3. Various types of transport problems: (*a*) ballistic transport as in the two-slit experiment; (*b*) transport via a chaotic cavity; (*c*) transport via diffusive cavity, as in weak localization experiments. One should consider also the case of ergodic motion via a diffusive cavity (not plotted).

probability of crossing the obstacle, or more generally being transmitted via some cavity, can be written as a (double) sum over classical trajectories. Thus,

$$\sum_{ab} A_a A_b^* \exp\left(i\frac{S[x_a] - S[x_b]}{\hbar} + i\frac{S_F[x_a, x_b]}{\hbar} - \frac{S_N[x_a, x_b]}{\hbar^2} \right) \tag{34}$$

where $S[x_a]$ is the classical action for the classical trajectory x_a, and A_a is the corresponding classical amplitude. Each pair a and b of classical trajectories constitutes a stationary-phase point of the exact path-integral expression. However, the notation is somewhat misleading since once the friction functional is switched on, a and b can no longer be considered independent indexes. In particular, strictly speaking, the diagonal terms are no longer truly diagonal. However, here comes a very important observation. In the case of the DLD model, assuming that x_a and x_b are well separated with respect to the microscopic scale ℓ, one has $S_F[x_a, x_b] = 0$. Thus we conclude that friction has no effect on the interference part of the sum (34), in consistency with our discussion of the scattering mechanism in the previous section. On the other hand, friction will have an effect on the diagonal terms of the sum, as required by the correspondence principle. It should be emphasized that in the case of the ZCL model, friction may have a non-negligible effect also on the interference terms, again in consistency with our discussion of the smearing mechanism.

The suppression of an off-diagonal term in (34) is given by the dephasing factor $\exp(-S_N[x_a, x_b]/\hbar^2)$. Using the definitions of appendix D it is possible to transform expression (21) into an integral over the Fourier components of the motion involved. Consequently, in most cases of interest, the dephasing factor can be cast into the form $\exp(-t/\tau_\varphi)$, where

$$\frac{1}{\tau_\varphi} = \frac{1}{\hbar^2} \int_0^\infty g(k)\, dk \int_0^\infty \frac{d\omega}{\pi} \phi(\omega) P(k, \omega). \tag{35}$$

The domain of integration is illustrated in figure 4. The power spectrum $P(k, \omega)$ of the motion is calculated in appendix C for ballistic, chaotic and diffusive motion. Various results that can be derived by using the above expression will be presented in later sections.

One should be very careful in the physical interpretation of $\exp(-t/\tau_\varphi)$. If we have two unrelated trajectories a and b, and we also have static disorder, then we will have 'statistical' dephasing that reflects the effect of averaging over realizations of the disorder. Consider for example a two-slit experiment: the interference pattern on the screen will be distorted for any particular realization of a static disorder, and will be washed away completely upon averaging over different realizations. This mechanism is non-effective if a
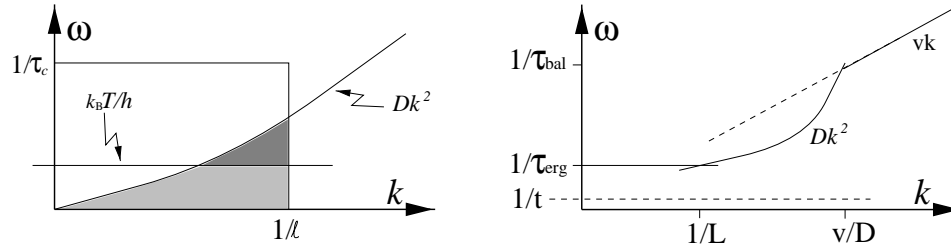
Figure 4. The (k, ω) plane. Left plot: the shaded regions indicate those environmental modes that are effective in the dephasing process. The darker region indicates a possible excess contribution due to ZPF. The curve Dk^2 illustrates the frequency-span of $P(k, \omega)$. Right plot: the power spectrum of an ergodic motion via a diffusive cavity is concentrated under the plotted curves. See further details in appendix D.

and b are related by time reversal, for which static disorder gives $S_N[x_a, x_b] = 0$. However, some suppression is also expected in the latter case, via the classical-like amplitudes $|A_a|^2$. In the case of a dynamical environment the role of elastic scattering is taken by inelastic scattering events. Therefore we have genuine dephasing, irrespective of whether a and b are related by time reversal. In some typical circumstances, the dephasing factor can be re-interpreted as imparting the probability of 'leaving a trace' in the environment (see appendix C). We take now the limit of zero temperature. Assuming for simplicity that a and b are related by time reversal, it follows from the definition of the influence functional that $\exp(-S_N[x_a, x_b]/\hbar^2)$ can be interpreted as the probability of exciting at least one of the oscillators along the way.

Up to now we have encountered, depending on the physical circumstances, two possible interpretations for the dephasing factor. The 'statistical' interpretation holds in the case of static disorder, while the 'leaving-trace' interpretation holds in typical cases of a dynamical environment. Let us keep $T = 0$ and take the limit $E \to 0$, where E is the available energy of the particle. In order to make the following argument more illuminating, one may assume that the particle has a small but finite kinetic energy E, and that all the oscillators have relatively large frequencies, such that $E \ll \omega_\alpha$. Consequently, all the scattering events are elastic. On the other hand $1/\tau_\varphi$ has a non-vanishing value, which implies that the off-diagonal terms are being suppressed. Obviously, the 'leaving-trace' interpretation of the dephasing factor no longer holds. One wonders whether $\exp(-S_N/\hbar^2)$ acquires, under such circumstances, a somewhat different physical meaning, as in the case of static disorder. Maybe it now has the probability of being scattered elastically (rather than inelastically) along the way, thus leading to dephasing of the 'statistical' type. A perturbative treatment of the scattering process leads to the conclusion that the probability of being elastically scattered by a zero-temperature oscillator is proportional to c_α^4. This is because the elastic scattering off an oscillator involves a second-order process of virtual emission followed by absorption of a quanta $\hbar\omega_\alpha$. At the same time $S_N[x_a, x_b]$ is proportional to c_α^2 to leading order. Therefore, $\exp(-S_N/\hbar^2)$ cannot have the desired physical significance. We therefore must conclude that something goes wrong with the present semiclassical approach once a low-energy particle is concerned. We shall return to this point later on.

6. Dephasing at high temperatures

As a first step in the application of the semiclassical approach, it is interesting to recover results (31) and (33) for the dephasing rate. These results hold at high temperatures, such

that the WNA is applicable. The noise functional in such circumstances takes the following simple form

$$S_N[r] = 2\eta k_B T \int_0^t [w(0) - w(r(t'))] \, dt' \tag{36}$$

where $r = x_a - x_b$. If the two trajectories are well separated with respect to the microscopic scale ℓ, then one indeed recovers the universal result (31). Recall that $w(0) \equiv \ell^2$. At the other extreme limit $\ell \to \infty$, that corresponds to the ZCL model, one obtains

$$S_N[r] = 2\eta k_B T \int_0^t r(t')^2 \, dt' \equiv 2\eta k_B T r_\perp^2 \cdot t. \tag{37}$$

In the case of the two-slit experiment, $r \sim d$, and one recovers (33). Obviously, it is not a universal result. For example, if the separation between the two trajectories grows linearly, then the dephasing time will be proportional to $1/T^{1/3}$ rather than to $1/T$.

The generalized DLD model with $-2 < \sigma < 0$, requires special care since $w(0)$ diverges. We are still considering high temperatures, such that the WNA is applicable. Using the notations of appendix D, expression (36) can be cast into the form

$$S_N[r] = vt \int w(r) P(r) \, dr = vt \int_0^\infty g(k) \, dk \, P(k). \tag{38}$$

The latter expression converges for any $-2 < \sigma$. In the ZCL limit, where ℓ is much larger than the average transverse distance between the trajectories, one may use the small-k approximation of appendix D, and then (37) is recovered. In the DLD case, where ℓ is very small scale, the integration should be split into the domains $k < k_\perp$ and $k > k_\perp$. The wavenumber $k_\perp \equiv 1/r_\perp$ is associated with the transverse distance $r_\perp$ between the two trajectories. Thus,

$$r_\perp \approx \begin{cases} d & \text{for ballistic motion } t < \tau_{\text{bal}} \\ \sqrt{Dt} & \text{for diffusive motion } \tau_{\text{bal}} < t < \tau_{\text{erg}} \\ L & \text{for ergodic motion } \tau_{\text{erg}} < t. \end{cases} \tag{39}$$

Note that ergodic motion refers to either chaotic or diffusive trajectories with $\tau_{\text{erg}} < t$. Again we use the notation of appendix D. For $0 < \sigma$ the integration is dominated by the upper cut-off $k \sim 1/\ell$. One then recovers the standard result (31). On the other hand, for $-2 < \sigma < 0$, the integration is dominated by the $k \sim k_\perp$ modes. The wavelength of these modes is like the transverse distance between the trajectories. One obtains then

$$S_N[r] = 2\eta k_B T \ell^2 \left(\frac{\ell}{r_\perp} \right)^\sigma \cdot t \qquad \text{WNA for } -2 < \sigma < 0. \tag{40}$$

For non-ergodic diffusive motion $r_\perp$ depends on t and therefore the dephasing factor is of the type $\exp(-(t/\tau_\varphi)^{(2-\sigma)/2})$. In all other cases we have a simple exponential dephasing factor.

Obviously (38) is a special case of the general formula (35). Note that (35) should be multiplied by $\hbar^2 t$ in order to become a proper expression for S_N. By comparing (35) to (38) one can reveal the actual condition for the validity of the WNA. Within the interval $0 < k < 1/\ell$, the power spectrum of the motion occupies a frequency range $\omega < \omega_{\text{cl}}$, where the effective cut-off is

$$\omega_{\text{cl}} = \begin{cases} v/\ell & \text{if } k = 1/\ell \text{ is located within the ballistic regime} \\ D/\ell^2 & \text{if } k = 1/\ell \text{ is located within the diffusive regime.} \end{cases} \tag{41}$$

As long as $\omega_{\text{cl}} < k_B T/\hbar$ one can use the WNA in estimating the integral in (35). One observes that under such circumstances (35) reduces, after multiplication by $\hbar^2 t$, to (38).

7. Dephasing at zero temperatures

As the temperature becomes low, such that $k_B T /\hbar < \omega_{cl}$, the dephasing rate becomes larger than the value which is predicted by the WNA. This is due to the ZPF in the frequency zone $k_B T /\hbar < \omega$. The temperature becomes effectively zero if $k_B T /\hbar < 1/t$. Time longer than t is required in order to resolve such low temperatures. One can then use the zero temperature limit of $\phi(\tau)$ for which $\phi(\omega) = \hbar\eta\omega$. This power spectrum corresponds to the ZPF of the environmental modes.

For either ballistic or chaotic motion, The integral (35) is dominated by (k, ω) modes that are concentrated below the curve $\omega = vk$. One obtains

$$\frac{1}{\tau_\varphi} \approx \frac{C}{(1+\sigma)\pi} \times \frac{1}{\hbar}\eta\ell v \qquad \text{for ballistic or chaotic motion.} \qquad (42)$$

Similarly, for diffusive motion the integral is dominated by (k, ω) modes that are concentrated below the curve $\omega = Dk^2$. One obtains

$$\frac{1}{\tau_\varphi} = \frac{C}{(2+\sigma)\pi} \ln\left(1 + \left(\frac{\ell v}{D}\right)^4\right) \times \frac{1}{\hbar}\eta D \qquad \text{for diffusive motion.} \qquad (43)$$

In both cases most of the contribution comes from modes with large wavenumber, namely $k \sim 1/\ell$.

The validity of the present semiclassical approach, which is based on the stationary-phase approximation is limited. Common wisdom [9] is that the applicability of the semiclassical approach is restricted to circumstances in which the energy transfer between the particle and the environment is much smaller than the particle's available energy. Technically, this is equivalent to the assumption of uncoupled wave equations†. The *coupled* wave equations for the particle oscillators system can be *uncoupled* provided certain conditions are satisfied. Then we can treat the particle as moving with constant velocity v and solve for the oscillators. It turns out that this reduction requires the assumption of small-energy transfer. Therefore, one should anticipate problems once oscillators with ω_α larger than E are involved. In the latter case, there is no justification for thinking of the particle as decoupled from the bath, moving with some constant velocity, capable of exciting oscillators along the way. Therefore the corresponding factor $\exp(-S_N/\hbar^2)$ loses its physical significance. From the above it follows that a reasonable condition for the validity of our semiclassical approach is

$$\hbar\omega_{cl} < E \qquad \text{definition of large energy.} \qquad (44)$$

On physical grounds this is the condition for being able to leave a trace 'along the way'. The semiclassical significance of this condition can be further illuminated. For ballistic-like motion it is equivalent to the condition of small momentum transfer $\hbar/\ell < p$. This is precisely the condition for the applicability of the semiclassical methods for the scattering of the particle by the oscillators. More generally, one can define a quantum mechanical version of $P(k, \omega)$ as the FT of the correlator of the Heisenberg picture operator $\exp(ikx(t))$. The quantal $P(k, \omega)$ coincides with the classical $P(k, \omega)$ only for $\omega < E/\hbar$. Larger classical frequencies are not supported by the effectively banded energy spectrum. Our practical conclusion is that the contribution of ZPF should be included if non-thermal motion with large energy is concerned. It should be excluded in case of thermal motion with $E \sim k_B T$.

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† I thank Uzy Smilansky for pointing out this observation.

8. Dephasing due to thermal noise

The possible contribution of ZPF to the dephasing has been discussed in the previous section. Now we are interested in the the thermal noise contribution (TNC). The TNC can be calculated using (35) with $\phi(\omega)$ replaced by $\phi_T(\omega)$. See (23) for the definition of the latter. As the temperatures are raised various regimes are encountered:

$$
\begin{aligned}
\text{extremely low temperatures} &\quad\Leftrightarrow\quad && 1/t < k_B T/\hbar < 1/\tau_\perp \\
\text{low temperatures} &\quad\Leftrightarrow\quad && 1/\tau_\perp < k_B T/\hbar < \omega_{\text{cl}} \\
\text{high temperatures} &\quad\Leftrightarrow\quad && \omega_{\text{cl}} < k_B T/\hbar.
\end{aligned}
\tag{45}
$$

The timescale $\tau_\perp$ corresponds to the transverse distance $r_\perp$ of (39). It equals d/v for ballistic motion, and τ_{erg} for ergodic motion. In case of non-ergodic diffusive motion, where $\tau_\perp = t$, the extremely low temperature regime is absent. At high temperatures we can use the WNA as discussed in previous sections. The low and the extremely low temperature regimes are discussed in the next two paragraphs.

When considering *extremely low* temperatures it is useful to define the critical exponent $\sigma_c = 1$ for ballistic motion, and $\sigma_c = 2$ for diffusive motion. If $\sigma < \sigma_c$ the integration in (35) is dominated by $k \sim k_\perp$ and one obtains the result

$$
\frac{1}{\tau_\varphi} = C' \frac{2\eta k_B T}{\hbar^2} \ell^2 \times \left(\frac{\ell}{r_\perp}\right)^\sigma \times \left(\frac{k_B T/\hbar}{1/\tau_\perp}\right).
\tag{46}
$$

For $\sigma_c < \sigma$ the integration is dominated by $k \sim 1/\ell$ and one obtains the result

$$
\frac{1}{\tau_\varphi} = C' \frac{2\eta k_B T}{\hbar^2} \ell^2 \times \left(\frac{k_B T/\hbar}{\omega_{\text{cl}}}\right).
\tag{47}
$$

In the formulae above C' is a numerical factor of order unity. Note that in both cases, disregarding the possible contribution of ZPF, the dephasing rate is proportional to T^2. This should be contrasted with the high-temperature behaviour where, disregarding the special case of non-ergodic diffusive motion, the dephasing rate is proportional to T.

At *low* (but not extremely low) temperatures it is useful to define a temperature-dependent wavenumber as follows:

$$
k_T = \begin{cases} (k_B T/\hbar v) & \text{in the ballistic regime} \\ (k_B T/\hbar D)^{1/2} & \text{in the diffusive regime.} \end{cases}
\tag{48}
$$

For $\sigma < 0$ the integration in (35) is dominated by $k \sim k_\perp$, and one can use the WNA result (40). Thus, for $\sigma < 0$, in the absence of ZPF contribution, we can trust the WNA at both high and low temperatures, and an actual crossover is expected only when extremely low temperatures are involved. For $0 < \sigma$ the integration is dominated by $k \sim k_T$, and one obtains the result

$$
\frac{1}{\tau_\varphi} = C' \frac{2\eta k_B T}{\hbar^2} \ell^2 \times (\ell k_T)^\sigma.
\tag{49}
$$

If $\sigma_c < \sigma$, the integration is dominated by $k \sim 1/\ell$, and one again obtains (47). Thus, for $0 < \sigma < \sigma_c$, there is a non-trivial low-temperature regime where the TNC to the dephasing rate is proportional to a non-universal power of T. Below this intermediate-temperature regime the TNC is proportional to T^2. Above this intermediate-temperature regime we can trust the WNA and the dephasing rate is proportional to T.

9. Manifestation of effective static disorder

We shall consider in this section quantal Brownian motion in $2 < d$ dimensions, that is described by the DLD model with short range spatial correlations. For concreteness let us assume that the fluctuations of the effective stochastic potential satisfy

$$\langle \mathcal{U}(x'', t'')\mathcal{U}(x', t') \rangle = \phi(t'' - t') \cdot \ell^2 \exp\left(-\frac{1}{2} \left(\frac{x'' - x'}{\ell} \right)^2 \right). \qquad (50)$$

If we are interested in the dynamics over a *finite* time interval t, then obviously all the Fourier components in the frequency regime $|\omega| < 1/t$ will have the same effect as static disorder. Let us denote by $\bar{\mathcal{U}}(x, t)$ the 'static' component of the effective stochastic potential. The variance of $\bar{\mathcal{U}}$ is determined by the product $\phi(\omega = 0) \cdot (1/t)$. Recall that $\phi(\omega = 0) \equiv \nu$. Consequently we have

$$\langle \bar{\mathcal{U}}(x'', t'')\bar{\mathcal{U}}(x', t') \rangle = W^2 \exp\left(-\frac{1}{2} \left(\frac{x'' - x'}{\ell} \right)^2 \right) \qquad (51)$$

where $W^2 = \nu \ell^2 / t$. We would like to find out whether this effective static disorder will manifest itself. A similar question has been raised in [10]. One should not take for granted that the effect of low-frequency fluctuations is completely masked by the incoherent effect of the high-frequency modes.

The first obvious step is to calculate the 'statistical' dephasing rate due to the presence of the (effective) disorder. We can use (35) with the formal substitution $\phi(\tau) = W^2/\ell^2$, thus obtaining

$$\frac{1}{\tau_\varphi^0} = \frac{1}{\hbar^2} \frac{W^2}{\ell^2} \int_0^\infty g(k) \, dk \, P(k, \omega = 0). \qquad (52)$$

For ballistic-like motion one obtains

$$\frac{1}{\tau_\varphi^0} = \frac{1}{\hbar^2} W^2 \frac{\ell}{v} = \left(\frac{\nu \ell^3}{\hbar^2 v} \right) \cdot \frac{1}{t}. \qquad (53)$$

In order to have a non-vanishing effect we should have $\tau_\varphi^0 \ll t$. This condition can be cast into the form $\xi \ll vt$ where $\xi = (\hbar v)^2/(\ell W^2)$ is the mean free path. Still another way to express this condition is $\tau_\varphi^{\text{WNA}} \ll \ell/v$, where $\tau_\varphi^{\text{WNA}}$ is given by (31). On the other hand, the actual dephasing time should satisfy $\ell/v < \tau_\varphi$, otherwise coherent effects due to the scattering by the (effective) disordered potential will not manifest themselves. Thus, we come to the conclusion that the following condition should be satisfied in order to have manifestation of coherent effects due to scattering by the effective static disorder:

$$\tau_\varphi^{\text{WNA}} \ll \frac{\ell}{v} \ll \tau_\varphi. \qquad (54)$$

Obviously, this condition can be satisfied only in the low-temperature regime.

10. Dephasing versus dissipation, concluding remarks

It is quite striking that the friction coefficient η is not affected by quantum mechanical effects. Having the Boltzmann picture in mind it is not anticipated that all the quantum mechanical scattering events will conspire to give the classical result. Still, this conclusion follows from the FV formalism quite easily. In order to have a clear physical picture, let us consider the time evolution of a Gaussian wavepacket using either the propagator (26)

or the equivalent master equation (27). Clearly, the dissipation rate dE/dt is the same as in the classical picture, with small transient corrections. Thus, also in the quantum mechanical picture, the damping process is characterized by the time constant $\tau_\eta = (\eta/m)^{-1}$.

Consider a ballistic-like motion with non-thermal energy, such that $\hbar/\ell \ll p$. Consequently the equivalent condition (44) is satisfied, and we can trust semiclassical considerations as far as dephasing is concerned. We can cast the universal WNA result (31) and the ZPF result (42) into the form

$$\frac{1/\tau_\varphi}{1/\tau_\eta} \approx \begin{cases} (\ell/\lambda_T)^2 & \text{high-}T \text{ (universal)} \\ (\ell/\lambda_B) & \text{low-}T \end{cases} \tag{55}$$

where λ_T is the thermal wavelength, and λ_B is the de Broglie wavelength. We see that for the above *non-thermal motion* the dephasing time is always shorter than the damping time. The dephasing rate is linear in T at high temperatures, and saturates at low temperatures.

For *thermal motion* the latter statement is no longer true. For the low-temperature regime the ZPF contribution should be excluded, and therefore the dephasing rate goes to zero, while the damping time remains finite. One should differentiate between various physical circumstances. At extremely low temperatures, assuming ergodic motion, the dephasing rate is proportional to T^2. At higher temperatures we can trust the WNA provided $\sigma < 0$. Otherwise, there is an intermediate low-temperature regime where the dephasing rate is proportional to a non-trivial power of T.

In this paper we have introduced a systematic derivation of a general formula for the dephasing rate, equation (35), that holds in all physical circumstances, including the ZCL limit. Expressions similar to equation (35), that incorporate integration over the (k, ω) environmental modes, are encountered frequently in the literature, starting from the well known work in [11]. Whenever diffusive motion of electrons is concerned (see appendix E), our expressions agree with well known results [7]. However, most publications avoid a straightforward application of the FV formalism, and introduce at some stage heuristic considerations in order to obtain a convergent result. There is a 'zoo' of cut-offs that are introduced in performing the (k, ω) integration, and some of them are questionable. For example, it is customary to take $1/\tau_\varphi$ itself as a lower cut-off for the ω integration. The upper cut-off is sometimes $k_B T/\hbar$, sometimes Fermi energy, sometimes the kinetic energy that corresponds to the drift velocity, sometimes the inelastic scattering rate, and so on. The role of Fermi statistics in the determination of the various cut-offs is usually left unclear. There are similar ambiguities in the determination of the the proper cut-offs for the k integration. Our derivation has led to a proper definition for the power spectrum of the motion, and has made it unnecessary to introduce *ad hoc* cut-offs into the calculations. Furthermore, in the analysis of dephasing at the limit of zero temperature, we were successful in reducing our considerations to the level of 'one-particle physics', thus avoiding a complicated discussion of the role played by the Pauli exclusion principle.

Appendix A. Useful identity

We first cite the well known identity

$$\frac{1}{\sqrt{2\pi\,\delta t}}\exp\left(+\frac{i}{2\delta t}(x-x_0)^2\right)=\left[1+\frac{i}{2}\delta t\,\partial_x^2+\mathcal{O}(\delta t^2)\right]\delta(x-x_0).$$

Both sides of this identity should be interpreted as kernels of operators. When applied to wavefunctions, the left-hand side corresponds to the free-motion propagator, and the second term in the right-hand side corresponds to the free-motion Hamiltonian. We shall explain now how to derive the following related identity:

$$\frac{1}{2\pi\,\delta t}\exp\left(\frac{i}{\delta t}(R-R_0)(r-r_0)-i\eta\cdot(R-R_0)\right)$$
$$=[1+i\delta t\,\partial_R\partial_r+\eta\delta t\,\partial_r+\mathcal{O}(\delta t^2)]\delta(R-R_0)\delta(r-r_0). \tag{A1}$$

Both sides of the latter identity should be interpreted as kernels of operators that operate on phase-space functions. We start the derivation by writing an equality that follows from the first identity via simple replacements:

$$\frac{1}{\sqrt{2\pi\,\delta t}}\exp\left(\pm\frac{i}{2\delta t}\left(x_\pm\mp\frac{1}{2}\eta\delta t\right)^2\right)=\left[1+\frac{i}{2}\delta t\,\partial_\pm^2+\mathcal{O}(\delta t^2)\right]\delta\left(x_\pm\mp\frac{\eta}{\delta}t\right).$$

Upon multiplication it follows that

$$\frac{1}{\sqrt{2\pi\,\delta t}}\exp\left(\pm\frac{i}{2\delta t}(x_+^2-x_-^2)-i\frac{1}{2}\eta(x_++x_-)\right)$$
$$=\left[1+\frac{i}{2}\delta t\,\partial_+^2\right]\left(\delta(x_+)-\frac{1}{2}\eta\delta t\,\delta'(x_+)\right)$$
$$\cdot\left[1-\frac{i}{2}\delta t\,\partial_-^2\right]\left(\delta(x_-)+\frac{1}{2}\eta\delta t\,\delta'(x_+)\right).$$

Simplification of the right-hand side gives

$$=\left[1+\frac{i}{2}\delta t\,(\partial_+^2-\partial_-^2)+\frac{1}{2}\eta\delta t\,(\partial_+-\partial_-)+\mathcal{O}(\delta t^2)\right]\delta(x_+)\delta(x_-).$$

The replacements $x_+\to(x''-x_0'')$ and $x_-\to(x'-x_0')$, followed by the transformation to the variables $R=(x''+x')/2$ and $r=(x''-x')$, gives the desired result. Note that the derivation holds also in the case where η is replaced by some function of r.

Appendix B. Derivation of the master equation

The Wigner function $\rho(R,P)$ is the Fourier transform of the reduced-probability density matrix $\rho(R,r)$ in the variable $r\rightsquigarrow P$. The path-integral expression for the kernel $\mathcal{K}(R,r|R_0,r_0)$ is of the type $\int\int\mathcal{D}R\mathcal{D}r$ with obvious endpoint conditions, and its integrand is essentially the same as in (19). See [2] for more details. The infinitesimal-time kernel equates simply to the integrand of the path-integral expression, namely

$$\mathcal{K}_{\delta t}=\exp\left(\frac{i}{\delta t}\frac{m}{\hbar}(R-R_0)(r-r_0)+i\frac{\eta}{m}w'(r)\cdot(R-R_0)-\delta t\frac{\nu}{\hbar^2}(w(0)-w(r))\right)$$
$$=\left[1+\delta t\left(i\frac{\hbar}{m}\partial_R\partial_r-\frac{\eta}{m}w'(r)\partial_r-\frac{\nu}{\hbar^2}(w(0)-w(r))\right)+\mathcal{O}(\delta t^2)\right]$$
$$\delta(R-R_0)\delta(r-r_0)$$

where the second line is obtained by employing the identity of appendix A. Thus, to leading order, the kernel $\mathcal{K}_{\delta t}$ has the same effect as operating with a differential operator of the type $1 + \delta t \mathcal{L}$. Consequently, we find that $\rho(R, r)$ satisfies a master equation of the form $\partial \rho / \partial t = \mathcal{L} \rho$. This master equation is easily transformed into the Wigner representation, where any derivative ∂_r is replaced by multiplication with $iP/\hbar$. Similarly, any multiplication by a (real symmetric) function $\tilde{G}(r)$ is transformed into a convolution with a (real symmetric) kernel $G(P - P')$. For the convolution $\int G(P - P') \rho(P') \, dP'$ we use the notation $G \star \rho$. A kernel $G(P - P')$ and a function $\tilde{G}(r)$ are related by a Fourier transform with the convention

$$G(P - P') = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} G(r) \cos\left(\frac{P - P'}{\hbar} r\right) dr.$$

In the above expression we have modified the standard FT convention, by including the factor $1/(2\pi\hbar)$. This has been done in order to have properly normalized kernels, with the measure dP rather than $dP/(2\pi\hbar)$. Note also that the FT of $w'(r)$ equals to the $i\hbar\partial_P$ derivative of the kernel G_F that corresponds to the real symmetric function $w'(r)/r$.

Appendix C. Dephasing and inelastic scattering

Consider the interference contribution of two trajectories a and b that are related by time reversal. Assuming short range interaction with the environmental modes we can prove that dephasing is related to the probability for leaving a trace in the environment. This statement is true at any temperature. 'Short range interaction' means that $0 < \sigma$ and that ℓ is a small scale. 'Leaving a trace' means that at least one of the oscillators has changed its quantum-mechanical state. It follows that under such circumstances dephasing-rate is equal to the inelastic scattering rate.

Recall the definition of the influence functional. For simplicity we consider first a zero-temperature bath, meaning that all the bath-oscillators are initially in the ground state. These oscillators are driven by the motion of the particle. The excited states of the bath will be denoted by $|\{n_\alpha\}\rangle$. The evolution operator of a driven oscillator will be denoted by $U_\alpha[x]$.

$$F[x_a, x_b] \equiv \sum_{\{n_\alpha\}} \langle \{n_\alpha\} | \prod_\alpha U_\alpha[x_b] | \{0\} \rangle \langle \{n_\alpha\} | \prod_\alpha U_\alpha[x_a] | \{0\} \rangle^\star.$$

If a and b are related by time reversal, then one can consider only those oscillators that are located 'along the loop'. Each oscillator along the loop is characterized by its natural frequency ω_α and by the time t_α at which $x_a(t) \approx x_\alpha$. Consequently we have the following expression for the influence functional:

$$F[x_a, x_b] = \sum_{\{n_\alpha\}} e^{-i \sum_\alpha n_\alpha \omega_\alpha \cdot (2t_\alpha - t)} P(\{n_\alpha\} | \{0\}) \tag{C1}$$

where the excitation probability is $P(\{n_\alpha\} | \{0\}) \equiv \prod |\langle n_\alpha | U_\alpha[x_a] | 0 \rangle|^2$. We would like to argue that $F[x_a, x_b] = P(\{0\} | \{0\})$. Indeed, the summation in (C1) contains *one-oscillator* excitations for which $\sum n_\alpha = 1$, *two-oscillator* excitations for which $\sum n_\alpha = 2$ and so on. Let us consider all the one-oscillator excitations that involves ω_α in the range $[\omega, \omega + \delta\omega]$. Each of these excitations contributes the same $P(\{n_\alpha\} | \{0\})$, but with a different phase factor $\exp(i2\omega_\alpha t_\alpha)$. By construction of the DLD model (equation (12)), the summation over the phase factors will lead to a zero contribution. The meaning of 'zero contribution' is as follows: either we average over realizations of x_α, or else we recall that the influence functional appears inside a path-integral expression. The summation over paths will have

the same effect like an averaging procedure, and therefore we will have indeed a 'zero contribution' in the mathematically-generalized sense. Similar argumentation applies to other subsets of excitations. If the bath is initially in thermal equilibrium, rather than in zero temperature, than the relation $F[x_a, x_b] = P(\{0\}|\{0\})$ can be generalized into

$$F[x_a, x_b] = \exp(-S_N[x_a, x_b]/\hbar^2) = P \text{ (leaving no trace)}. \tag{C2}$$

In order to prove the latter statement one should use the same procedure as above with $|\{0\}\rangle$ replaced by some arbitrary initial preparation $|\{n_\alpha^0\}\rangle$. Then the result should be thermally averaged.

Appendix D. Statistical characterization of the trajectories

Having a pair a and b of trajectories, we can define the function

$$P_{ab}(r, \tau) = \langle \delta(r - (x_a(t' + \tau) - x_b(t'))) \rangle \tag{D1}$$

where the average is over t' within the time interval $[0, t]$ which is considered. We shall use the notations $P_{aa} = P_\parallel$ and $P_{ab} = P_\perp$ for $a \neq b$. It is also useful to define the functions

$$P(r, \tau) \equiv P_\parallel(r, \tau) - P_\perp(r, \tau)$$
$$P(r) \equiv P(r, 0). \tag{D2}$$

The FT of $P(r)$ will be denoted by $P(k)$, and the double FT of $P(r, \tau)$ will be denoted by $P(k, \omega)$. The above functions all appear within integrals where 'isotropic' integration over r or k is being performed. Therefore it is convenient to average all these functions over all orientations, and so to have functions that depend on either $r = |r|$ or $k = |k|$ respectively. A useful notation is $\mathrm{Cos}(r) \equiv \langle \cos(\mathbf{\Omega} \cdot r) \rangle$, where the average is over the orientation of the unit vector $\mathbf{\Omega}$. The function $\mathrm{Cos}(r)$ depends only on $|r|$. It equals $\cos(r)$ in one dimension, regular Bessel function $\mathcal{J}_0(r)$ in two dimensions, and $\mathrm{sinc}(r)$ in three dimensions.

We shall distinguish now between *ballistic* trajectories as in figure 3(a), *chaotic* trajectories as in figure 3(b), and *diffusive* trajectories as in figure 3(c). We can also add to this list the case of diffusive trajectories that cover ergodically the whole available space. In case of a ballistic trajectory $P_\parallel(r, \tau) = \delta(r - \bar{r})$, leading to $P_\parallel(k, \tau) = \mathrm{Cos}(k\bar{r})$, where $\bar{r} = v\tau$. Similar expressions hold for $P_\perp$, where $\bar{r} \approx (d^2 + (v\tau)^2)^{-1/2}$, and d is the transverse distance between the slits. For a chaotic trajectory $P_\parallel(r, \tau)$ starts as in the ballistic case, while $P_\perp(r, \tau)$ is uniform in r. The ergodic time is given essentially by the ballistic time, namely $\tau_{\mathrm{erg}} = \tau_{\mathrm{bal}} = L/v$, where L is the linear dimension of the cavity. For $\tau_{\mathrm{erg}} < \tau$ both $P_\parallel(r, \tau)$ and $P_\perp(r, \tau)$ become uniform in r. For diffusive motion $P_\parallel(r, \tau)$ is as for the ballistic case as long as $\tau < \tau_{\mathrm{bal}}$ where $\tau_{\mathrm{bal}} = D/v^2$. At larger times it becomes a Gaussian, and consequently $P_\parallel(k, \tau) = \exp(-Dk^2\tau)$, where D is the diffusion coefficient. The latter expression as well as the approximation $P_\perp(r, \tau) \approx P_\parallel(r, t)$ hold for non-ergodic diffusive motion. After the ergodic time $\tau_{\mathrm{erg}} = L^2/D$, both $P_\parallel(r, \tau)$ and $P_\perp(r, \tau)$ become uniform in r.

In the following paragraphs we discuss the various k regimes of the function $P(k, \omega)$. We shall distinguish between the small-k regime and the large-k regime. In the case of diffusive trajectories the large-k regime will be further divided into a ballistic regime ($v/D < k$) and a diffusive regime ($k < v/D$), where v/D is the inverse of the mean free path. We turn first to define the notions of small and large k. The transverse distance between two trajectories has a distribution $P_\perp(r)$, and we can define a typical value $r_\perp = (\langle r^2 \rangle)^{1/2}$. See equation (39). The associated wavenumber is $k_\perp \equiv 1/r_\perp$. Large k means from now on

$k_\perp \ll k$. For large k we may use the approximation

$$P(k, \omega) \approx P_\parallel(k, \omega) \qquad \text{for large } k \text{ (meaning } k_\perp \ll k\text{).} \qquad \text{(D3)}$$

For either ballistic, diffusive or chaotic trajectories we have

$$P(k, \omega) \approx \frac{1}{vk} \hat{B} \left(\frac{\omega}{vk} \right) \qquad \text{in the ballistic regime} \qquad \text{(D4)}$$

where $\hat{B}$ is the FT of the Cos function. This rectangular-like scaling function has a unit width and a unit normalization. In the case of diffusive motion we also have

$$P(k, \omega) \approx \frac{2Dk^2}{(Dk^2)^2 + \omega^2} \qquad \text{in the diffusive regime.} \qquad \text{(D5)}$$

Diffusive regime means here any large k that satisfies $k < v/D$. The validity of the latter approximation is further restricted by the condition $\omega < v^2/D$. The collision frequency $\omega \sim v^2/D$ should be used as a cut-off to the slow $1/\omega^2$ power-law decay. In our calculations we shall assume that $1/\ell \ll v/D$, meaning that the motion is diffusive also on the spatial scale ℓ.

For small k, meaning $k \leqslant k_\perp$, the transverse term cannot be ignored, and we can no longer use the approximation (D3). We can define a timescale $\tau_\perp$ that corresponds to $k_\perp$. It equals d/v for ballistic motion, t for diffusive motion, and τ_{erg} for ergodic motion. For a given small k, the function $P(k, \tau)$ is concentrated within $\tau < \tau_\perp$, since $P_\parallel$ and $P_\perp$ are not identical there. Consequently

$$P(k, \omega) \approx \tau_\perp \cdot P(k) \qquad \text{for } \omega < 1/\tau_\perp \text{ in the small-}k \text{ regime.} \qquad \text{(D6)}$$

The function $P_\perp(k)$ is the FT of $P_\perp(r)$, and therefore we have for $P(k) = 1 - P_\perp(k)$ the following small-k approximation

$$P(k) \approx \tfrac{1}{2} r_\perp^2 k^2 = \begin{cases} \tfrac{1}{2} d^2 k^2 & \text{for ballistic motion} \\ Dt \cdot k^2 & \text{for diffusive motion} \\ \tfrac{1}{2} L^2 k^2 & \text{for ergodic motion.} \end{cases} \qquad \text{(D7)}$$

For large k we have $P(k) \approx 1$, as implied by (D3) and the definition (D2).

Appendix E. Friction constant for electrons in metal

The effect of electron–electron Coulomb interaction can be analysed by considering the motion of a single electron under the influence of a fluctuating electrostatic potential $\mathcal{U}(x, t)$ that is created by the other electrons. Thus the electron experiences an interaction with a fluctuating field that is characterized by $\langle \mathcal{U}(x'', t'')\mathcal{U}(x', t') \rangle$. It is well known, using the fluctuation–dissipation theorem, that for diffusive electrons the corresponding power spectrum is

$$\tilde{w}(k)\phi(\omega) = \frac{e^2}{\sigma_d} \cdot \frac{1}{k^2} \omega\hbar \coth\left(\frac{\hbar\omega}{2k_B T} \right) \qquad \text{for } \omega < \frac{1}{\tau_c}, \ |k| < \frac{1}{\ell}.$$

The ohmic behaviour is cut off by the Drude collision frequency $1/\tau_c$. The elastic mean free path is $\ell = v\tau_c$, where v is the Fermi velocity. For $k < 1/\ell$ the power spectrum $P(k, \omega)$ of the diffusive motion is concentrated below the Drude cut-off frequency. The ballistic regime $1/\ell < k$ is of no interest because its contribution is suppressed due to the Drude cut-off. One observes that these fluctuations correspond to the generalized DLD model (15) with $\sigma = d - 2$. Having observed that the mean free path is the physical (effective) cut-off

for the spatial fluctuations (this statement is true for $\sigma < 4$), it follows from the convention $|w''(0)| = 1$ that the friction constant (up to dimensionless factor of order unity) is given by the expression

$$\eta = \frac{e^2}{\sigma_d} \left(\frac{1}{\ell}\right)^d = \frac{1}{D}\Delta_\ell. \tag{E1}$$

Here e is the charge of an electron, and σ_d is the conductivity defined for $d = 1, 2, 3$ dimensions. The last equality is obtained by using the Einstein relation in order to express σ_d in terms of D. The notation Δ_ℓ stand for the mean level spacing within a cube whose volume is ℓ^d.

References

[1] Wilkinson M 1988 *J. Phys. A: Math. Gen.* **21** 4021
     Jarzynski C 1995 *Phys. Rev. Lett.* **74** 2937
[2] Cohen D 1997 *Phys. Rev. Lett.* **78** 2878
     Cohen D 1997 *Phys. Rev.* E **55** 1422
[3] Caldeira A O and Leggett A J 1983 *Physica* **121A** 587
     Grabert H, Schramm P and Ingold G L 1988 *Phys. Rep.* **168** 115
[4] Feynman R P and Vernon F L Jr 1963 *Ann. Phys., NY* **24** 118
[5] Bulgac A, Dang G D and Kusnezov D 1998 *Phys. Rev.* E **58** 196
[6] Cohen D 1991 *Phys. Rev.* A **44** 2292
[7] Imry Y 1997 *Introduction to Mesoscopic Physics* (Oxford: Oxford University Press) and references therein
[8] Cohen D and Fishman S 1989 *Phys. Rev.* A **39** 6478
[9] Alder K and Winther A 1975 *Electromagnetic Excitation, Theory of Coulomb Excitation With Heavy Ions*
         (Amsterdam: North-Holland)
[10] Oppen F and Stern A 1997 *Phys. Rev. Lett.* **79** 114
[11] Chakravarty S and Schmid A 1986 *Phys. Rep.* **140** 193
[12] Mohanty P, Jariwala E M and Webb R A 1997 *Phys. Rev. Lett.* **77** 3366
     Golubev D S and Zaikin A D 1997 *Preprint* cond-mat/9710079
     Golubev D S and Zaikin A D 1997 *Preprint* cond-mat/9712203
[13] Cohen D and Imry Y 1998 *Preprint* cond-mat/9807038