

Non Perturbative Destruction of Localization in the Quantum Kicked Particle Problem

Doron Cohen

Department of Physics, Technion - Israel Institute of Technology, Haifa 32000, Israel
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The angle coordinate of the Quantum Kicked Rotator problem is treated as if it were an extended coordinate. A new mechanism for destruction of coherence by noise is analyzed using both heuristic and formal approach. Its effectiveness constitutes a manifestation of long-range non-trivial dynamical correlations. Perturbation theory fails to quantify certain aspects of this effect. In the perturbative case, for sufficiently weak noise, the diffusion coefficient \mathcal{D} is just proportional to the noise intensity ν . It is predicted that in some generic cases one may have a non-perturbative dependence $\mathcal{D} \propto \nu^\alpha$ with $0.35 < \alpha < 0.38$ for arbitrarily weak noise. This work has been found relevant to the recently studied ionization of H-atoms by a microwave electric field in the presence of noise [13]. **Note added (a):** Borgonovi and Shepelyansky (Physica D 109, 24 (1997)) have adopted this idea of *non-perturbative* transport, and have demonstrated that the same effect manifests itself in the tight-binding Anderson model with the *same* exponent α .

Note added (b): The recent interest in the work reported here comes from the experimental work by the *Austin* group (Klappauf, Oskay, Steck and Raizen, PRL 81, 1203 (1998)), and by the *Auckland* group (Ammann, Gray, Shvachuck and Christensen, PRL 80, 4111 (1998)). In these experiment the QKP model is realized literally. However, the novel effect of *non-perturbative transport*, reported in this Letter, has not been tested yet.

The most striking manifestation of quantum mechanical effects on classical chaos is dynamical localization which leads to suppression of chaos. Consider for example a particle that is confined to move in a one dimensional space whose length is L and that is subject to a kicking potential with period T . Classically, the motion of the particle becomes ergodic in space but diffusive in momentum¹. Thus, the kinetic energy of the particle grows like $\langle p^2 \rangle \sim D_0 t$, where the diffusion coefficient D_0 depends on the strength of the kicking potential. Quantum mechanically it is found that diffusion in momentum is suppressed². This is due to localization of the Floquet eigenstates in momentum³. A standard argumentation⁴ leads to the following expression for the localization length

$$\ell = \frac{2\pi}{L} \hbar \xi = \frac{TL}{2\pi\hbar} D_0 \quad (1)$$

where ℓ is measured in units of p while ξ is the dimensionless localization length. The prototype problem for the investigation of dynamical localization is the Quantum Kicked Rotator (QKR) Problem^{2,3}. In this problem the particle is kicked by a $\cos x$ potential and periodic boundary conditions are imposed over $[0, 2\pi]$. However, we may consider x to be an extended variable and impose periodic boundary conditions over $[0, 2\pi\mathcal{M}]$ where \mathcal{M} is an integer and the limit $\mathcal{M} \rightarrow \infty$ is taken. We obtain then a new problem to be entitled 'The Quantum

Kicked Particle (QKP) Problem'. It is *not* correct to use (1) with $L = 2\pi\mathcal{M}$ to obtain $\ell = T \frac{D_0}{\hbar} \mathcal{M} \rightarrow \infty$ since due to the translational symmetry of $\cos x$ the localization length ℓ is the same as in the QKR problem ($\mathcal{M} = 1$) irrespective of \mathcal{M} . However, it is evident that any dislocation in the periodic structure of the kicking potential will result in $\ell \rightarrow \infty$. Therefore we expect localization in the QKP problem to be extremely sensitive to any generic perturbation. We shall discuss in this letter the effect of noise on localization in the QKP problem. In conclusion we shall explain why this problem should be considered a prototype example⁵ for the recently studied noise-induced diffusive-ionization of a highly excited H-atom that is subject to a monochromatic microwave electric field^{6,7}.

We are considering in this letter the quantized version of the classical standard map with noise, namely

$$\begin{aligned} x_t &= x_{t-1} + p_{t-1} \\ p_t &= p_{t-1} + K \sin x_t + f_t \end{aligned} \quad (2)$$

It is implicit that the dynamical behavior should be averaged over realizations of the sequence f_t such that $\langle f_t \rangle = 0$ and

$$\langle f_t f_{t'} \rangle = \nu \delta_{t,t'} \quad (3)$$

*This Letter has been submitted to PRL. Eventually, the comprehensive paper Ref.[10] that contains the reported results, as well as other results, has been published first. Consequently there was no longer basis for the publication of this Letter.

Following Ott, Antonsen and Hanson⁸ we assume that the one-step propagator that generates this map is

$$\hat{U} = \exp \left[-\frac{i}{\hbar} (K \cos \hat{x} + \hat{V}_{int}) \right] \exp \left[-\frac{i}{\hbar} \frac{1}{2} \hat{p}^2 \right] \quad (4)$$

where V_{int} is the interaction term with the noise source. Consider first the standard QKR case in which x is an angle variable. The interaction term must respect then the 2π spatial periodicity of x . Possible choices that correspond to the classical map (2) are $\hat{V}_{int} = \sqrt{2\nu} \sin(\hat{x} + \varphi(t))$ where $\varphi(t)$ is a random phase⁸, and⁹ $\hat{V}_{int} = \int d\varphi f_\varphi(t) \sqrt{2} \sin(\hat{x} + \varphi)$ where $f_\varphi(t)$ satisfies $\langle f_\varphi(t) f_{\varphi'}(t') \rangle = \nu \frac{1}{2\pi} \delta(\varphi - \varphi') \delta_{t,t'}$. It may be shown that this QKR model is not sensitive to the detailed form of the interaction term¹⁰. In the QKP problem the map (2) describes the time evolution of a *particle*. A generic interaction term with the external noise source is not expected to respect the 2π spatial periodicity of the kicking potential. We may assume then e.g. a linear coupling scheme $V_{int} = -f_t \hat{x}$ where f_t satisfies (3). We shall see that in this QKP model the dynamical behavior is significantly different compared with the QKR model though both models correspond to the same map (2). From now on it is assumed that $1 \ll K$ which is the usual condition for being in the classically-chaotic regime of the standard map.

In the presence of strong noise diffusion in momentum is classical-like⁸ with coefficient $\frac{1}{2}K^2 + \nu$. If the noise is weak then classical-like diffusion lasts a characteristic time $t^* \approx 2\xi$ and then a crossover to slower diffusive behavior is observed. The asymptotic diffusion coefficient is defined as follows:

$$\mathcal{D} = \lim_{t \rightarrow \infty} \frac{\langle\langle (p(t) - p(0))^2 \rangle\rangle}{t} \quad (5)$$

where $\langle\langle \rangle\rangle$ denotes quantum statistical average over initial conditions and noise realizations (see Ref.10 for further details). In the absence of noise $\mathcal{D} = 0$ due to the localization effect. We shall use now a heuristic picture in order to determine \mathcal{D} in the presence of weak noise. Next we shall introduce a formal treatment and the limitations of both approaches will be pointed out.

A good way to gain insight of the effect of noise on coherence is to use Wigner's picture of the dynamics¹¹. Wigner's function $\rho(x, p)$ is defined on $[0, 2\pi\mathcal{M}] \times \frac{\hbar}{2\mathcal{M}}\mathcal{Z}$ where \mathcal{Z} are the integer numbers. Assuming that the particle is prepared in a \hat{U} -eigenstate, Wigner's function has details on spatial scale $\frac{1}{\xi}$ indicating a superposition of ξ momentum eigenstates. The effect of noise is to smear fine details of Wigner's function and thus to turn the superposition into a mixture¹¹. The coherence time t_c in the QKR problem is simply the time it takes for the noise to 'mix' neighboring momenta⁸ on momentum scale

\hbar , namely $t_c^{QKR} = (\hbar^2 \frac{1}{\nu})$, while in the QKP problem a shorter time scale exists, namely $t_c^{QKP} = (\frac{1}{\xi^2} \frac{1}{\nu})^{\frac{1}{3}}$ which is the time it takes to *spread* over spatial scale $\frac{1}{\xi}$. This spreading is absent in case of a rotator since it is associated with the noise-induced diffusion in the non-discrete momentum space. This diffusion is $\delta p \propto t^{\frac{1}{2}}$ while the associated spreading is $\delta x \propto t^{\frac{3}{2}}$. It is important to note that implicit in this heuristic picture is the underlying assumption that the kicks do not affect significantly the coherence time. This assumption has been shown to be incorrect in case of the QKR problem if the noise possesses long range correlations¹¹. Actually, we shall see that in the QKP problem the situation is similar, though the heuristic picture gives the right qualitative behavior.

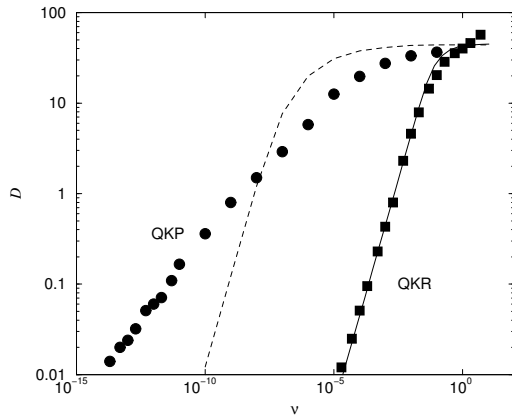


FIG. 1. The diffusion coefficient \mathcal{D} that has been determined by simulations as a function of ν in both the QKR (filled squares) and the QKP (filled circles) problems. The parameters are $K = 10$ and $\hbar = 2\pi \times 0.3/(\sqrt{5} - 1)$. Average over 100 realizations of the noise has been taken and the number of iterations was up to 10^4 for each realization. The smooth curves has been calculated using (10) and assuming the short time behavior in (7) to hold throughout the whole time domain. Either (8) or (9) were substituted. There are no fitting parameters. The numerical results in case of the QKP model (circles) definitely deviates from the calculated perturbative result (dashed line). Rather than having a perturbative behavior $\mathcal{D} \propto \nu$ we have a non-perturbative behavior $\mathcal{D} \propto \nu^\alpha$ with $0.35 < \alpha < 0.38$. This non-perturbative behavior persists for arbitrarily weak noise.

We proceed now to estimate \mathcal{D} . One may try to use the heuristic diffusion picture that is implicit in the work by Ott Antonsen and Hanson⁸. It is argued that for weak noise ($t^* \ll t_c$) the diffusion process in momentum space is similar to a random walk on a grid with spacing $\hbar\xi$ and hopping-probability $\frac{1}{t_c}$. The diffusion coefficient in the presence of weak noise is therefore of the order $(\hbar\xi)^2 \frac{1}{t_c}$ which upon using (1) leads to

$$\mathcal{D} \approx \frac{t^*}{t_c} D_0 \quad . \quad (6)$$

It follows that $\mathcal{D} \propto \nu^\alpha$ with $\alpha = 1$ for the QKR and $\alpha = \frac{1}{3}$ for the QKP. In Figure 1 the results of a numerical experiment are presented¹³. The observed behavior is indeed $\alpha = 1$ for the QKR but $0.35 < \alpha < 0.38$ for the QKP which deviates slightly from the heuristic value $\alpha = \frac{1}{3}$. We turn now to a formal analysis of the problem to overcome the natural limitations of the above heuristic picture. We shall try to explain the origin for the deviation from the heuristic result in case of the QKP, but we shall see that leading order perturbation theory cannot be trusted if we want to quantify this deviation.

In the absence of noise one may define the dispersion (energy) function $E(t) \equiv \langle \langle (p(t) - p(0))^2 \rangle \rangle$. This function is related to the momentum autocorrelation function¹⁰ via $E(t) = 2(C_p(0) - C_p(t))$. Its time derivative will be denoted by $D(t)$, and has the asymptotic value $\mathcal{D} = 0$ due to the localization effect. It has been found¹⁰ that dynamical correlations in the QKR problem decay exponentially on time scale t^* while on longer time scale a slower power-law decay is observed. Consequently

$$D(t) = \begin{cases} D_0 e^{-\frac{t}{t^*}} & \text{for } t < \mathcal{O}(t^*) \\ cD_0(\frac{t^*}{t})^{1+\beta} & \text{for } \mathcal{O}(t^*) < t \end{cases} \quad (7)$$

with $D_0 \approx \frac{1}{2}K^2$, $t^* \approx 2\xi$, $\beta \approx 0.75$ and $c \approx 0.5$. More details including analytical considerations may be found in Ref.10.

In the presence of noise coherence is destroyed. The decay probability $P(t)$ of a quasienergy eigenstate as a function of time may be calculated using leading order perturbative calculation^{9,10}. For the QKR the decay rate is constant, namely

$$\dot{P}(t) = \frac{1}{\hbar^2} \nu \quad (8)$$

For the QKP one obtains

$$\dot{P}(t) = \frac{1}{\hbar^2} \nu \sum_{\tau=-t}^t C_p(\tau)(t - |\tau|) \quad (9)$$

For $t^* \ll t$ the behaviour is roughly $P(t) \approx \nu \xi^{2+\beta} t^{3-\beta}$ in the latter case. These results hold as long as $P(t) \ll 1$. However, if we assume that $P(t)$ is a function of a single scaled variable $\frac{t}{t_c}$ then the perturbative result suggests that for the QKR problem $t_c^{QKR} = (\hbar^2 \frac{1}{\nu})$ which is the inverse of the decay rate and agrees with our heuristic expectation. For the QKP one obtains $t_c^{QKP} \approx (\frac{1}{\xi^{2+\beta} \nu})^{\frac{1}{3-\beta}}$ that coincide with the heuristic result only if we assume very strong dynamical correlations ($\beta \rightarrow 0$) which is not correct since $\beta \approx 0.75$. Note that the latter results are as exact as leading order perturbation theory permits.

We consider now diffusion in presence of weak noise ($t^* \ll t_c$) using a formal approach. A derivation¹⁰ which

is based on leading order perturbation theory leads to the result

$$\mathcal{D} \approx \sum_{t=0}^{\infty} \dot{P}(t)D(t) \quad (10)$$

This expression may be trusted only if it is dominated by the short-time terms (those with $t \ll t_c$). This would be always the case if dynamical correlations possessed a short-range nature. Specifically, if $D(t)$ decayed exponentially on the relatively short time scale t^* , then the sum in (10) would be dominated by the terms in its head whose number is of the order t^* . Evidently \mathcal{D} should be proportional then to the intensity of the noise. A non-trivial dependence of the form $\mathcal{D} \propto \nu^\alpha$ with $\alpha \neq 1$ is therefore a manifestation of long range dynamical correlations. The sum in (10) is necessarily dominated then by the long-time terms and consequently the perturbative estimate for \mathcal{D} cannot be trusted any more.

In case of the QKR model one easily finds that in spite of the power law decay of the long-time terms, yet the sum (10) is dominated by the short time terms. Substitution of (7) and (8) into (10) leads then to the heuristic formula (6). Figure 1 illustrates comparison of the numerical results (filled squares) with the analytical estimate (smooth curve, no fitting parameters). We turn now to discuss the QKP case. Here the behaviour of the terms in the sum (10) that satisfy $t^* \ll t \ll t_c$ is

$$\dot{P}(t)D(t) = (3-\beta)cD_0 \frac{(t^*)^{1+\beta}}{(t_c)^{3-\beta}} t^{1-2\beta} \quad (11)$$

where we have used (7) and (9). This behaviour (provided $\beta \leq 1$) indicates that most of the contribution to \mathcal{D} in (10) comes from the long-time terms with t which is of the order t_c . This observation is supported by the comparison of the numerical¹² results (Figure 1, filled circles) with analytical estimate that takes into account only the short-time contribution (dashed curve, no fitting parameters). One may try to use the following extrapolation scheme in order to estimate the dominant contribution of the long-time terms to the diffusion coefficient : (a) To assume that nevertheless (10) holds, (b) To assume that (9) holds for $t \leq t_c$ while $\dot{P}(t) = 0$ for $t_c < t$. One obtains then $\mathcal{D} = c'(\frac{t^*}{t_c})^{1+\beta} D_0$ instead of (6) where c' is a prefactor of order unity. Consequently $\mathcal{D} \propto \nu^\alpha$ with $\alpha = \frac{1+\beta}{3-\beta}$. This result differs for $0 < \beta$ from the heuristic one. It predicts that α is larger than $\frac{1}{3}$, namely $\alpha \approx 0.78$. Unfortunately, the latter value does not agree with the numerical results which leads to the conclusion that leading order perturbation theory is not sufficient in order to obtain a quantitative description of the effect.

The QKP problem is a prototype example that illustrates destruction of coherence via a *spreading mechanism*. This mechanism is operative in systems where the noise does not respect a *symmetry* that is responsible for the localization. In the QKP problem, due to translational symmetry of the kicking potential, only states that have finite momentum separation are coupled by the kicks. This feature is shared by the recently studied highly excited H-atom that is subject to a monochromatic microwave electric field⁶. The high energy levels of the undriven H-atom are very dense, but *only photon-distant states are coupled by the interaction with the field*. It follows that this problem reduces locally to a generalized QKP-problem with finite \mathcal{M} rather than QKR-problem. Generic noise will induce diffusion to neighbouring levels that play no significant role in the dynamics if the noise is absent. If the time scale that is required for this diffusion is much less than t_c^{QKP} then we may expect that the spreading mechanism for destruction of coherence will become effective. Our results therefore suggests that if the H-atom is prepared in a *very* high excited state, then a new behaviour which is different from the one that has been reported in Ref.7 may be found. Namely, the ionization time will not be in general inverse proportional to the variance of the noise. This subject obviously deserves a systematic study. Indeed Fishman and Shepelyansky¹³ have considered the effect of noise on ionization and pointed out that there are indications for the manifestation of the non-perturbative mechanism in experiments on Rubidium atoms that have been carried out recently by the Munich group¹⁴.

Note added (c): Further discussion of the perturbative and the non-perturbative mechanisms for destruction of coherence, in a more general context, may be found in the paper "Quantal Brownian Motion - Dephasing and Dissipation" (D. Cohen, *J. Phys. A* **31**, 8199 (1998)). It should be emphasized that the essential ingredient for the manifestation of the non-perturbative mechanism is the possibility for exchange of relatively small quanta of momentum between the particle and the environment. It should be possible to realize such type of noise in eg Raizen's experiments by introducing a noisy field with small q components (q = wavenumber in the relevant direction). The emphasis on 'symmetry breaking' in the above 1991 version of the Letter is somewhat misleading.

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