

# Non-equilibrium steady state (NESS) of sparse systems

Doron Cohen

Ben-Gurion University

$$\mathcal{H}_{\text{total}} = \{E_n\} - f(t)\{V_{nm}\} + F(t)\{W_{nm}\} + \textit{Bath}$$

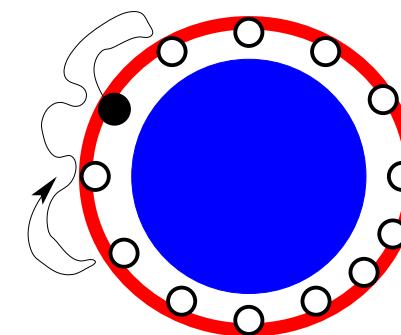
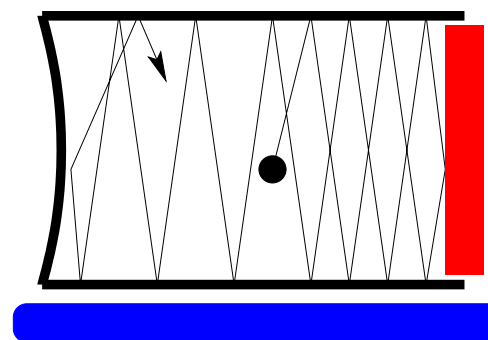
Daniel Hurowitz (BGU) [1,2]

Saar Rahav (Technion) [2]

Alex Stotland (BGU) [3,4]

Lou Pecora (NRL) [3]

Nir Davidson (Weizmann) [4]



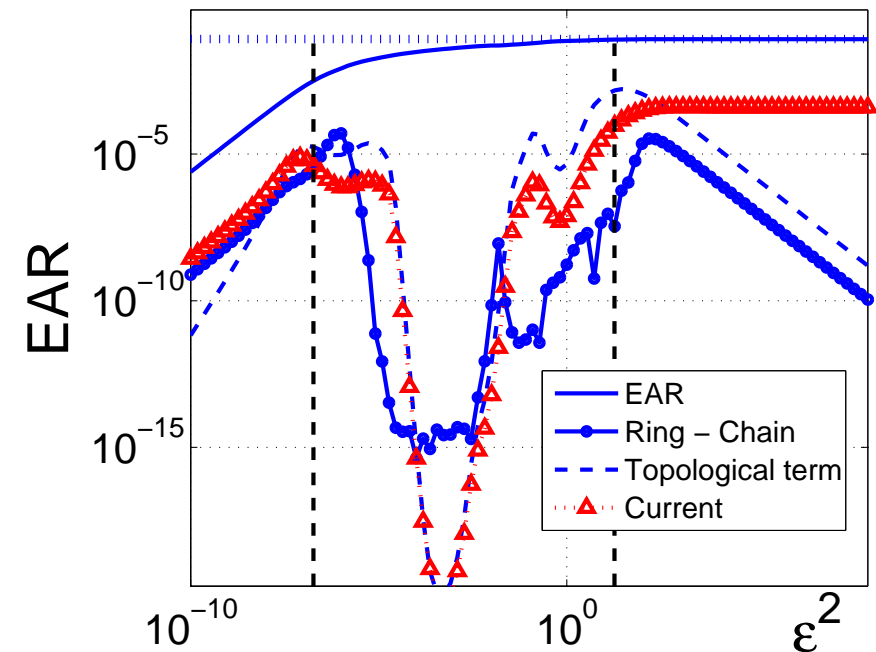
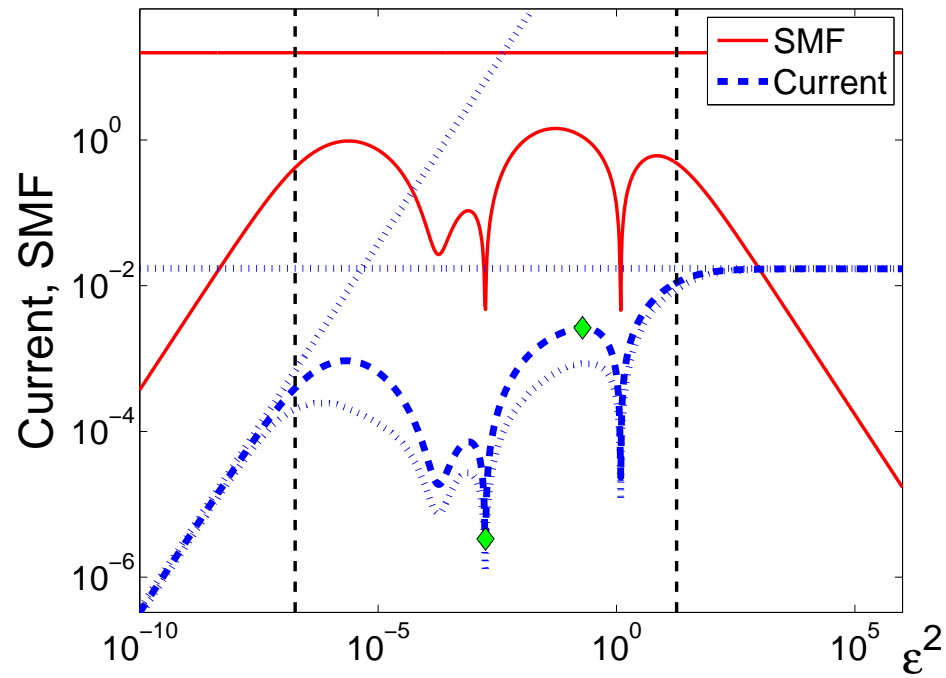
- [1] Hurowitz, Cohen (EPL 2011)
- [2] Hurowitz, Rahav, Cohen (arXiv 2011)
- [3] Stotland, Pecora, Cohen (EPL 2010, PRE 2011)
- [4] Stotland, Cohen, Davidson (EPL 2009)

<http://www.bgu.ac.il/~dcohen>

\$DIP, \$BSF, \$ISF

## Some results for the Ring

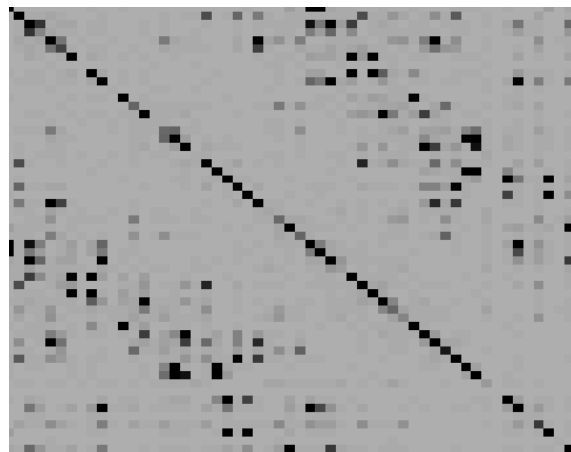
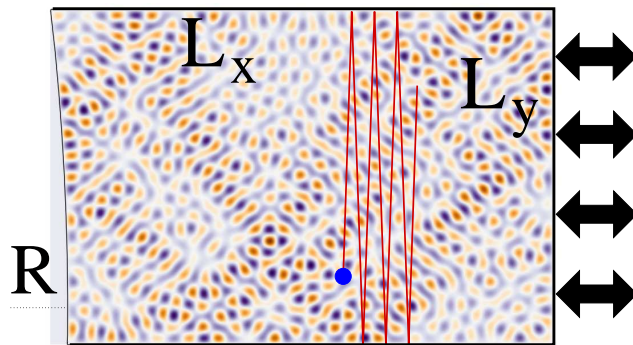
3 regimes:    LRT regime,    Sinai regime,    Saturation regime



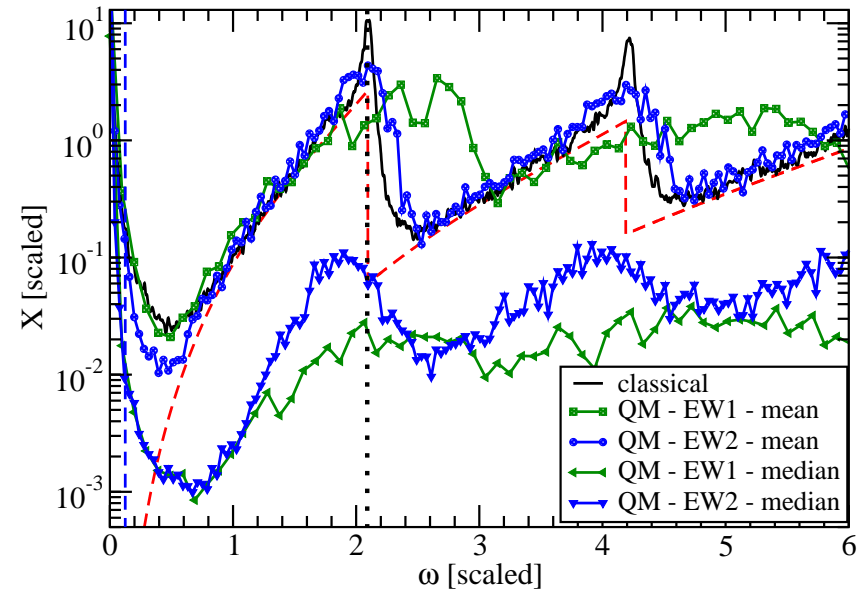
$$I \sim \frac{1}{N} \bar{w} \exp \left[ -\frac{\epsilon_n}{2} \right] 2 \sinh \left( \frac{\epsilon_\sigma}{2} \right)$$

# Sparsity

$$\mathcal{H}_{\text{total}} = \{E_n\} - f(t)\{V_{nm}\}$$



$$|V_{nm}|^2 \approx (2\pi\rho)^{-1} \tilde{C}_{cl}(E_n - E_m)$$



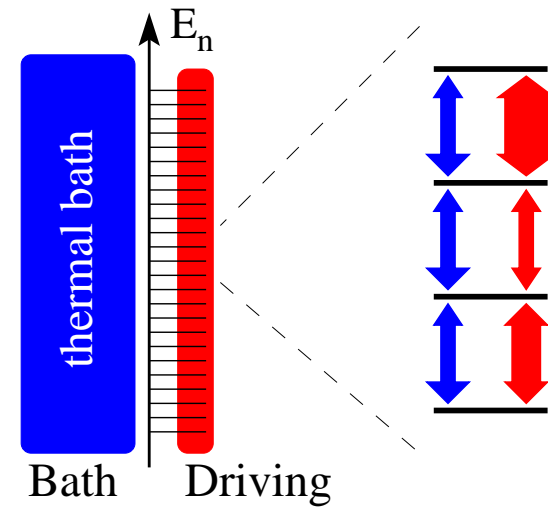
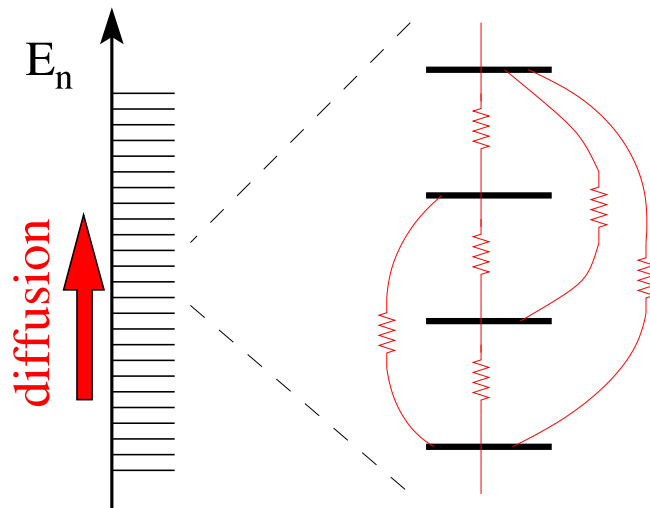
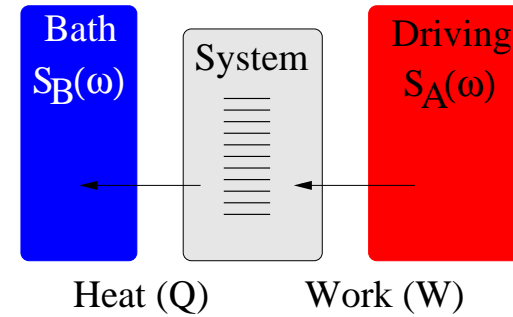
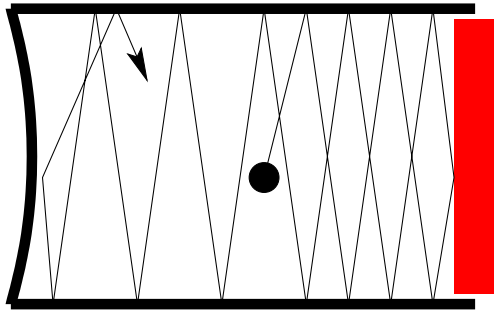
[median  $\ll$  mean]

# Driven system + Bath

$$\mathcal{H}_{\text{total}} = \{E_n\} - f(t)\{V_{nm}\} + F(t)\{W_{nm}\} + \text{Bath}$$

$\varepsilon^2 \equiv$  Driving intensity

$T_B \equiv$  Bath temperature

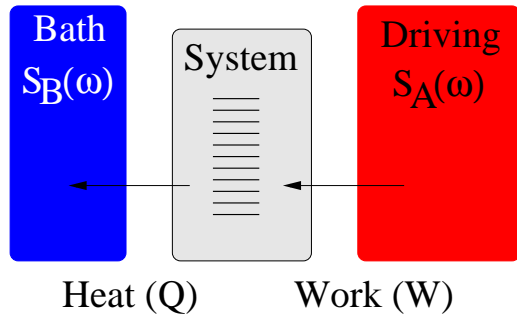


## Main statements

$$\mathcal{H}_{\text{total}} = \{E_n\} - f(t)\{V_{nm}\} + F(t)\{W_{nm}\} + \text{Bath}$$

$\varepsilon^2 \equiv$  Driving intensity

$T_B \equiv$  Bath temperature



Sparsity implies a **glassy** NESS

We can define NESS **effective temperature**

$D(\varepsilon)$  exhibit **LRT** to **SLRT** crossover

Quantum saturation

SLRT requires **resistor network calculation**

RMT perspective - **non Gaussian matrices**

Generalized **VRH** in energy space

Quantum **anomaly** in the theory of response.

$$\dot{W} = \text{rate of heating} = \frac{D(\varepsilon)}{T_{\text{system}}}$$

$$\dot{Q} = \text{rate of cooling} = \frac{D_B}{T_B} - \frac{D_B}{T_{\text{system}}}$$

Hence:

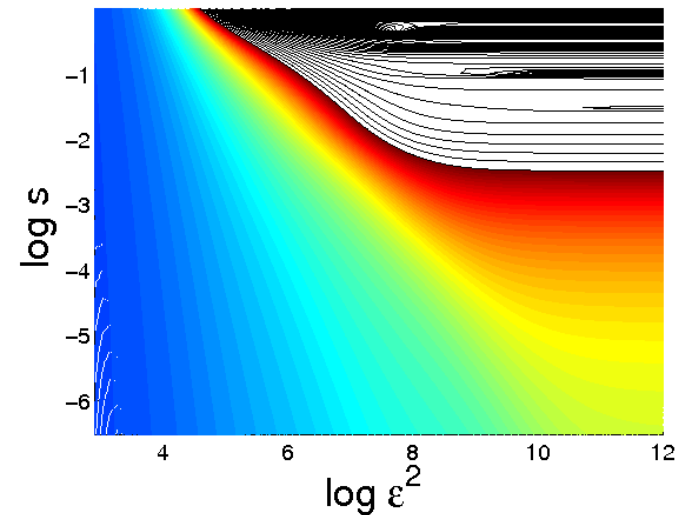
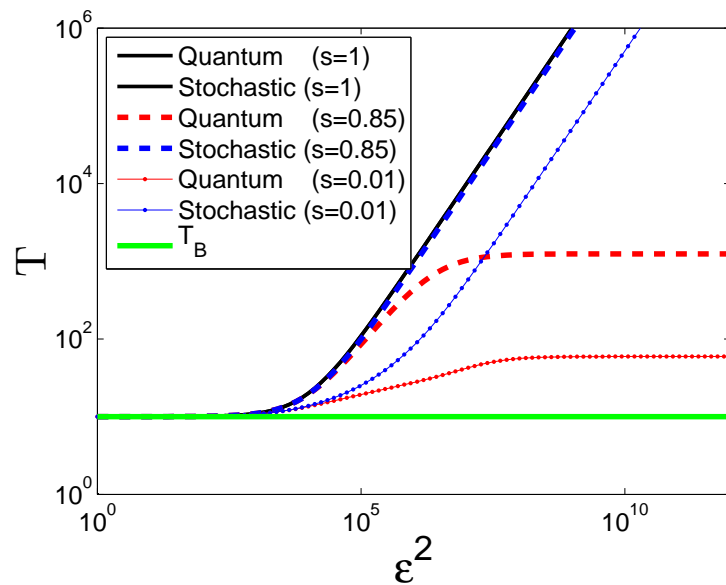
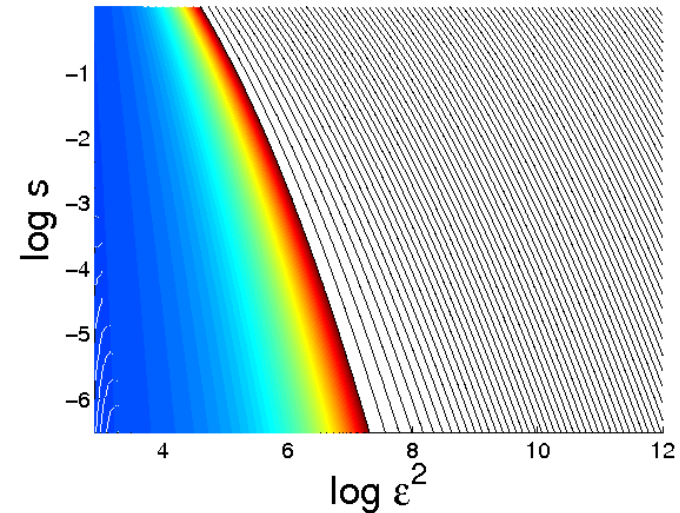
$$T_{\text{system}} = \left(1 + \frac{D(\varepsilon)}{D_B}\right) T_B$$

## Steady state temperature

$$\frac{d\rho}{dt} = -i[\mathcal{H}, \rho] - \frac{\varepsilon^2}{2}[V, [V, \rho]] + \mathcal{W}^\beta \rho$$

$$\frac{p_n}{p_m} = \exp\left(-\frac{E_n - E_m}{T_{nm}}\right)$$

$$T_{\text{system}} = \text{average}[T_{nm}]$$



# Quantum NESS for toy model with n.n. transitions

$$\frac{d\rho}{dt} = -i[\mathcal{H}, \rho] - \frac{\varepsilon^2}{2}[V, [V, \rho]] + \mathcal{W}^\beta \rho$$

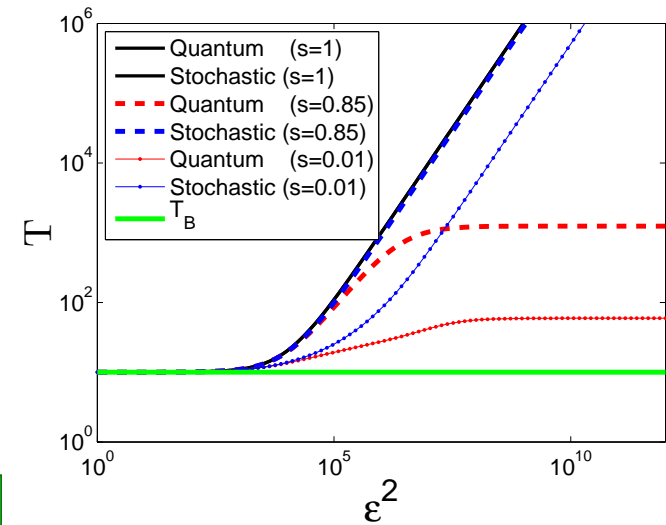
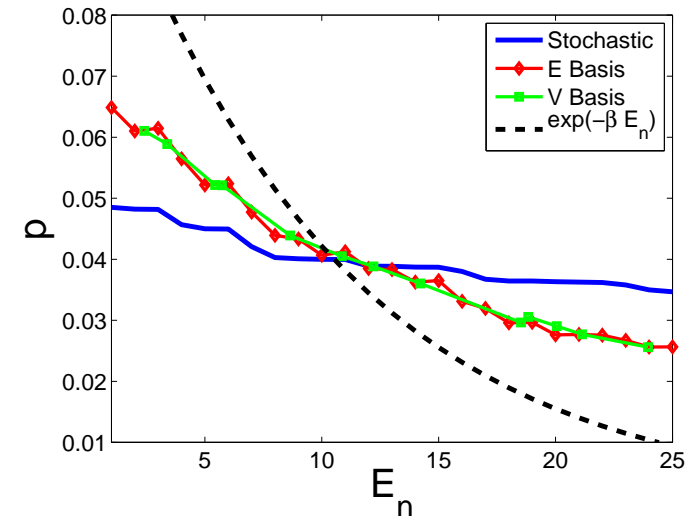
For very strong driving,  
the NESS is a mixture of  $V$  eigenstates:

$$p_r \sim \exp(-\langle E \rangle_r / T_B)$$

leading to:

$$p_n \sim \exp(-E_n / T_\infty)$$

$$T_B < T_\infty < \infty \quad [\text{depends on the sparsity}]$$



## How the temperature is defined

$$\mathcal{H}_{\text{total}} = \{E_n\} - f(t)\{V_{nm}\} + F(t)\{W_{nm}\} + \text{Bath}$$

The sources temperature:  $T_A = \infty$

$$\tilde{S}_A(\omega) \equiv \text{FT} \langle \dot{f}(t)\dot{f}(0) \rangle$$

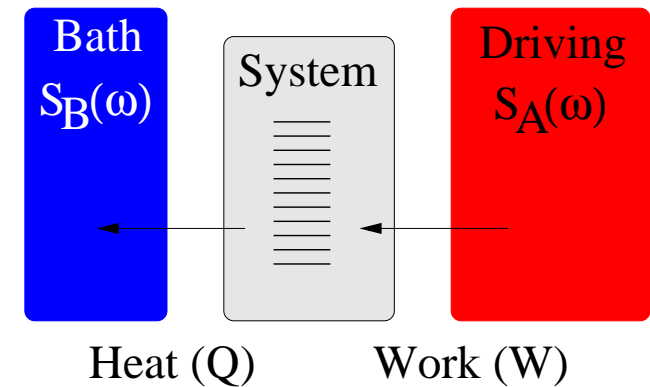
The bath temperature:  $T_B$

$$\tilde{S}_B(\omega)/\tilde{S}_B(-\omega) = \exp(-\omega/T_B)$$

Temperature of the system?

$$\dot{W} = \text{rate of heating} = \frac{D(\varepsilon)}{T_{\text{system}}}$$

$$\dot{Q} = \text{rate of cooling} = \frac{D_B}{T_B} - \frac{D_B}{T_{\text{system}}}$$



$$\frac{p_n}{p_m} = \exp\left(-\frac{E_n - E_m}{T_{nm}}\right)$$

$$T_{\text{system}} = \text{average}[T_{nm}]$$



## Digression - derivation of the cooling rate formula

$$\dot{Q} = \text{cooling rate} = - \sum_{n,m} (E_n - E_m) w_{nm}^{\beta} p_m$$

$$p_n - p_m = \text{occupation imbalance} = \left[ 2 \tanh \left( - \frac{E_n - E_m}{2T_{nm}} \right) \right] \bar{p}_{nm}$$

$$w_{nm}^{\beta} - w_{mn}^{\beta} = \text{up/down transitions imbalance} = \left[ 2 \tanh \left( - \frac{E_n - E_m}{2T_B} \right) \right] w_{nm}^{\beta}$$

$$\dot{Q} = \frac{1}{2} \sum_{n,m} (E_n - E_m)^2 \frac{w_{nm}^{\beta}}{T_B} \bar{p}_{nm} - \frac{1}{2} \sum_{n,m} (E_n - E_m)^2 \frac{w_{nm}^{\beta}}{T_{nm}} \bar{p}_{nm} = \frac{D_B}{T_B} - \frac{D_B}{T_{\text{system}}}$$

definition of the diffusion coefficient:  $D_B \equiv \overline{\left[ \frac{1}{2} \sum_n (E_n - E_m)^2 w_{nm}^{\beta} \right]}$

definition of effective system temperature:  $\frac{1}{T_{\text{system}}} \equiv \overline{\left[ \frac{1}{T_{nm}} \right]}$

## Stochastic NESS for toy model with n.n. transitions

$$D_B = w_\beta \Delta_0^2 \quad [\text{for use in the rate of cooling formula}]$$

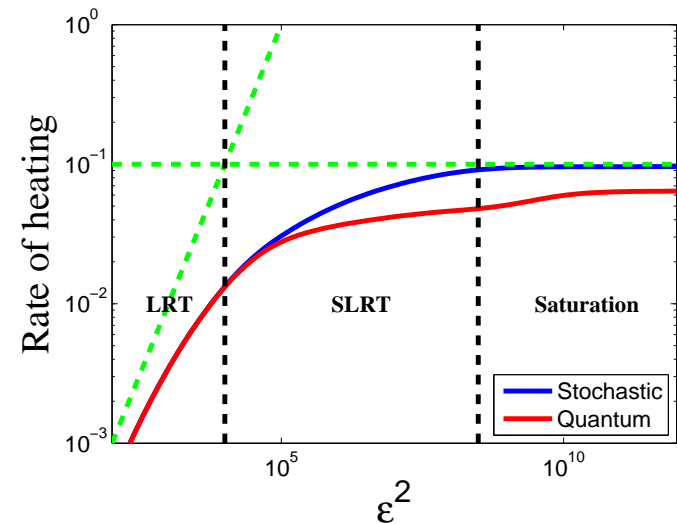
$$T_{\text{system}} = \left[ \left( \frac{1}{T_n} \right) \right]^{-1} = \left[ \left( \frac{w_\beta}{w_\beta + w_n} \right) \right]^{-1} T_B \quad [w_n \propto \text{strength of the driving}]$$

$$\dot{W} = \frac{D(\varepsilon)}{T_{\text{system}}}$$

$$D(\varepsilon) = \left[ \left( \frac{w_n}{w_\beta + w_n} \right) \right] \left[ \left( \frac{1}{w_\beta + w_n} \right) \right]^{-1} \Delta_0^2$$

$$D_{[\text{LRT}]} = \overline{w_n} \Delta_0^2 \quad [\text{weak driving}]$$

$$D_{[\text{SLRT}]} = \left[ \overline{1/w_n} \right]^{-1} \Delta_0^2 \quad [\text{strong driving}]$$



## Digression: random walk and the calculation of the diffusion coefficient

$w_{nm}$  = probability to hop from  $m$  to  $n$  per step.

$$\text{Var}(n) = \sum_n [w_{nm}t] (n - m)^2 \equiv 2Dt$$

For n.n. hopping with rate  $w$  we get  $D = w$ .

The continuity equation:

$$\frac{\partial p_n}{\partial t} = -\frac{\partial}{\partial n} J_n$$

Fick's law:

$$J_n = -D \frac{\partial}{\partial n} p_n$$

The diffusion equation:

$$\frac{\partial p_n}{\partial t} = D \frac{\partial^2}{\partial n^2} p_n$$

If we have a sample of length  $N$  then

$$J = -\frac{D}{N} \times [p_N - p_0]$$

$D/N$  = inverse resistance of the chain

If the  $w$  are not the same:

$$\frac{D}{N} = \left[ \sum_{n=1}^N \frac{1}{w_{n,n-1}} \right]^{-1}$$

Hence, for n.n. hopping

$$D = \langle\langle w \rangle\rangle_{\text{harmonic}}$$

FGR:  $w_{nm} \sim |V_{nm}|^2$

$$D = \langle\langle |V_{nm}|^2 \rangle\rangle$$

## Digression: the Fermi golden rule picture

Master equation:

$$\frac{dp_n}{dt} = - \sum_m w_{nm} (p_n - p_m)$$

The Hamiltonian in the standard representation:

$$\mathcal{H} = \{E_n\} - f(t) \{V_{nm}\}$$

The transformed Hamiltonian:

$$\tilde{\mathcal{H}} = \{E_n\} - \dot{f}(t) \left\{ \frac{iV_{nm}}{E_n - E_m} \right\} \quad \tilde{S}(\omega) \equiv \text{FT} \langle \dot{f}(t) \dot{f}(0) \rangle = 2\pi \overline{|\dot{f}|^2} \delta_0(E_n - E_m)$$

The FGR transition rate due to the low frequency noisy driving:

$$w_{nm} = \left| \frac{V_{nm}}{E_n - E_m} \right|^2 \tilde{S}(E_n - E_m) \equiv \pi \varrho^3 g_{nm} \overline{\dot{f}^2}$$

The LRT / SLRT formula

$$\mathbf{D} = \text{average} \left[ \frac{1}{2} \sum_n (E_n - E_m)^2 w_{nm} \right] = \pi \varrho \langle \langle |V_{nm}|^2 \rangle \rangle \times \overline{\dot{f}^2} \equiv \mathbf{G} \overline{\dot{f}^2}$$

## The resistor network calculation

$$\mathcal{H} = \{E_n\} - f(t)\{V_{nm}\}$$

$$g_s \equiv \frac{\langle\langle |V_{nm}|^2 \rangle\rangle_s}{\langle\langle |V_{nm}|^2 \rangle\rangle_a}$$

$$D = G \overline{\dot{f}^2}$$

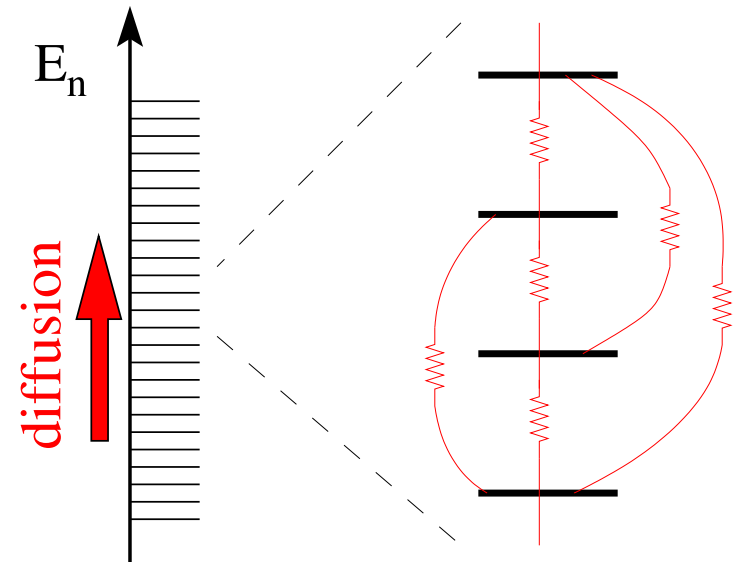
$$G_{\text{LRT}} = \pi \rho \langle\langle |V_{nm}|^2 \rangle\rangle_a$$

$$G_{\text{SLRT}} = \pi \rho \langle\langle |V_{nm}|^2 \rangle\rangle_s$$

LRT applies if the driven transitions are slower than the environmental relaxation

$$g_{nm} = 2\rho^{-3} \frac{|V_{nm}|^2}{(E_n - E_m)^2} \delta_0(E_n - E_m)$$

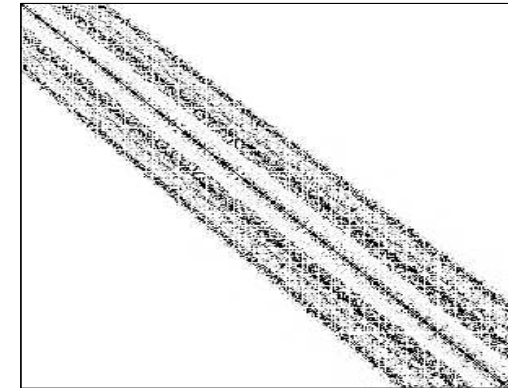
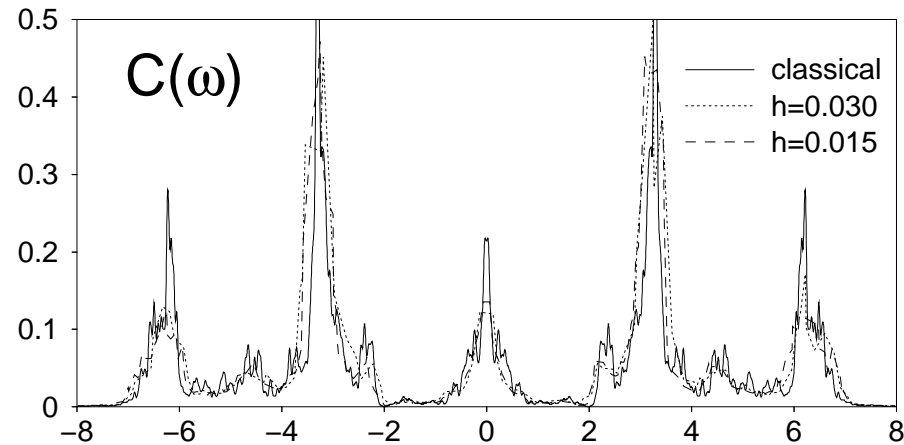
$$\langle\langle |V_{nm}|^2 \rangle\rangle_s \equiv \text{inverse resistivity}$$



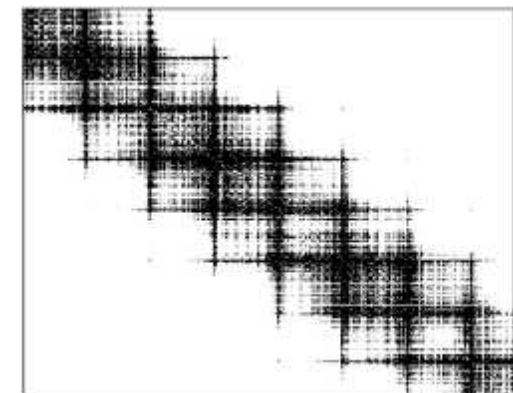
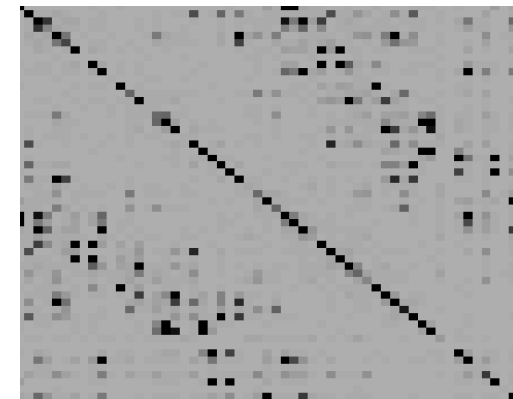
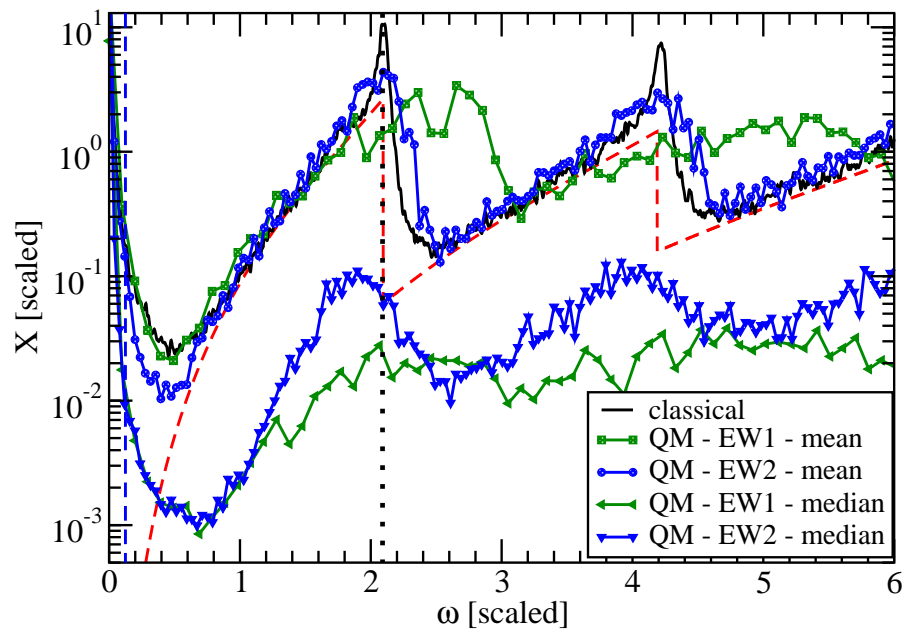
# Bandprofile, sparsity and texture

$$|V_{nm}|^2 \approx (2\pi\rho)^{-1} \tilde{C}_{cl}(E_n - E_m)$$

Hard Qchaos

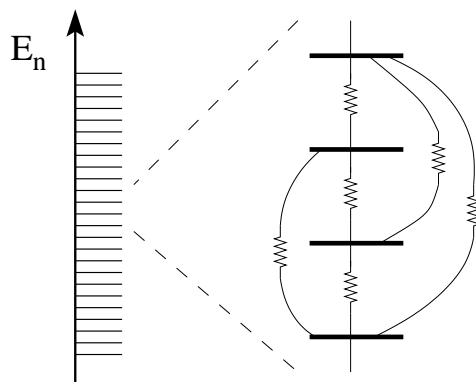
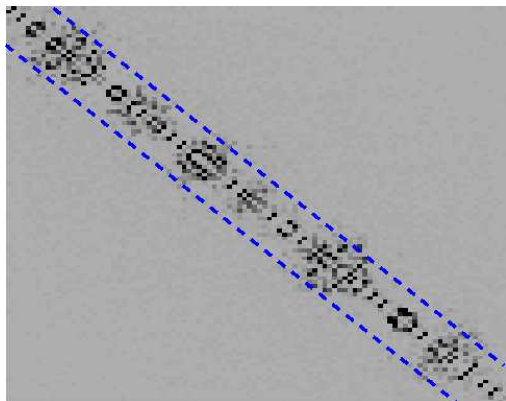


Weak Qchaos



[median  $\ll$  mean]

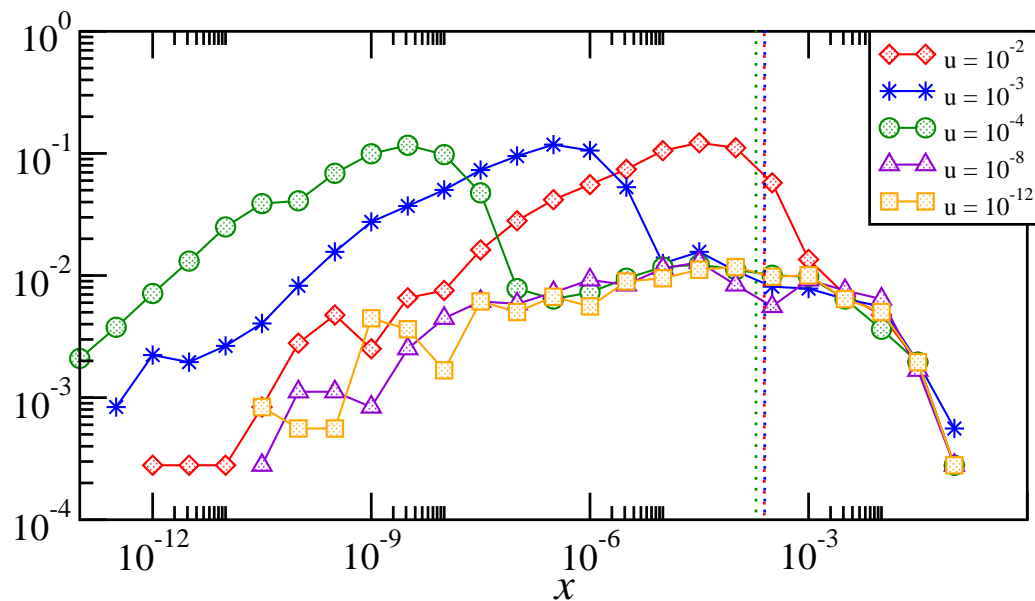
$\{|V_{nm}|^2\}$  as a random matrix  $\mathbf{X} = \{x\}$



$$s[\mathbf{X}] \equiv \frac{\text{PN}[\mathbf{X}]}{\text{PN}[\mathbf{X}_{\text{unf}}]} = \text{sparsity}$$

$$g_s[\mathbf{X}] \equiv \frac{\langle\langle \mathbf{X} \rangle\rangle_s}{\langle\langle \mathbf{X} \rangle\rangle_a} = \text{connectivity}$$

Histogram of  $x$  :



$x \sim \text{LogNormal}$

For a random sparse matrix:

$$s, g_s \ll 1$$

For a uniform (along diagonals):

$$s = g_s = 1$$

For a Gaussian matrix:

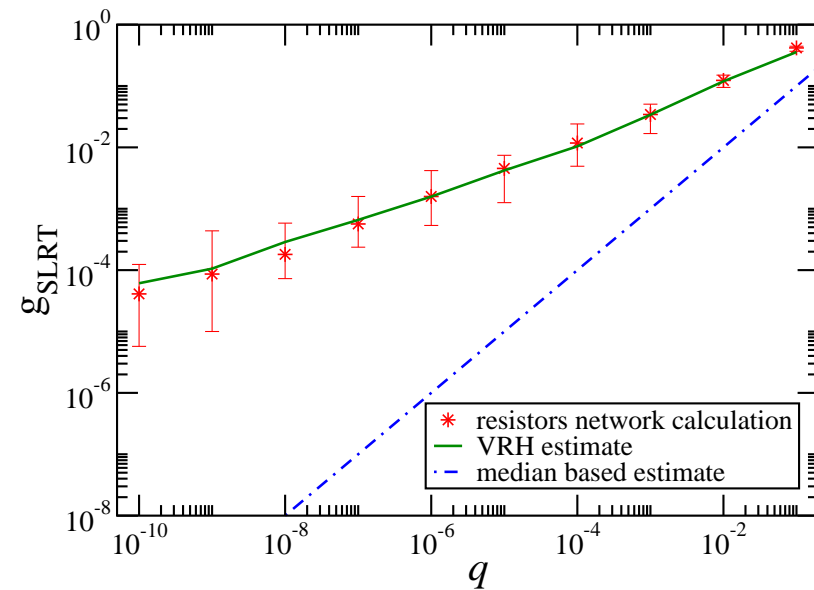
$$s = 1/3, g_s \sim 1$$

## RMT modeling, generalized VRH approx scheme

- log-normal distribution  $q$
- finite bandwidth  $b$

$$s = q^2 = (\text{median}/\text{mean})^2$$

$$g_s \approx q \exp \left[ 2\sqrt{-\ln q \ln(b)} \right]$$





## Digression: Generalized VRH

Definition of the typical matrix element for a range  $\omega$  transition:

$$\left(\frac{\omega}{\Delta}\right) \text{Prob}\left(x > x_\omega\right) \sim 1$$

In the standard-like case (ring with strong disorder):

$$x_\omega \approx v_F^2 \exp\left(\frac{\Delta_l}{|\omega|}\right) \quad [\text{corresponding to a log-box distribution}]$$

An example for the power spectrum of the driving:

$$\tilde{S}(\omega) \propto \exp\left(-\frac{|\omega|}{T}\right) \quad [\text{here the temperature } T \iff \omega_c]$$

Generalized VRH estimate:

$$D_{\text{SLRT}} \approx \int x_\omega \tilde{S}(\omega) d\omega \quad [\text{should be contrasted with}] \quad D_{\text{LRT}} = \int \tilde{C}(\omega) \tilde{S}(\omega) d\omega$$

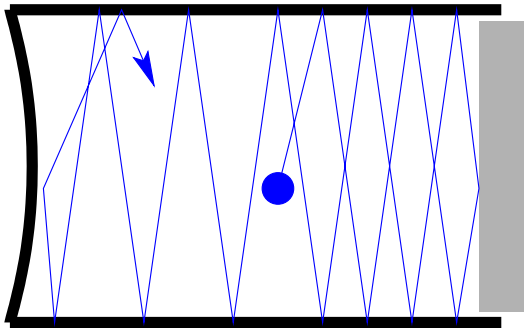
In the standard-like case (ring with strong disorder):

$$D_{\text{SLRT}} \approx \int \exp\left(\frac{\Delta_l}{|\omega|}\right) \exp\left(-\frac{|\omega|}{T}\right) d\omega$$

# The rate of heating: LRT and SLRT predictions

$$\mathcal{H}_{\text{total}} = \{E_n\} - f(t)\{V_{nm}\}$$

$f(t)$  = low freq noisy driving



↪ diffusion in energy space:

$$D_0 = \frac{4}{3\pi} \frac{M^2 v_E^3}{L_x} \overline{\dot{f}^2}$$

↪ energy absorption:

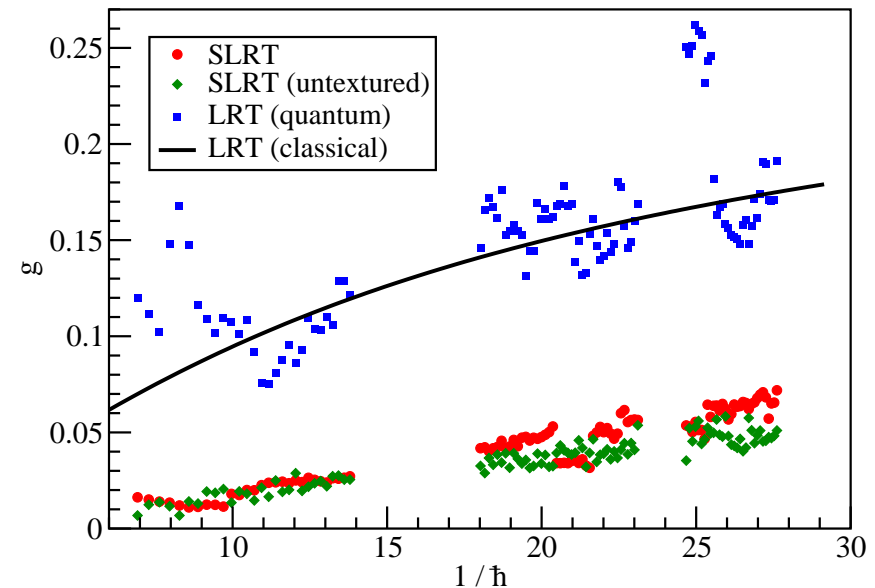
$$\dot{E} = (\text{particles/energy}) \times D$$

## Beyond the “Wall Formula”

[Beyond the “Drude Formula”]

$$D_{\text{LRT}} = g_c D_0 \quad \text{[“classical”]}$$

$$D_{\text{SLRT}} = g_s D_{\text{LRT}} \quad \text{[“quantum”]}$$



LRT applies if the driven transitions are slower than the environmental relaxation, else SLRT applies

## SLRT vs LRT

$$\mathcal{H}_{\text{total}} = \mathcal{H} + f(t)V$$

$$\tilde{C}(\omega) = \text{FT} \langle V(t)V(0) \rangle$$

$$\tilde{S}(\omega) = \text{FT} \langle \dot{f}(t)\dot{f}(0) \rangle$$

Kubo formula:

$$D = \int \tilde{C}(\omega)\tilde{S}(\omega)d\omega$$

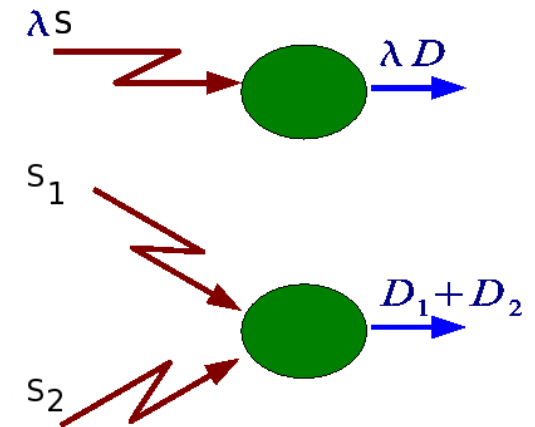
SLRT example:

$$D = \left[ \int R(\omega) [\tilde{S}(\omega)]^{-1} d\omega \right]^{-1}$$

Linear response implies

$$\tilde{S}(\omega) \mapsto \lambda\tilde{S}(\omega) \quad \implies \quad D \mapsto \lambda D$$

$$\tilde{S}(\omega) \mapsto \sum_i \tilde{S}_i(\omega) \quad \implies \quad D \mapsto \sum_i D_i$$



## Perspective and references

The classical LRT approach: Ott, Brown, Grebogi, Wilkinson, Jarzynski, Robbins, Berry, Cohen

The Wall formula (I): Blocki, Boneh, Nix, Randrup, Robel, Sierk, Swiatecki, Koonin

The Wall formula (II): Barnett, Cohen, Heller [1] - regarding  $g_c$

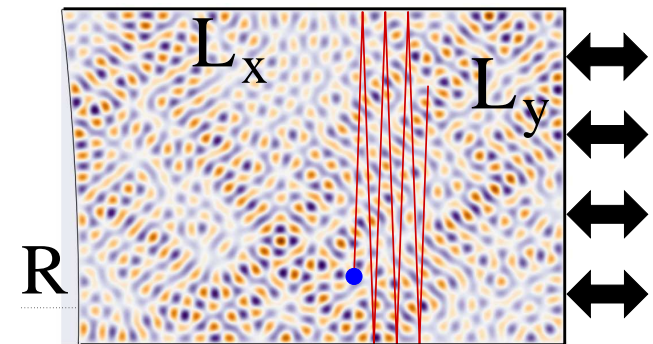
Semi Linear response theory: Cohen, Kottos, Schanz... [2-6]

Billiards with vibrating walls: Stotland, Cohen, Davidson, Pecora [7,8] - regarding  $g_s$

Sparsity: Austin, Wilkinson, Prosen, Robnik, Alhassid, Levine, Fyodorov, Chubykalo, Izrailev, Casati

$$u = (t_R / t_L)^{-1} = (R/L)^{-1} = \text{deformation}$$

$$\hbar = \lambda_E / L = 2\pi / (k_E L) = \text{function of } E$$



[1] A. Barnett, D. Cohen, E.J. Heller (PRL 2000, JPA 2000)

[2] D. Cohen, T. Kottos, H. Schanz (JPA 2006)

[3] S. Bandopadhyay, Y. Etzioni, D. Cohen (EPL 2006)

[4] M. Wilkinson, B. Mehlig, D. Cohen (EPL 2006)

[5] A. Stotland, R. Budoyo, T. Peer, T. Kottos, D. Cohen (JPA/FTC 2008)

[6] A. Stotland, T. Kottos, D. Cohen (PRB 2010)

[7] A. Stotland, D. Cohen, N. Davidson (EPL 2009)

[8] A. Stotland, L.M. Pecora, D. Cohen (EPL 2010, PRE 2011)

[9] D. Hurowitz, D. Cohen (EPL 2011)

## Conclusions

(\*) Wigner ( $\sim 1955$ ): The perturbation is represented by a random matrix whose elements are taken from a **Gaussian** distribution.

Not always...

1. “**weak quantum chaos**”  $\implies$  log-wide distribution, “**sparsity**” and “**texture**”
2. The heating process  $\sim$  a **percolation** problem.
3. Resistors network calculation to get  $G_{\text{SLRT}}$ .
4. Generalization of the **VRH estimate**.
5. **SLRT** applies if the driving is stronger than the background relaxation.
6. The **stochastic NESS** resembles a **glassy phase** (wide distribution of microscopic temperatures).
7. Definition of effective **NESS temperature**, and extension of the **FDT phenomenology**.
8. For very strong driving - **quantum saturation** of the NESS temperature ( $T \rightarrow T_\infty$ ).
9. Applications: beyond the “**Drude formula**” and beyond the “**Wall formula**”.