The non-equilibrium steady-state of a mesoscopic glassy system: interplay of resistor-network theory and Sinai physics

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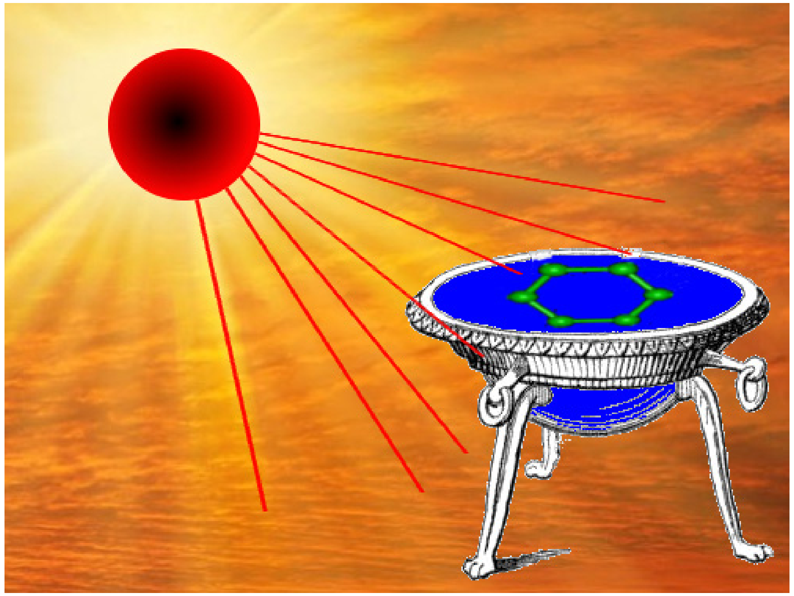
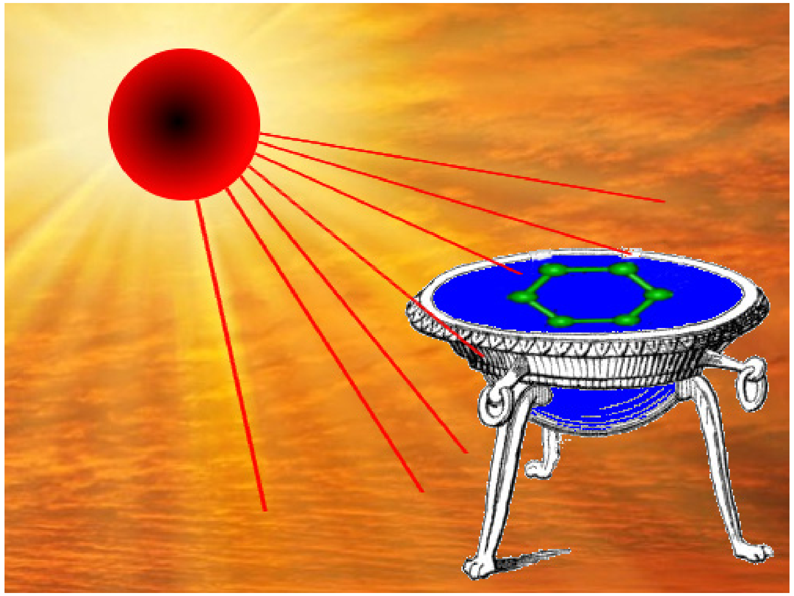
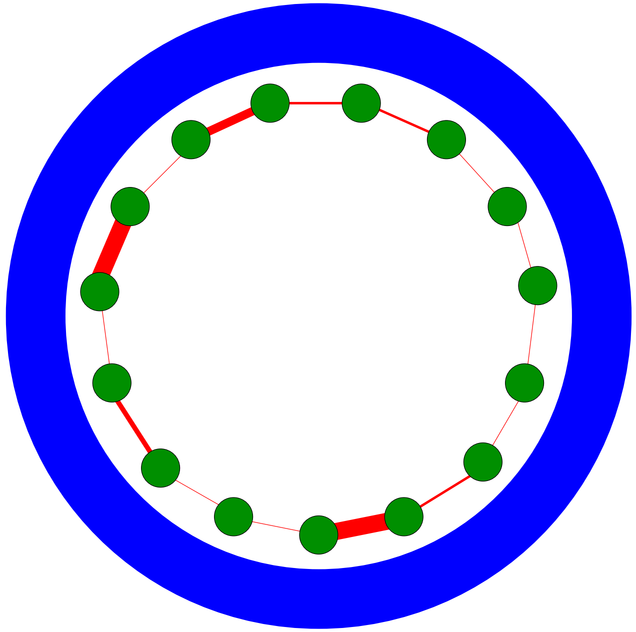


[1] Daniel Hurowitz, DC (EPL 2011, arXiv 2014)

[2] Daniel Hurowitz, Saar Rahav, DC (EPL 2012, PRE 2013)

Mesoscopic glassy systems





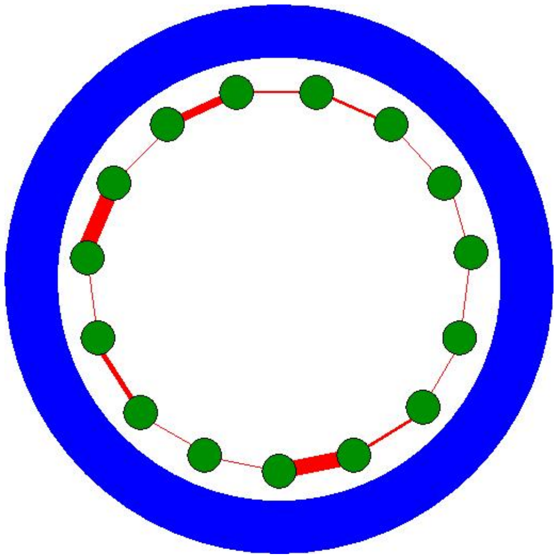
$$w_{\vec{n}} = w^{\beta}_{\vec{n}} + \nu g_n$$

In our study we consider systems that are "sparse" or "glassy", meaning that many time scales are involved.

Standard thermodynamics does not apply to such systems.

Minimal model of a "glassy" mesoscopic system

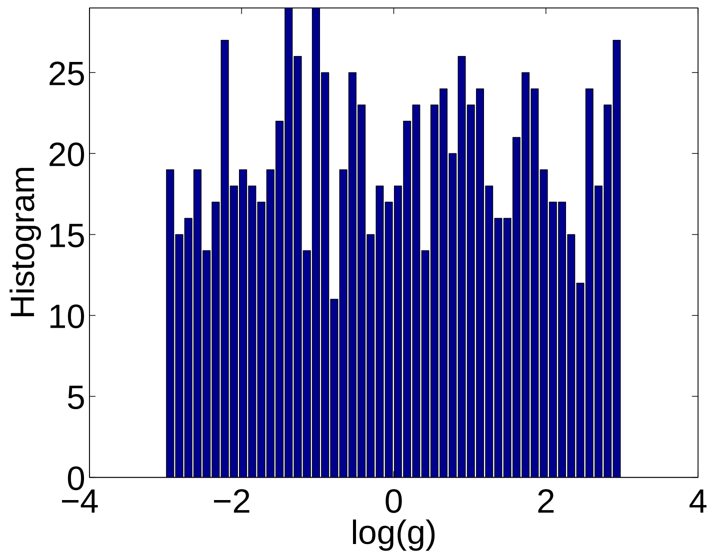
System + Bath + Driving



$$w_{\vec{n}} \;=\; w^{\beta}_{\vec{n}} \;+\; \nu g_n$$

$$g_n \;=\; \text{couplings}$$

Histogram of couplings



$\longleftarrow$ σ =few decades $\longrightarrow$

"sparsity" = log wide distribution of couplings

$$w^{\nu}_{n} = \nu g_n$$

corresponds to $T_A = \infty$

$$\frac{w^{\beta}_{\vec{n}}}{w^{\beta}_{\overleftarrow{n}}} = \exp\left[-\frac{E_n - E_{n-1}}{T_B}\right]$$

corresponds to $T_B =$ finite

The stochastic potential and the SMF

Steady state rate equations:

$$I = w_{\overrightarrow{n}} p_n - w_{\overleftarrow{n}} p_{n+1}$$

Stochastic field:

$$\mathcal{E}(x_n) \equiv \ln\left[\frac{w_{\overrightarrow{n}}}{w_{\overleftarrow{n}}}\right] \approx -\left[\frac{1}{1+g_n\nu}\right]\frac{E_n - E_{n-1}}{T_B}$$

Stochastic potential:

$$V(x) = -\int^x \mathcal{E}(x')dx' \approx \sum_n \left[\frac{1}{1+g_n\nu}\right]\frac{E_n - E_{n-1}}{T_B}$$

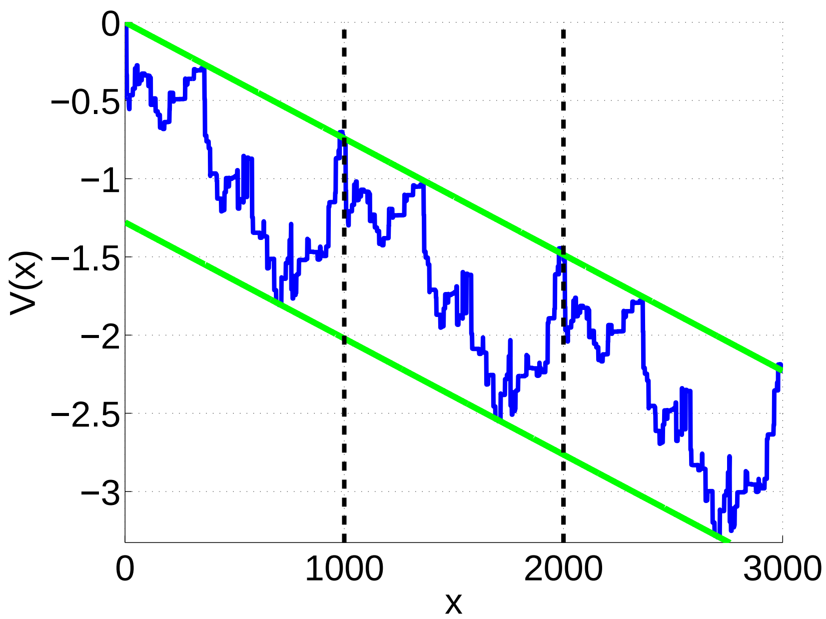
Stochastic Motive Force:

$$\mathcal{E}_{\circlearrowleft} \equiv \ln\left[\frac{\prod_n w_{\overrightarrow{n}}}{\prod_n w_{\overleftarrow{n}}}\right] = \oint \mathcal{E}(x)\, dx \qquad \text{[0 if no driving]}$$

Telescopic correlations:

$$\mathcal{E}(x_n) \sim \Delta_n \equiv (E_n - E_{n+1})$$

Yet... we have sparsely distributed couplings



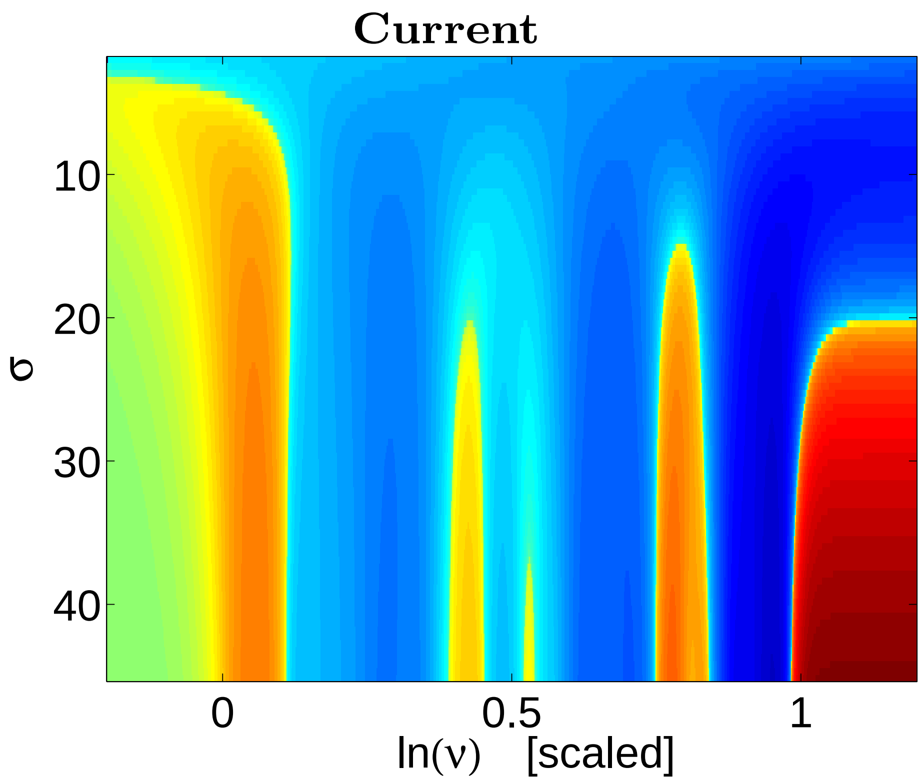
Sinai's Random-Walk [1982]:

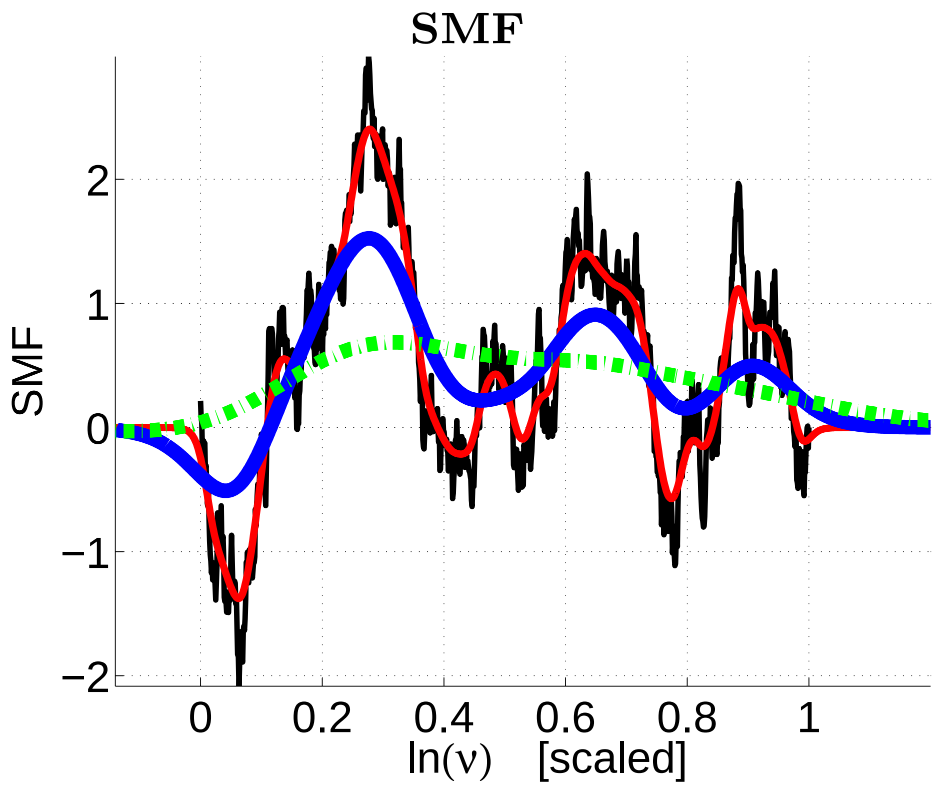
Random, Uncorrelated & non symmetric transition rates

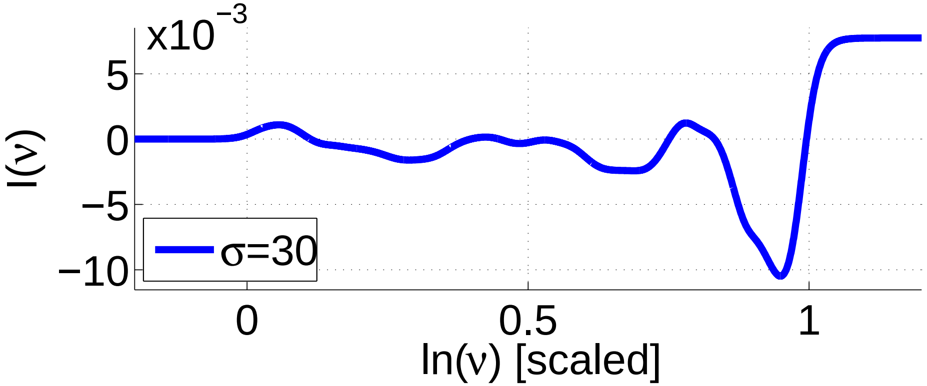
$\rightsquigarrow$ Buildup of activation barrier $B \sim \sqrt{N}$

$\rightsquigarrow$ Exponentially low current $I \sim e^{-\sqrt{N}}$

Current vs Driving intensity







$$I(\nu) \quad \sim \quad \frac{1}{N}\, w_\varepsilon\, \mathrm{e}^{-B}\, 2\sinh\left(\frac{\mathcal{E}_{\circlearrowleft}}{2}\right)$$

$\mathcal{E}_{\circlearrowleft}$ - Stochastic Motive Force

B - Effective Activation Barrier

The number of sign changes depends on the sparsity $\approx \sqrt{\pi\sigma}$

SMF vs Driving intensity

Stochastic Motive Force

$$\mathcal{E}_{\circlearrowleft}(\nu) \;\approx\; -\sum_{n=1}^{N}\left[\frac{1}{1+g_n\nu}\right]\frac{E_n - E_{n-1}}{T_B}$$

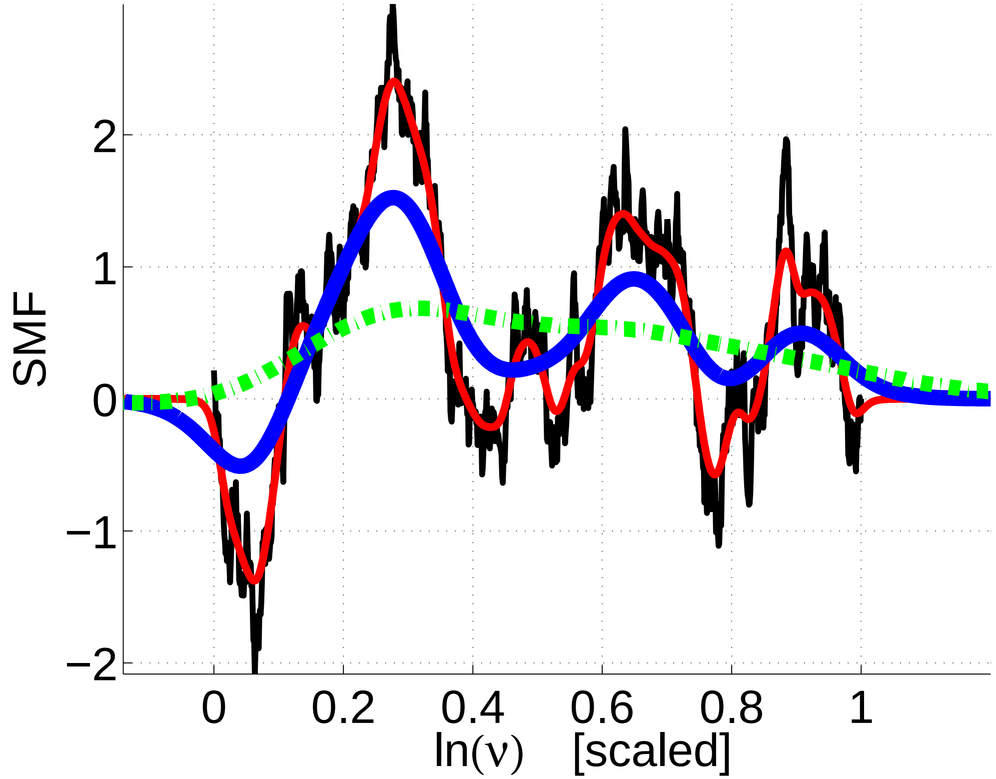
$$\tau \equiv \frac{1}{\sigma}\ln(g_{\mathsf{max}}\nu), \quad \tau_n = \frac{1}{\sigma}\ln\left(\frac{g_{\mathsf{max}}}{g_n}\right)$$

$$\sigma \;=\; \ln\frac{g_{\mathsf{max}}}{g_{\mathsf{min}}}, \qquad \text{[log-width of distribution]}$$

Coarse grained random walk:

$$\mathcal{E}_{\circlearrowleft}(\tau) = -\sum_{n=1}^{N} f_\sigma(\tau-\tau_n)\frac{E_n - E_{n-1}}{T_B}$$

$$f_\sigma(t) \;\equiv\; [1+\mathrm{e}^{\sigma t}]^{-1} \qquad \text{["step" function]}$$

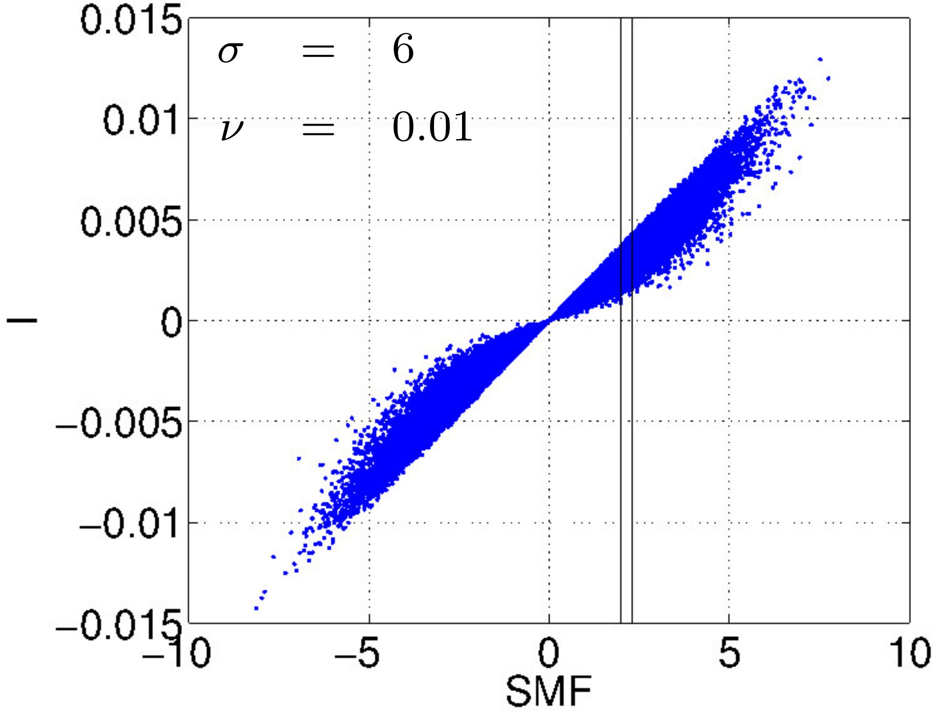


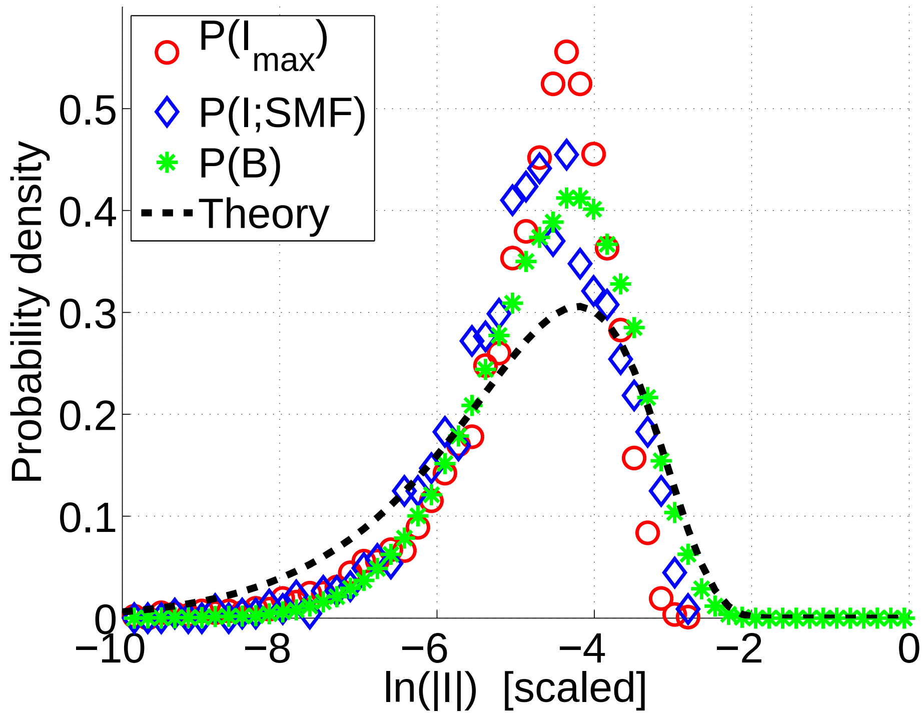
Sinai regime: $\frac{1}{g_{\mathsf{max}}} < \;\nu\; < \frac{1}{g_{\mathsf{min}}}$

$0 < \;\tau\; < 1$

Expected number of sign changes $\approx \sqrt{\pi\sigma}$

Statistics of the Current in the Sinai Regime





Single barrier approximation for the current

$$I(\nu) \sim \frac{1}{N} w_\varepsilon \, e^{-B} \, 2\sinh\left(\frac{\mathcal{E}_{\circlearrowleft}}{2}\right)$$

$\mathcal{E}_{\circlearrowleft}$ - Stochastic Motive Force

B - Effective Activation Barrier

Barrier distribution

$$\text{Prob}\,\{\text{barrier} < B\} \sim \exp\left[-\frac{1}{2}\left(\frac{\pi\sigma_B}{2B}\right)^2\right]$$

$$\sigma_B^2 = 2\Delta^2 N \, \frac{\ln(g_{\max}\nu)}{\sigma}$$

Barrier Statistics

Activation Barrier $\equiv$ Occupation range of a random walk.

$B \approx \frac{1}{2}\left[\max\{U\} - \min\{U\}\right] \equiv 2R$

- Joint probability that a RW occupies the interval $[x_a, x_b]$:

$$P_t(x_a, x_b) \equiv \text{Prob}(x_a < x(t') < x_b), \quad t' \in [0, t]$$

$$f(x_a, x_b) = -\frac{d}{dx_a}\frac{d}{dx_b}P_t(x_a, x_b)$$

- Make the transformation $X = \frac{x_a + x_b}{2}, \quad R = x_b - x_a$
- A random walk process occupies range R:

$$f(R) = \partial_R^2\left[R\ P_t(R)\right]$$

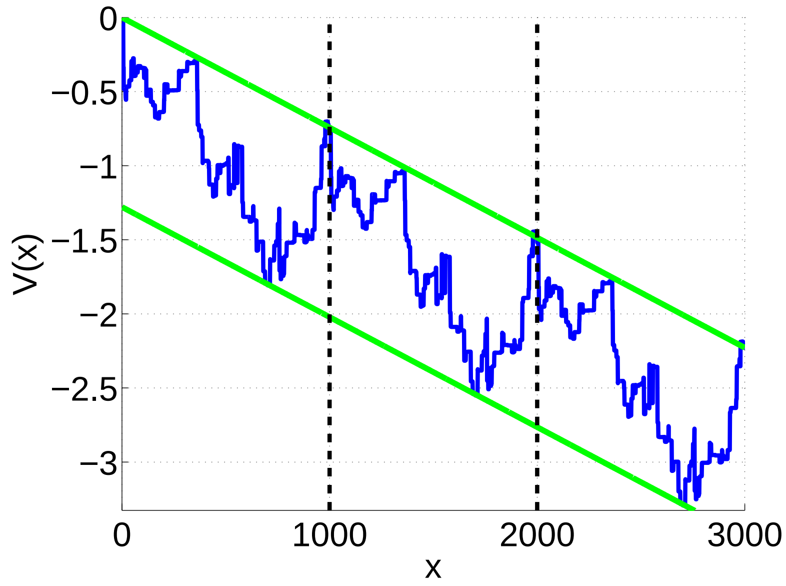
Survival probability of a diffusion process with initial uniform distribution: $P_t(R)$

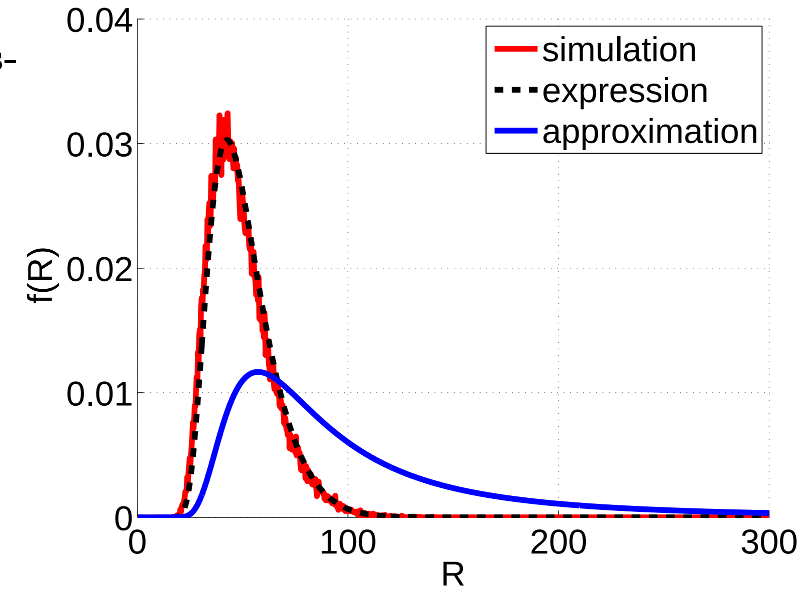
- Solution to diffusion equation

$$\rho_t(x) = \sum_{n=1,3,5,\dots}^{\infty} \exp\left[-D\left(\frac{\pi n}{R}\right)^2 t\right]\frac{4}{\pi n R}\sin\left(\frac{\pi n}{R}x\right)$$

$$P_t(R) = \int_0^R \rho_t(x)\ dx = \sum_{n=1,3,5,\dots}^{\infty}\frac{8}{\pi^2 n^2}\exp\left[-D\left(\frac{\pi n}{R}\right)^2 t\right]$$

$$P_t(r) \approx \exp\left(-\frac{1}{2}\left(\frac{\pi\sigma}{R}\right)^2\right)$$





The NFT-based Einstein relation

The rate equation can be regarded as a probabilistic description of Sinai-type random walk process.

Drift: $\langle x \rangle = vt$

Diffusion: $\mathrm{Var}(x) = 2Dt$

Einstein: $\frac{v}{D} = s \quad \rightsquigarrow \quad \frac{\mu}{D} = \frac{1}{T}$

$s \equiv$ entropy-production-per-distance

Optional notations:

SMF: $\mathcal{E}_{\circlearrowleft} = Ns$

Bias field: $s = F/T$

Mobility: $v = \mu F$

Drift current: $I = (1/N)v$

Analysis:

Time dependent solution of the rate equation $(d/dt)\boldsymbol{p} = W\boldsymbol{p}$

$$p_n(t) \approx \frac{1}{L}\sum_{k,\nu} C_{k,\nu}\, \mathrm{e}^{-\lambda_\nu(k)t}\, \mathrm{e}^{ikn}$$

$$v = i\frac{\partial \lambda_0(k)}{\partial k}\bigg|_{k=0}$$

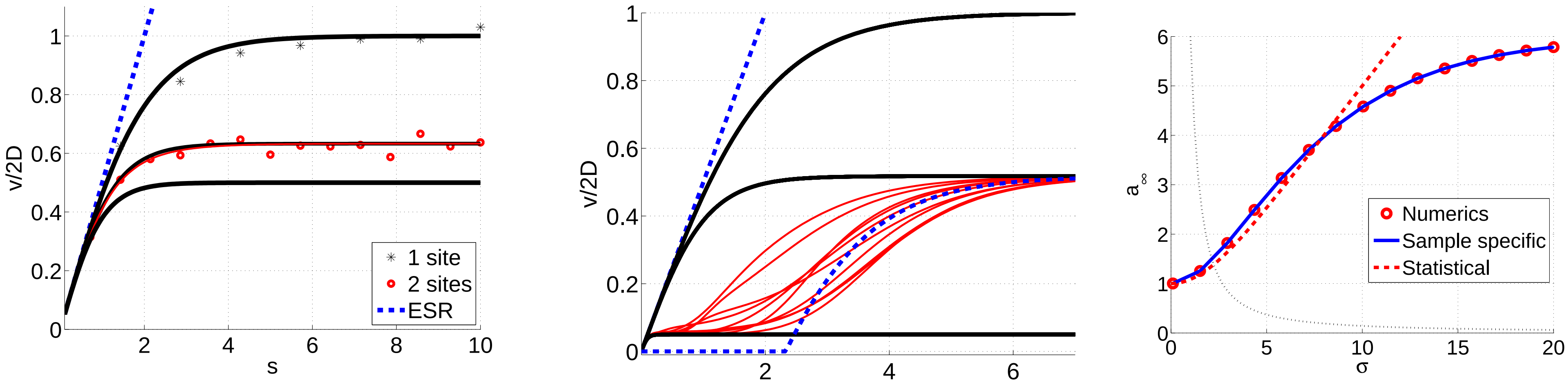
$$D = \frac{1}{2}\frac{\partial^2 \lambda_0(k)}{\partial k^2}\bigg|_{k=0}$$

Observations

$$\frac{v}{D} = f_\sigma(s) \qquad\qquad f_\sigma(s) = \frac{2}{a_s}\tanh\frac{a_s s}{2}$$

Without disorder a is the lattice constant $a^{(0)}$.
As a function of σ, the asymptotic value a_∞ varies from $a^{(0)}$ to $a^{(N)}$.
As a function of s the function a_s varies from $a^{(N)}$ to a_∞.

$$a_\infty = \left[\frac{\langle(1/w)^2\rangle}{\langle(1/w)\rangle^2}\right] a_0$$



NFT-based derivation

Define x as the winding number times the length of the ring.

$$\frac{P\left[x(-t)\right]}{P\left[x(t)\right]} = \exp\left[-\mathcal{S}[x]\right] \quad \rightsquigarrow \quad \frac{p(-x;t)}{p(x;t)} = \mathrm{e}^{-sx}$$

Gaussian approximation

$$p(x;t) \approx \overline{p}(x;t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left[-\frac{(x-vt)^2}{4Dt}\right] \quad \rightsquigarrow \quad \frac{v}{D} = s$$

The naive reasoning, based on CLT, is wrong, If we smear $p(x)$ we get

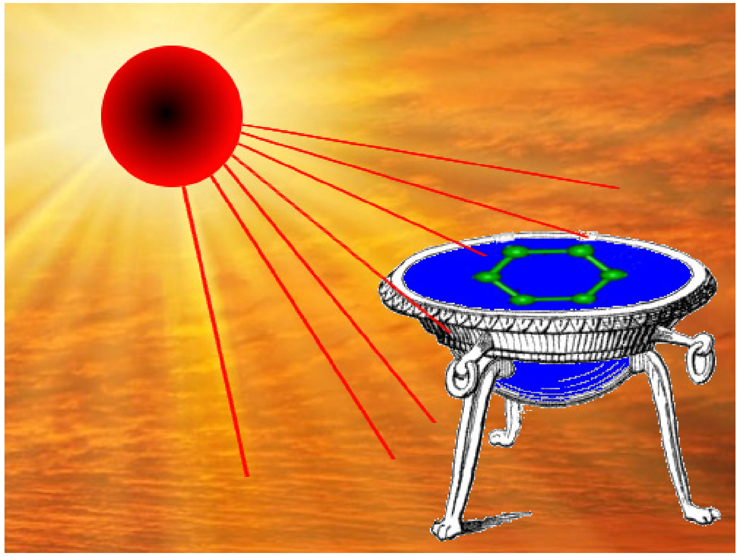
$$\frac{\overline{p}(-x;t)}{\overline{p}(x;t)} = \mathrm{e}^{-\overline{s}x} \quad \rightsquigarrow \quad \frac{v}{D} = \overline{s} = \frac{2}{a}\tanh\frac{as}{2}$$

Optionally we can derive $f_\sigma(s)$ directly from the exact expression:

$$p(x;t) = \int_{-\infty}^{\infty} dk \ \mathrm{e}^{ikx+\left(\overrightarrow{w}\mathrm{e}^{-ik}+\overleftarrow{w}\mathrm{e}^{ik}-(\overleftarrow{w}+\overrightarrow{w})\right)t}$$

$$\overline{p}(x;t) = \int_{-\infty}^{\infty} dk \ \mathrm{e}^{ik(x-(\overrightarrow{w}-\overleftarrow{w})t)-\frac{k^2}{2}(\overrightarrow{w}+\overleftarrow{w})t+\ \cancel{\mathcal{O}(k^3t)}}$$

Thermodynamics of a "glassy" system



$$w_{nm} = w^{\beta}_{nm} + w^{\nu}_{nm} = w^{\beta}_{nm} + \nu g_{nm}$$

Cold bath: $$\frac{w^{\beta}_{nm}}{w^{\beta}_{mn}} = \exp\left[-\frac{E_n - E_m}{T_B}\right]$$

Hot source: $$g_{nm} = g_{mn}$$

w^{ν} by themselves - induces diffusion / ergodization

w^{β} by themselves - leads to equilibrium

Combined - leads to **NESS**

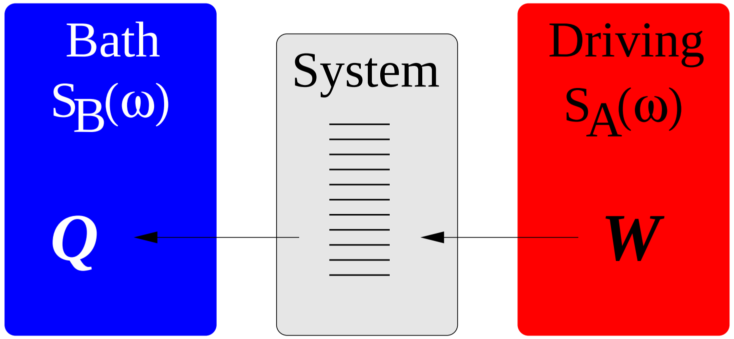
Linear response and traditional FD: $\nu \times \{g\} \ll \{w^{\beta}\}$

Glassy response and Sinai physics: [within a wide crossover regime]

Semi-linear response and Saturation: $\nu \times \{g\} \gg \{w^{\beta}\}$

FD relation for the rate of energy flow

$$w_{nm} \;=\; w^{\beta}_{nm} + w^{\nu}_{nm} \;=\; w^{\beta}_{nm} + \nu g_{nm}$$



$$\dot{W} \;=\; \text{rate of heating} \;=\; \frac{D(\nu)}{T_{\text{system}}}$$

$$\dot{Q} \;=\; \text{rate of cooling} \;=\; \frac{D_B}{T_B} - \frac{D_B}{T_{\text{system}}}$$

Hence at the NESS:

$$T_{\text{system}} \;=\; \left(1 + \frac{D(\nu)}{D_B}\right) T_B$$

$$\dot{Q} \;=\; \dot{W} \;=\; \frac{1/T_B}{D_B^{-1} + D(\nu)^{-1}}$$

Experimental way to extract response:

$$D(\nu) \;=\; \frac{\dot{Q}(\nu)}{\dot{Q}(\infty) - \dot{Q}(\nu)} D_B$$

$D(\nu)$ exhibits LRT to SLRT crossover

$$D(\nu) \;=\; \left[\overline{\left(\frac{w_n}{w_\beta + w_n}\right)}\right] \left[\overline{\left(\frac{1}{w_\beta + w_n}\right)}\right]^{-1}$$

$$D_{[\text{LRT}]} \;=\; \overline{g_n}\,\nu \qquad \text{[weak driving]}$$

$$D_{[\text{SLRT}]} \;=\; [\overline{1/g_n}]^{-1}\nu \qquad \text{[strong driving]}$$

Expressions above assume n.n. transitions only.

Summary of main results

1. Number of current sign changes is determined by log-width of coupling distribution,
   Expected number of sign changes $\approx \sqrt{\pi\sigma}$

2. The current in the Sinai regime may be estimate by a single barrier approximation,
   $I(\nu) \sim \frac{1}{N}\, w_{\varepsilon}\, \mathrm{e}^{-B}\, 2\sinh\left(\frac{\mathcal{E}_{\circlearrowleft}}{2}\right)$

3. Exact expression for (non-canonical) NESS occupation probability
   reflects crossover from Sinai spreading to resistor network picture.
   $p_n \;\propto\; \left(\frac{1}{w(x_n)}\right)_{\varepsilon} \mathrm{e}^{-(U(n)-U_{\varepsilon}(n))}$

4. Distribution of currents reflects Barrier statistics
   $\text{Prob}\{\text{barrier} < B\} \;\sim\; \exp\left[-\frac{1}{2}\left(\frac{\pi\sigma_{\mathrm{B}}}{2B}\right)^2\right]$

5. Modified Einstein relation $v/D = f_{\sigma}(s)$

6. FD phenomenology for the rate of energy flow through a "glassy" system.