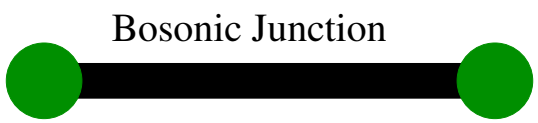
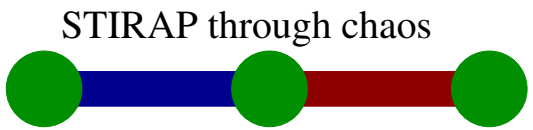
Metastability and thermalization in Bose-Hubbard circuits

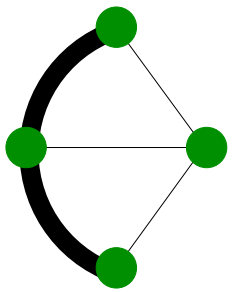
Doron Cohen, Ben-Gurion University

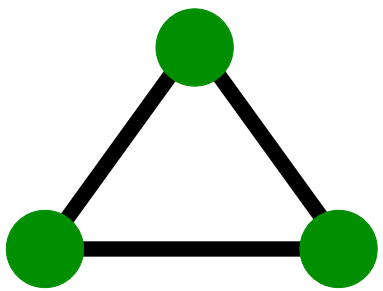
Circuits with condensed bosons are the building blocks for quantum Atomtronics. Such circuits will be used as QUBITs (for quantum computation) or as SQUIDs (for sensing of acceleration or gravitation). We study the feasibility and the design considerations for devices that are described by the Bose-Hubbard Hamiltonian. It is essential to realize that the theory involves "Quantum chaos" considerations.

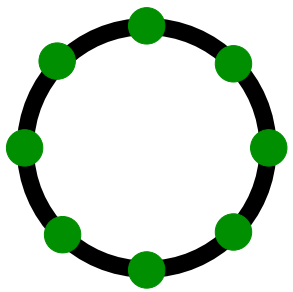
- The Bose-Hubbard Hamiltonian.
- Relevance of chaos for *Metastability and Ergodicity*.
- Thermalization and quantum localization.

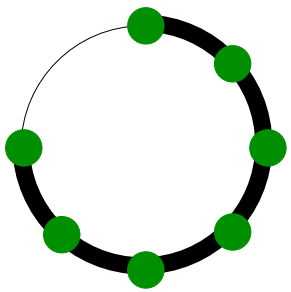












The Bose Hubbard Hamiltonian

The system consists of N bosons in M sites. Later we add a gauge-field Φ.

$$\mathcal{H}_{\rm BHH} = \frac{U}{2}\sum_{j=1}^{M} a_j^\dagger a_j^\dagger a_j a_j - \frac{K}{2}\sum_{j=1}^{M}\left(a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}\right)$$

$$u \equiv \frac{NU}{K} \qquad \text{[classical, stability, supefluidity, self-trapping]}$$

$$\gamma \equiv \frac{Mu}{N^2} \qquad \text{[quantum, Mott-regime]}$$

Dimer (M=2): Minimal BHH; Bosonic Josephson junction; Pendulum physics [1].
Driven dimer: Landau-Zener dynamics [2], Kapitza effect [3], Zeno effect [4], Standard-map physics [5].
Trimer (M=3): Minimal model for low-dimensional chaos; Many body STIRAP [6].
Triangular trimer (M=3): Minimal model with topology, Superfluidity [7], Stirring [...].
Larger rings (M>3) High-dimensional chaos; web of non-linear resonances [7].
Coupled subsystems (M>3): Minimal model for Thermalization [8,9].

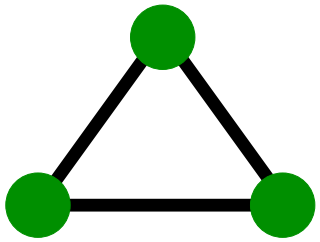
[1] Chuchem, Smith-Mannschott, Hiller, Kottos, AV, DC (PRA 2010).
[2] Smith-Mannschott, Chuchem, Hiller, Kottos, DC (PRL 2009).
[3] Boukobza, Moore, DC, AV (PRL 2010).
[4] Khripkov, AV, DC (PRA 2012)
[5] Khripkov, DC, AV (JPA 2013, PRE 2013).
[6] Dey, DC, AV (arXiv 2018).
[7] Geva Arwas et al (PRA 2014, SREP 2015, NJP 2016, PRB 2017, PRA 2017, arXiv 2018).
[8] Tikhonenkov, AV, Anglin, DC, (PRL 2013).
[9] Christine Khripkov, AV, DC (NJP 2015, PRE 2018).

AV = Ami Vardi

Quantum Chaos perspective on Metastability and Ergodicity

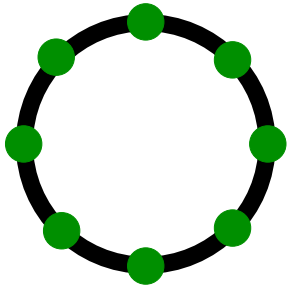
Stability of flow-states (I):

- Landau stability of flow-states ("Landau criterion")
- Bogoliubov perspective of dynamical stability
- KAM perspective of dynamical stability



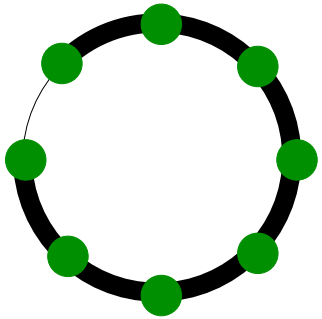
Stability of flow-states (II):

- Considering high dimensional chaos ($M > 3$).
- Web of non-linear resonances.
- Irrelevance of the the familiar Beliaev and Landau damping terms.
- Analysis of the quench scenario.



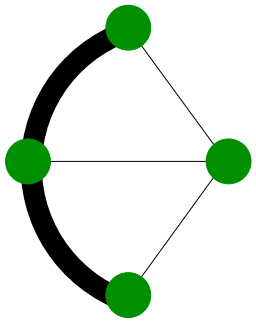
Coherent Rabi oscillations:

- The hallmark of coherence is Rabi oscillation between flow-states.
- Ohmic-bath perspective $\rightsquigarrow$ $\eta = (\pi/\sqrt{\gamma})$
- Feasibility of Rabi oscillation for $M < 6$ devices.
- Feasibility of of chaos-assisted Rabi oscillation.



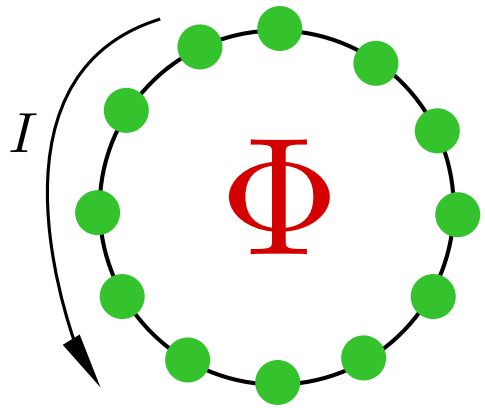
Thermalization:

- Spreading in phase space is similar to Percolation.
- Resistor-Network calculation of the diffusion coefficient.
- Observing regions with Semiclassical Localization.
- Observing regions with Dynamical Localization.



Bose Hubbard Ring

In the rotating reference frame we have a Coriolis force, which is like magntic field $\mathcal{B} = 2\mathsf{m}\Omega$. Hence is is like having flux



$$\mathcal{H} \;=\; \sum_{j=1}^{M}\left[\frac{U}{2}a_j^\dagger a_j^\dagger a_j a_j - \frac{K}{2}\left(\mathrm{e}^{i(\Phi/L)}a_{j+1}^\dagger a_j + \mathrm{e}^{-i(\Phi/L)}a_j^\dagger a_{j+1}\right)\right]$$

$$\mathcal{H} \;=\; \sum_k \epsilon_k(\Phi)\, b_k^\dagger b_k \;+\; \frac{U}{2L}\sum{}' b_{k_4}^\dagger b_{k_3}^\dagger b_{k_2} b_{k_1}$$

$$\Phi \;=\; 2\pi R^2 \mathsf{m}\,\Omega$$

For L=3 sites, using $b_k = \sqrt{n_k}e^{i\varphi_k}$, and $M = (n_1 - n_2)/2$, and $n = (n_1 + n_2)/2$

$$\mathcal{H}(\varphi, n; \phi, M) \;=\; \mathcal{H}^{(0)}(\varphi, n; M) \;+\; \left[\mathcal{H}^{(+)} + \mathcal{H}^{(-)}\right]$$

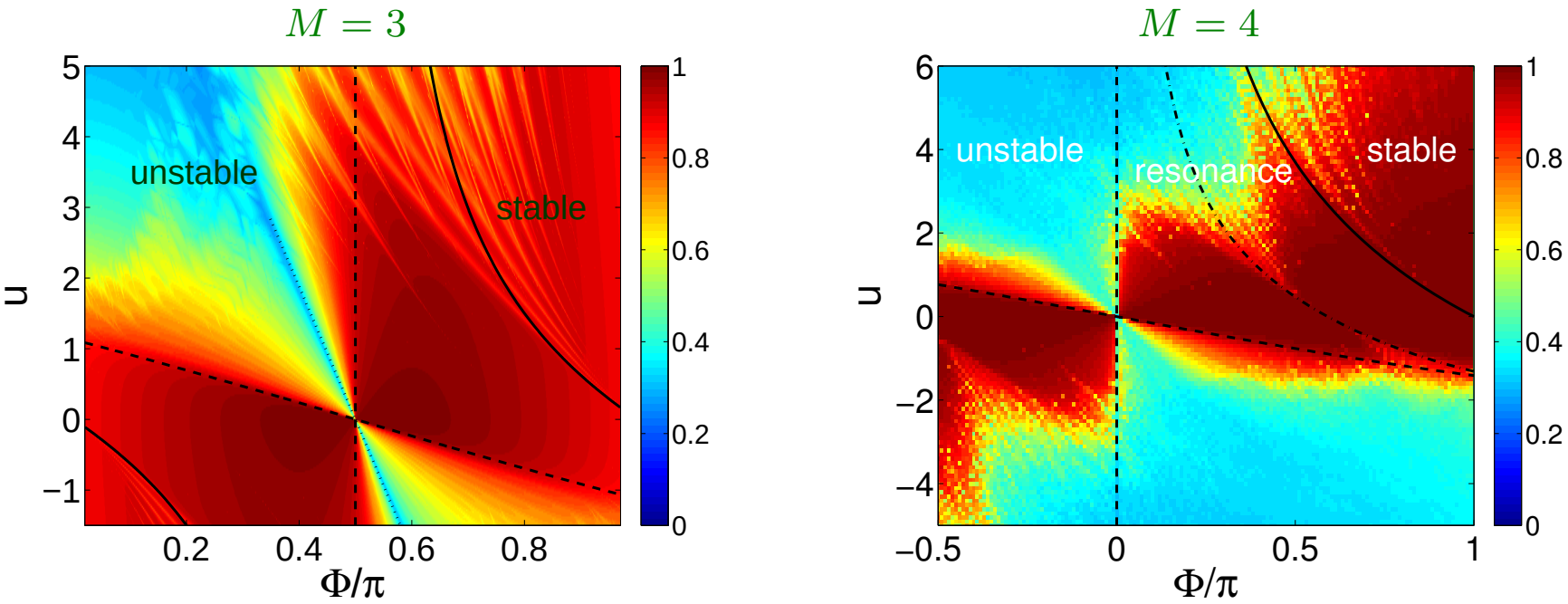
$$\mathcal{H}^{(0)}(\varphi, n; M) \;=\; \mathcal{E}n + \mathcal{E}_\perp M - \frac{U}{3}M^2 \;+\; \frac{2U}{3}(N-2n)\left[\frac{3}{4}n + \sqrt{n^2 - M^2}\cos(\varphi)\right]$$

$$\mathcal{H}^{(\pm)} \;=\; \frac{2U}{3}\sqrt{(N-2n)(n\pm M)}(n\mp M)\cos\left(\frac{3\phi\mp\varphi}{2}\right)$$

Flow-state stability regime diagram

The I of the maximum current state is imaged as a function of (Φ, u)

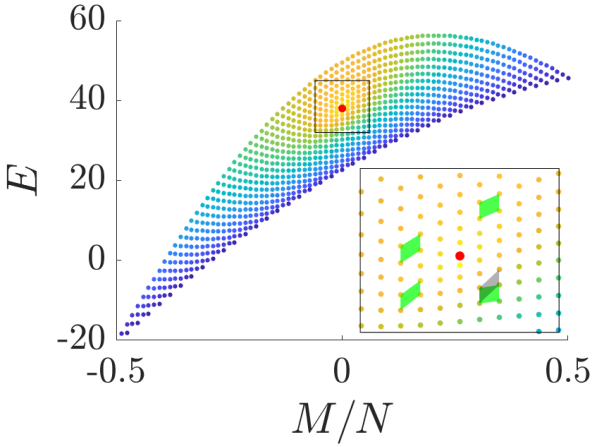
- solid lines = energetic stability borders (Landau)
- dashed lines = dynamical stability borders (Bogoliubov)

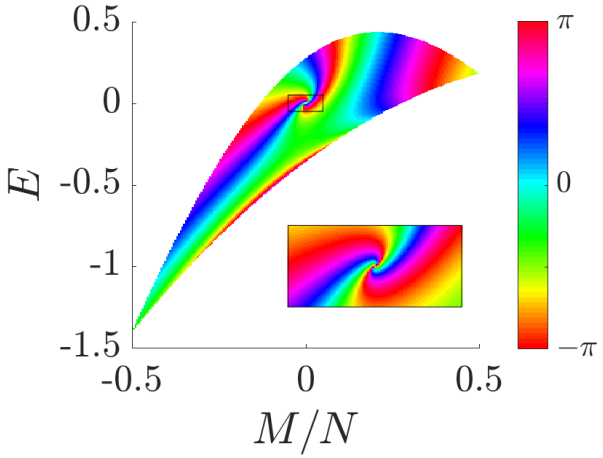


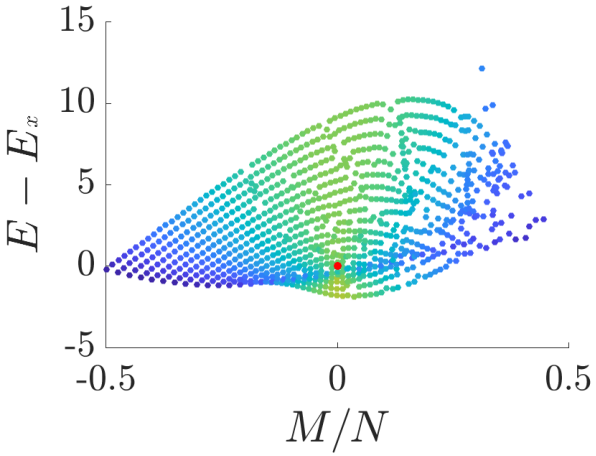
The traditional paradigm associates flow-states with stationary fixed-points in phase space. Consequently the Landau criterion, and more generally the Bogoliubov linear-stability-analysis, are used to determine the viability of superfluidity.

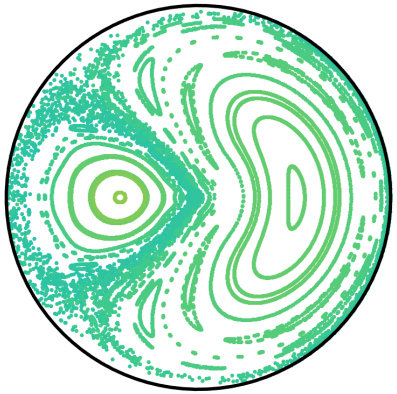
Arwas, Vardi, DC [**Scientific Reports 2015**]

Monodromy, Chaos, and Metastability of superflow









$M = (n_1 - n_2)/2$, constant of motion.
$n = (n_1 + n_2)/2$, conjugate phase φ.

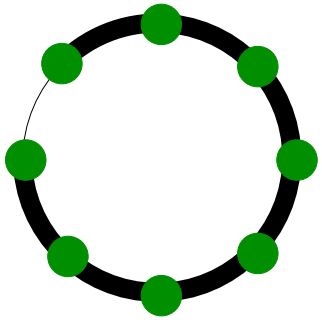
Section energy $E = E[\mathrm{SP}]$
Section coordinates (φ, n)

Regular separatrix $E = E_x(M)$

The swap transition

Arwas, DC **[arXiv 2018]**

SQUID-like geometry (weak link or barrier)



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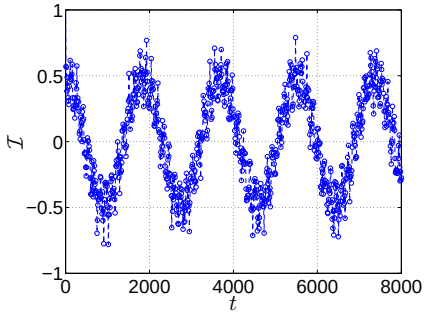
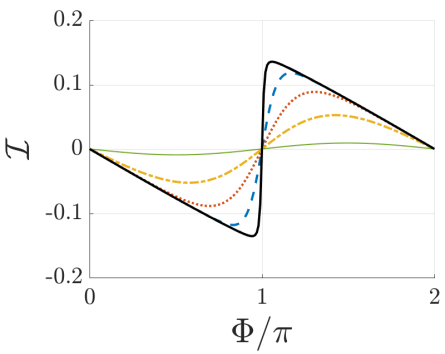
Arwas, DC [**New Journal of Physics 2016**]

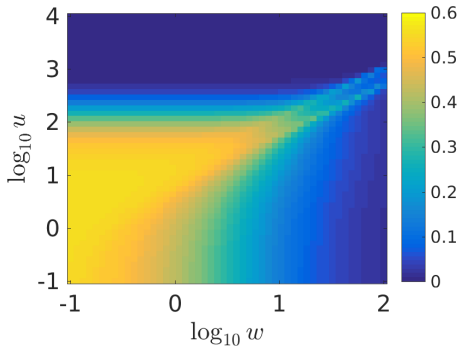


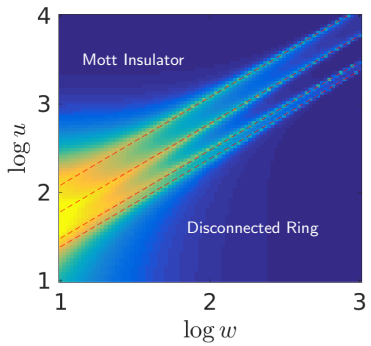
Resonant persistent currents ($\gamma \sim 1$):

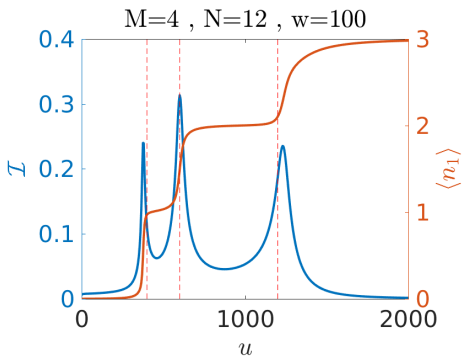
- Without optical lattice - non monotonic dependence on u
- With optical lattice - Mott transition; Quantum resonances

$$\max\left[I(\Phi)\right] \quad = \quad 2J\frac{N}{M}\,\alpha(w,u)$$









Arwas, DC, Hekking, Minguzzi [**PRA 2017**] - **Editors' Suggestion**

Strong localization and the exploration of phase space for quantum thermalization

[**1**] Christine Khripkov, Amichay Vardi, DC [**Physical Review E 2018**]
[**2**] Christine Khripkov, Amichay Vardi, DC [**New Journal of Physics 2015**]

[**3**] DC, Slava Yukalov, Klaus Ziegler [**Physical Review A 2016**]
[**4**] Igor Tikhonenkov, Amichay Vardi, James Anglin, DC [**Physical Review Letters 2013**]

Short version of talk prepared for *Celebrating 70th birthday of Rick Heller* (Cuernavaca 2017)

PHYSICAL REVIEW A VOLUME 35, NUMBER 3 FEBRUARY 1, 1987

Quantum localization and the rate of exploration of phase space

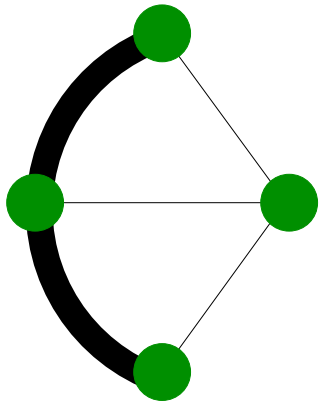
Eric J. Heller
Department of Chemistry and Department of Physics, University of Washington, Seattle, Washington 98195
(Received 29 August 1986)

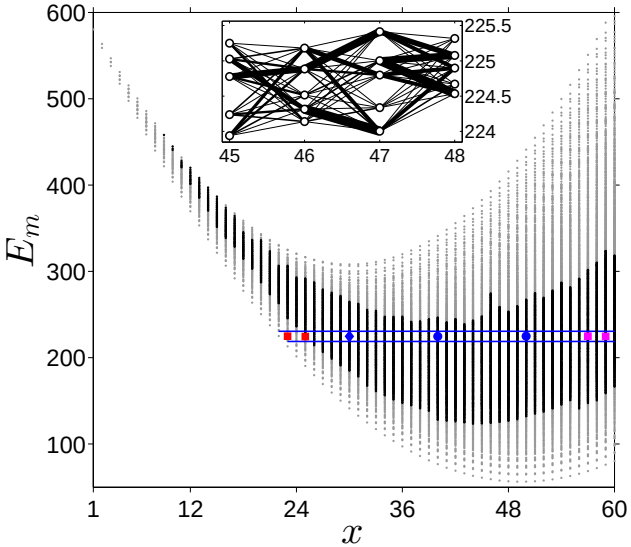
The minimal model for thermalization

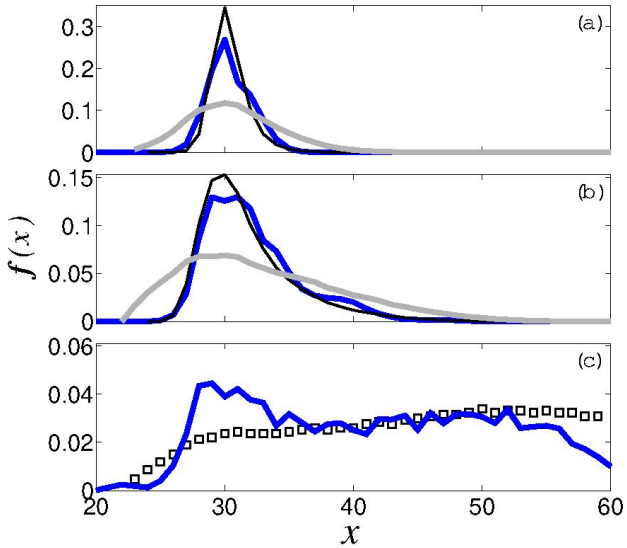
The FPE description makes sense if the sub-systems are chaotic.

Minimal model for a chaotic sub-system: BHH trimer.

Minimal model for thermalization: BHH trimer + monomer







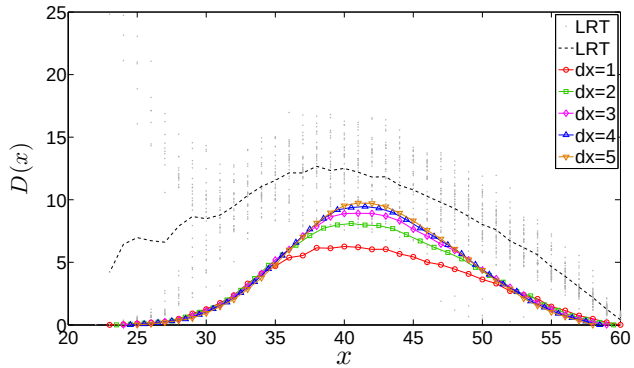
$N = 60 =$ number of particles

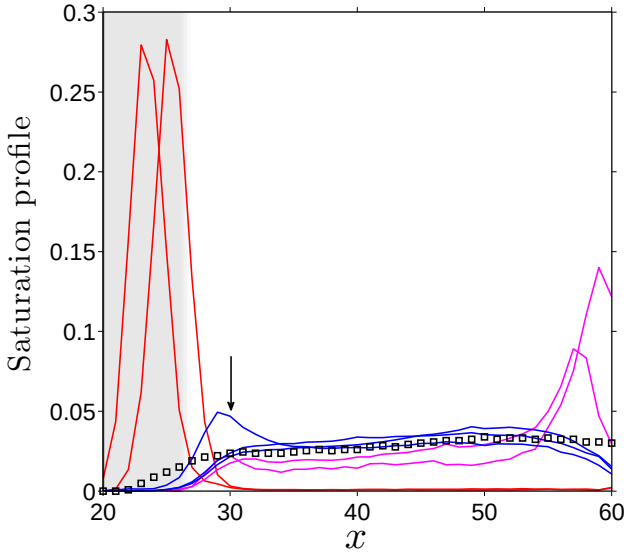
$x =$ occupation of the trimer

$N - x =$ occupation of the monomer

$f(x) =$ probability distribution

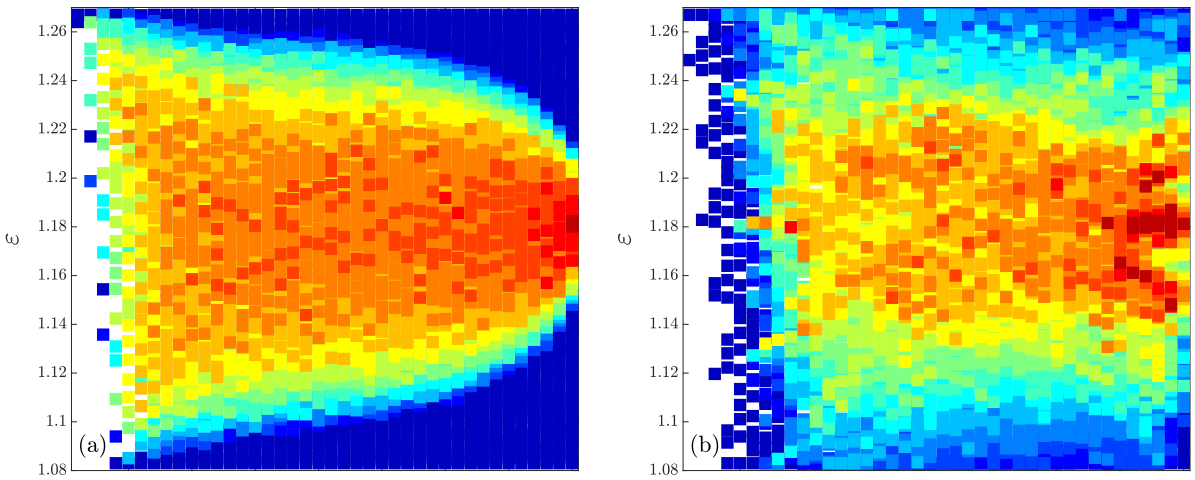
$$\frac{\partial f(x)}{\partial t} = \frac{\partial}{\partial x}\left[g(x)D(x)\frac{\partial}{\partial x}\left(\frac{f(x)}{g(x)}\right)\right]$$



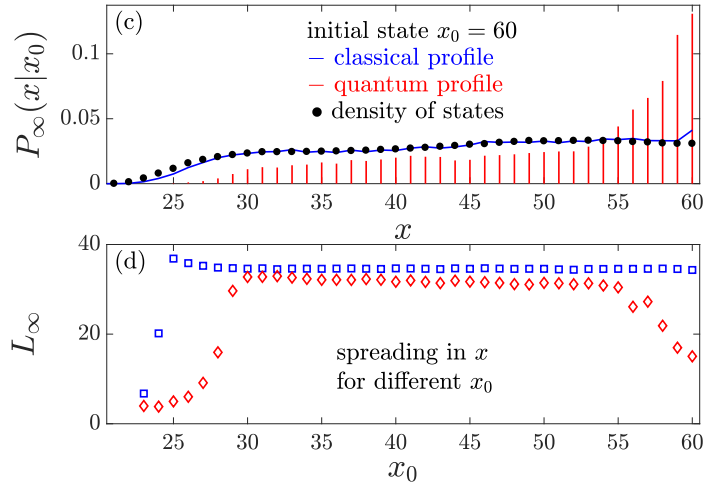


Dynamical localization

Here we start the simulation with $x_0 = 60$,
meaning that initially all the particles are in the trimer.
We plot the saturation profile $P_\infty(x, \varepsilon)$.



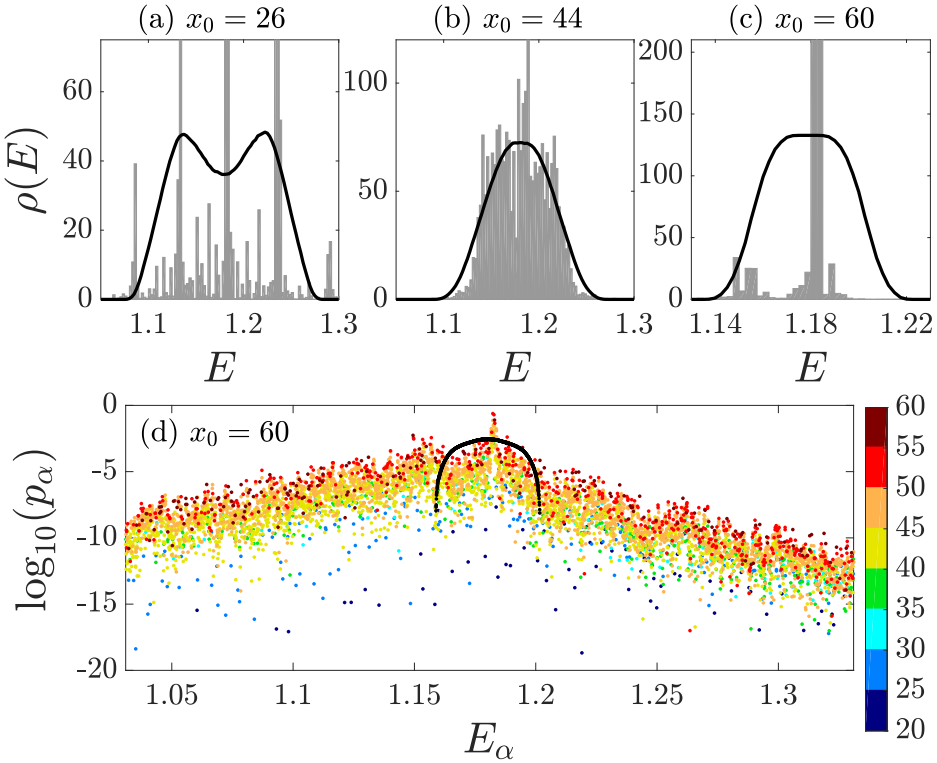
Classical Quantum

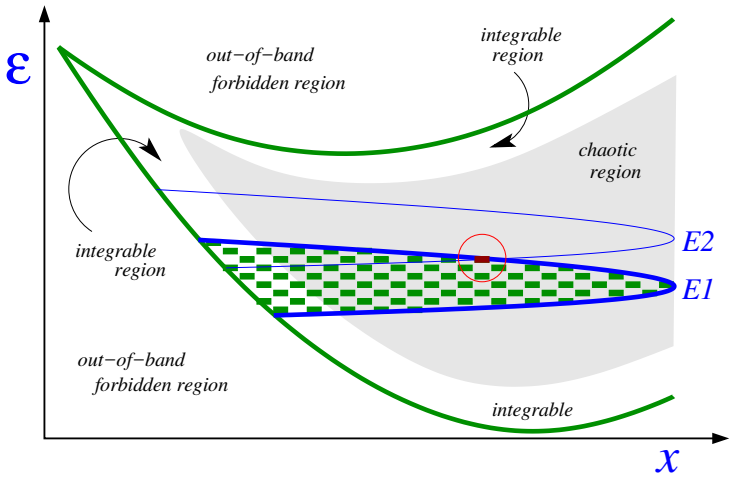


The LDOS of the initial preparation

We observed dynamical localization if we start with $x_0 < 30$ or with $x_0 > 55$.

Let us look on the LDOS of representative preparations:

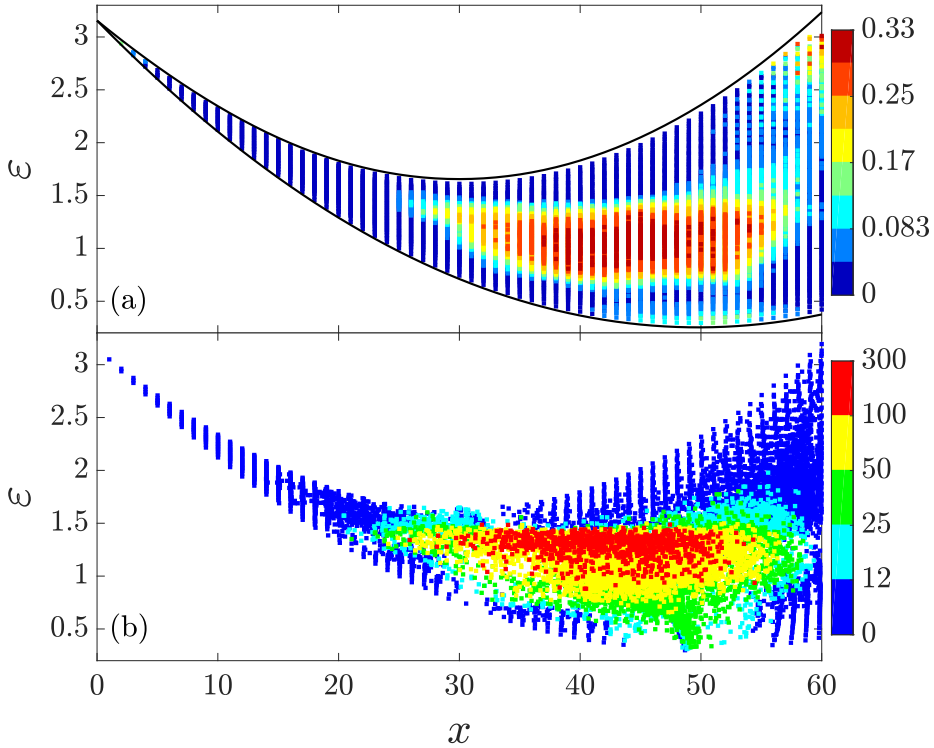




$$\mathcal{F}^{\rm qm} \equiv \frac{\mathcal{N}_\infty}{\mathcal{N}_E}$$

Localization does not always manifests as sparsity. It depends on the geometry of the $r = (x, \epsilon)$ space.

Localization of the eigenstates



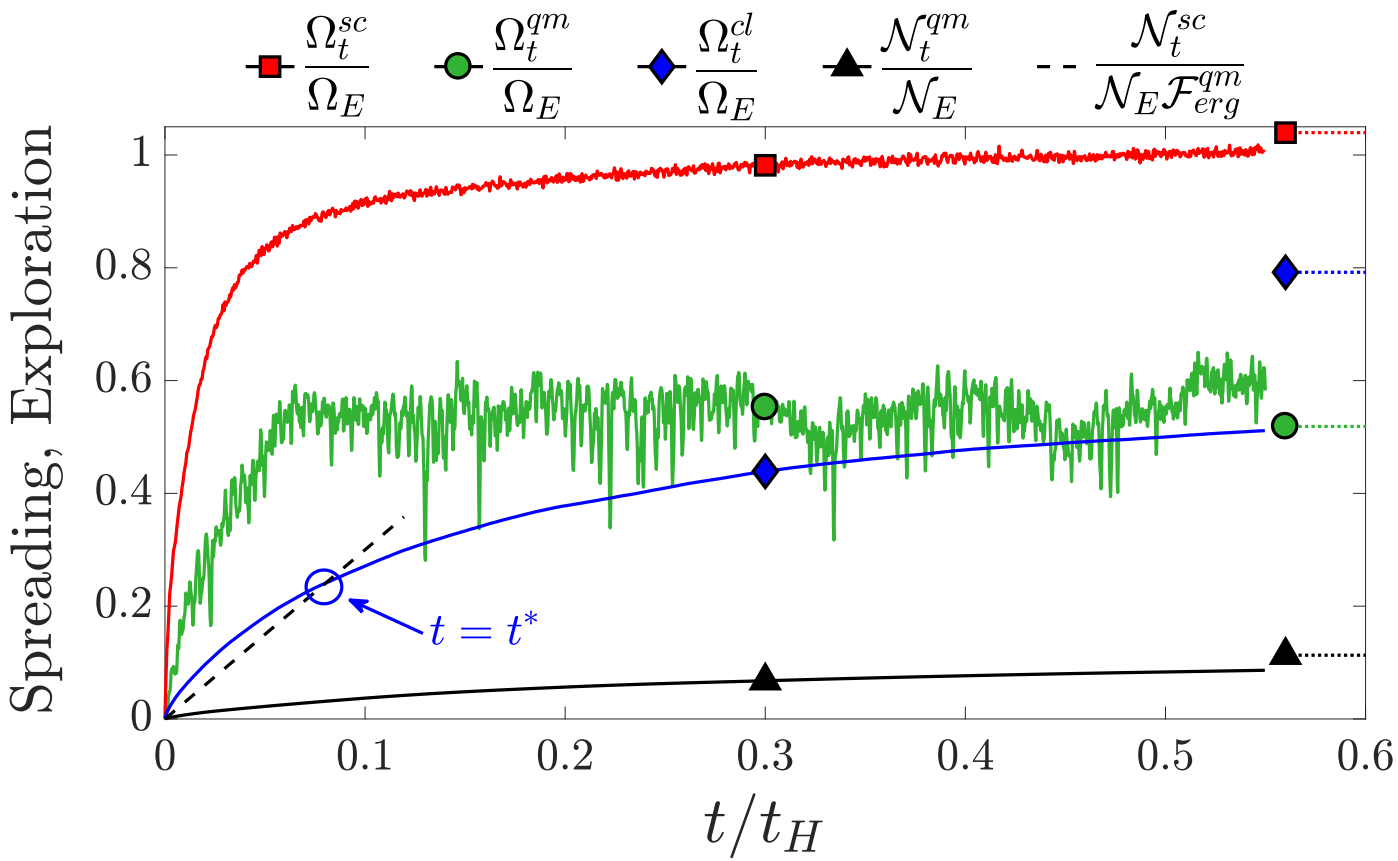
Upper panel:
The unperturbed states $|r\rangle = |x, \varepsilon\rangle$,
color-coded according to $\mathcal{F}^{\mathrm{qm}}$.
arranged by (x, ε).

Lower panel:
The perturbed states $|E_\alpha\rangle$,
color-coded according to $\mathrm{var}(x)_\alpha$,
sorted by $(\langle x\rangle_\alpha, \langle \varepsilon\rangle_\alpha)$.

Participation, Exploration, and Breaktime

We display $\Omega^{\rm cl}(t)$, and $\Omega^{\rm sc}(t)$, and $\Omega^{\rm qm}(t)$, and $\mathcal{N}(t)$.

The breaktime is determined by the intersection of the scaled $\mathcal{N}^{\rm sc}(t)$ with the scaled $\Omega^{\rm cl}(t)$.

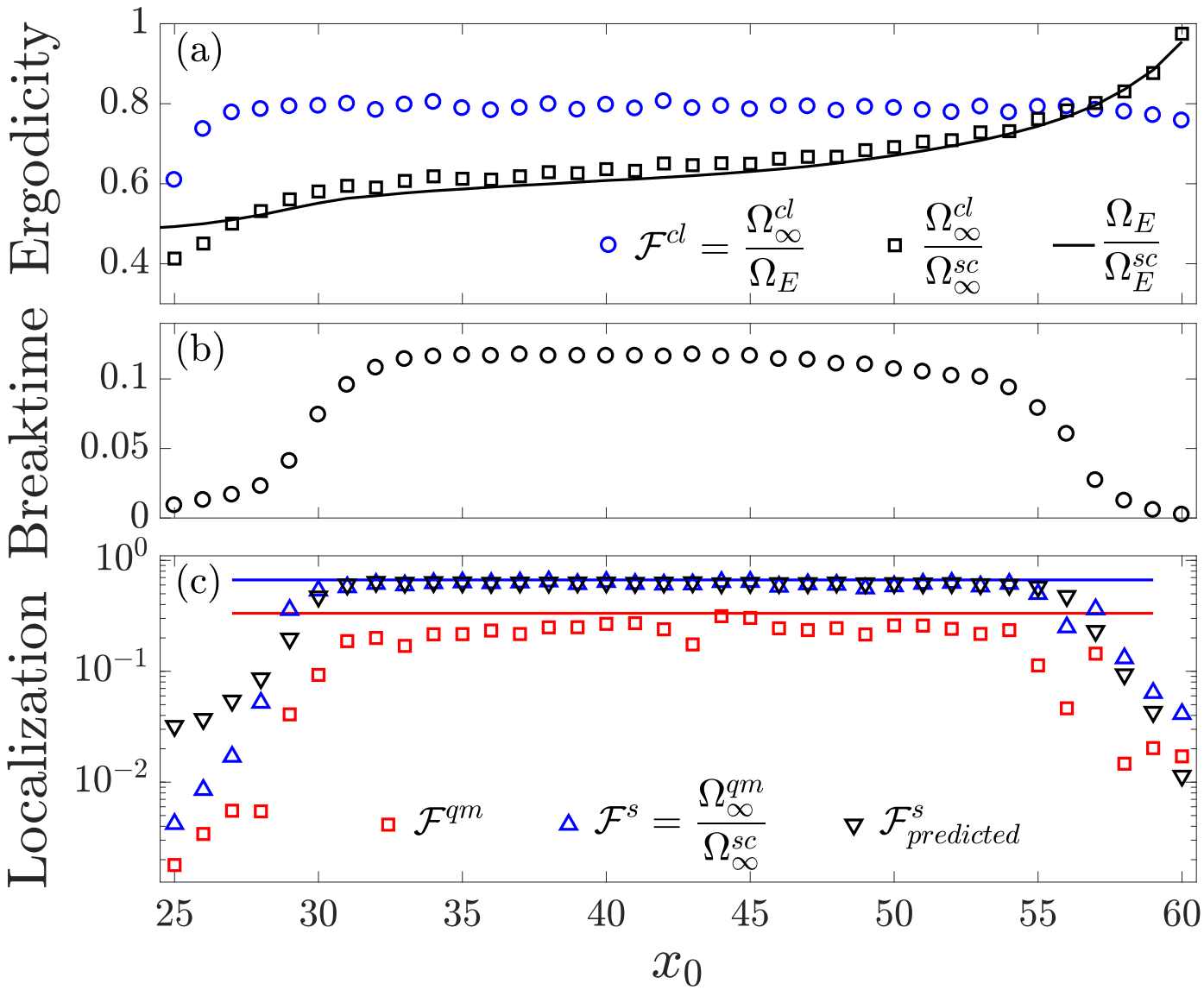


$$\mathcal{F}^{\rm s} \equiv \frac{\Omega_\infty^{\rm qm}}{\Omega_\infty^{\rm sc}}$$

Prediction:

$$\Omega_\infty^{\rm qm} \approx \mathcal{F}_{\rm erg}^{\rm qm}\, \Omega_{t^*}^{\rm sc}$$

Localization measures



$$\mathcal{F}^{\rm cl} \equiv \frac{\Omega_\infty^{\rm cl}}{\Omega_E}$$

$$\mathcal{F}^{\rm qm} \equiv \frac{\mathcal{N}_\infty^{\rm qm}}{\mathcal{N}_E}$$

$$\mathcal{F}^{\rm s} \equiv \frac{\Omega_\infty^{\rm qm}}{\Omega_\infty^{\rm sc}}$$

For ergodic system

$$\Omega_\infty^{\rm cl} \rightsquigarrow \Omega_E$$

$$\Omega_\infty^{\rm sc} \rightsquigarrow \sqrt{[\mathcal{N}_E(r_0)]^2 + \Omega_E^2}$$

The "classical exploration" notion of random walk

Spreading (semiclassical or quantum):

$$\Omega_t^{\mathrm{sc/qm}} \quad = \quad \left\{ \sum_r \left[P_t(r|r_0) \right]^2 \right\}^{-1} \quad \equiv \quad \mathrm{PN}\left\{ [r_0], t \right\}$$

Classical exploration:

$$\Omega_t^{\mathrm{cl}} \quad = \quad \mathrm{PN}\left\{ r_0, [0, t] \right\}$$

Which can be written as

$$\Omega_t^{\mathrm{cl}} \quad \equiv \quad \left\{ \mathrm{trace}\left[\overline{\rho}_{\mathrm{cl}}(t)^2 \right] \right\}^{-1}$$

Example:

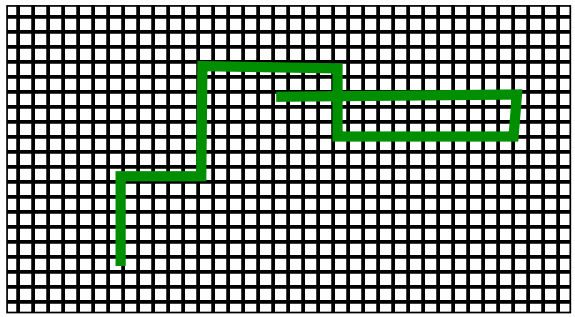
Random walk in 3D

$t = 100$ steps

explored volume ~ 99

spreading radius ~ 10

spreading volume ~ 1000



Classical exploration for random walk on a lattice [Montroll and Weiss 1965]:

$$\Omega_t \sim \sqrt{D_0 t} \qquad \text{for } d = 1$$

$$\Omega_t \sim \frac{v_0 t}{\log(t)} \qquad \text{for } d = 2$$

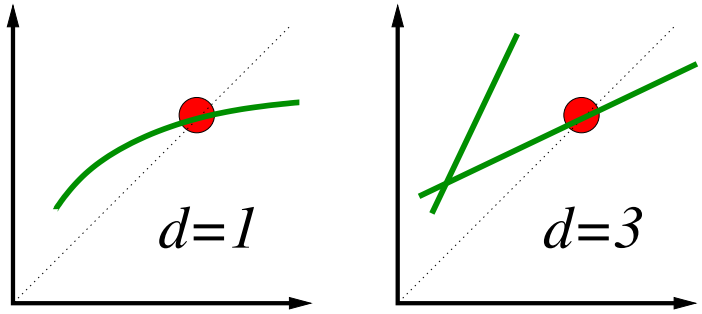
$$\Omega_t \sim v_0 t \qquad \text{for } d > 2$$

The breaktime concept

- Stationary view of strong localization: interference of trajectories.
- Scaling theory of localization: the importance of dimensionality.
- Dynamical view of strong localization: breakdown of quantum-classical correspondence.

$$t_H[\Omega] \;=\; \frac{2\pi}{\Delta_0} \;\propto\; \Omega$$

$$t \;\ll\; t_H[\Omega_t] \quad \rightsquigarrow \quad t^*$$



$$\Omega_t \;=\; \sqrt{D_0 t} \qquad \text{for } d=1 \quad \rightsquigarrow \quad \text{always localization}$$

$$\Omega_t \;=\; c_0 + v_0 t \qquad \text{for } d>2 \quad \rightsquigarrow \quad \text{mobility edge}$$

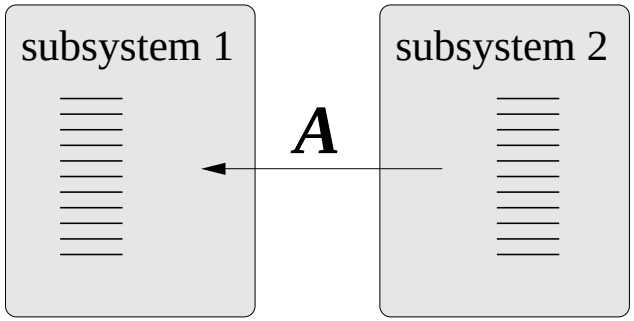
For diffusion in 1D we get $\xi = gD$, where g is the local DOS.

Chirikov, Izrailev, Shepelyansky [SovSciRevC 1981]; Shepelyansky [PhysicaD 1987];
Dittrich, *Spectral statistics for 1D disordered systems* [Phys Rep 1996];
DC, *Periodic Orbits Breaktime and Localization* [JPA 1998].

Manifestation of localization in thermalization?

$$\frac{\partial f(\varepsilon)}{\partial t} = \frac{\partial}{\partial \varepsilon}\left(g(\varepsilon)D(\varepsilon)\frac{\partial}{\partial \varepsilon}\left(\frac{1}{g(\varepsilon)}f(\varepsilon)\right)\right)$$

$g(\varepsilon)$ - local density of states



Rate of energy transfer [FPE version]:

$$A(\varepsilon) = \partial_\varepsilon D + (\beta_1 - \beta_2)D$$

For canonical preparation:

$$\langle A(\varepsilon)\rangle = \left(\frac{1}{T_1} - \frac{1}{T_2}\right)\langle D(\varepsilon)\rangle$$

Here we considered a Bose-Hubbard system where the diffusion is in x

$x =$ the occupation of subsystem 1

$N - x =$ the occupation of subsystem 2

Hurowitz, DC (EPL 2011) - MEQ version

Tikhonenkov, Vardi, Anglin, DC (PRL 2013)] - FPE version

Bunin, Kafri (JPA 2013) - NFT version

Khripkov, Vardi, DC (NJP 2015) - Resistor network calculation of $D(\varepsilon)$

Question: Do we have $\xi = g(\varepsilon)D(\varepsilon)$?

Phase space formulation of the QCC condition

We propose a generalized QCC condition for the purpose of breaktime determination:

Rough version:
$$t \quad < \quad \left[\frac{\Omega_t^{\rm cl}}{\Omega_E}\right] t_H$$

Refined version:
$$\mathcal{N}_t^{\rm sc} \quad < \quad \mathcal{F}_{\rm erg}^{\rm qm} \left[\frac{\mathcal{N}_E}{\Omega_E}\right] \Omega_t^{\rm cl}$$

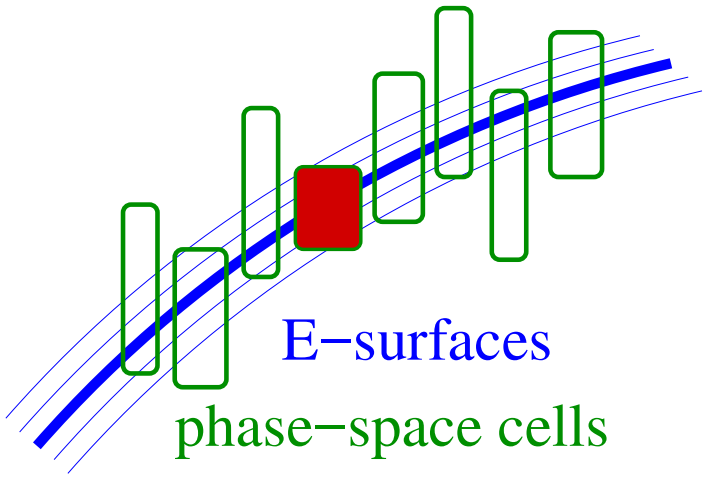
$\mathcal{N}_E$ = total number of states within the energy shell (r_0 dependent)

$\mathcal{F}_{\rm erg}^{\rm qm}$ = filling fraction for a quantum ergodic state, say $= 1/3$

Ω_E = number of cells that intersect an energy-surface

$\Omega_t^{\rm cl}$ = explored phase-space volume during time t (starting at r_0)

$\mathcal{N}_t^{\rm sc} \approx t/t_E$ = semiclassical number of participating-states during time t



$$\left|\langle r_j | E_\alpha \rangle\right|^2$$

It is unavoidable to use in the semiclassical analysis improper Planck cells. Namely, a chaotic eigenstate is represented by a microcanonical energy-shell of thickness $\propto \hbar^d$ and radius $\propto \hbar^0$. For some preparations it is implied that $\mathcal{N}_E \ll \Omega_E$ rather than $\mathcal{N}_E \sim \Omega_E$.

Cartoon: $\Omega_E = 8$, while $\mathcal{N}_E = 5$.

Proper Planck cell: $\Delta Q \Delta P > \hbar/2$ for each coordinate.

The "quantum exploration" notion of Heller

The LDOS: $\varrho(E) = \sum p_\alpha \delta(E - E_\alpha)$

Δ_0 = The mean level spacing

Δ_E = The width of the energy shell

$\mathcal{N}_E$ = States within the energy shell

$\mathcal{N}_\infty$ = Participating states

$\mathcal{F}^{\rm qm}$ = Localization measure

$$p_\alpha = \left|\langle E_\alpha|\psi\rangle\right|^2$$

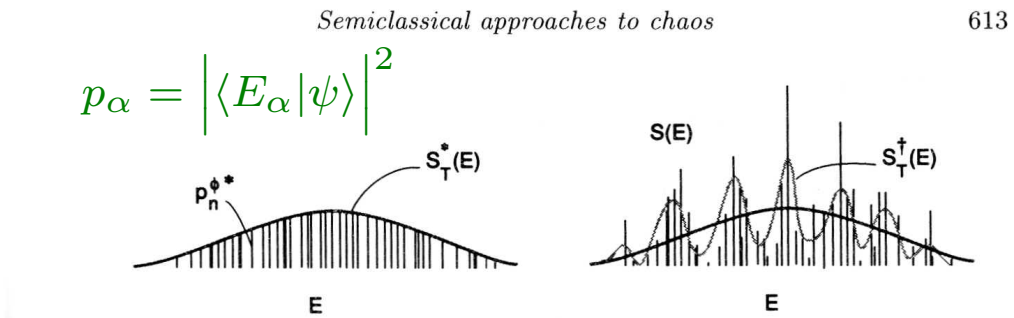


Fig. 38. Ideally ergodic (left) and typically found (right) spectral intensities and envelopes. Both spectra have the same low resolution envelope.

$$t_H = \frac{2\pi}{\Delta_0}, \qquad t_E = \frac{2\pi}{\Delta_E}, \qquad \mathcal{N}_E = \frac{\Delta_E}{\Delta_0}, \qquad \mathcal{N}_\infty = \left[\sum_\alpha p_\alpha^2\right]^{-1}, \qquad \mathcal{F}^{\rm qm} \equiv \frac{\mathcal{N}_\infty}{\mathcal{N}_E}$$

The number of states that participate in the dynamics up to time t is:

$$\mathcal{N}_t \equiv \left\{\text{trace}\left[\overline{\rho}(t)^2\right]\right\}^{-1} = \left[\frac{2}{t}\int_0^t \left(1 - \frac{\tau}{t}\right)\mathcal{P}(\tau)d\tau\right]^{-1} \qquad \overline{\rho}(t) \equiv \frac{1}{t}\int_0^t \rho(t')dt'$$

Short times: $\mathcal{N}_t^{\rm qm} \approx \mathcal{N}_t^{\rm sc} \approx t/t_E$ (based on the classical envelope)

Long times: $\mathcal{N}_t^{\rm cl} \to \mathcal{N}_\infty$ (due to the discreteness of the spectrum)