

Metastability and thermalization in Bose-Hubbard circuits

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Circuits with condensed bosons are the building blocks for quantum Atomtronics. Such circuits will be used as QUBITs (for quantum computation) or as SQUIDs (for sensing of acceleration or gravitation). We study the feasibility and the design considerations for devices that are described by the Bose-Hubbard Hamiltonian. It is essential to realize that the theory involves “Quantum chaos” considerations.

- The Bose-Hubbard Hamiltonian.
- Relevance of chaos for Metastability and Ergodicity.
- Thermalization and quantum localization.

Bosonic Junction

STIRAP through chaos
The Bose Hubbard Hamiltonian

The system consists of $N$ bosons in $M$ sites. Later we add a gauge-field $\Phi$.

$$\mathcal{H}_{\text{BHH}} = \frac{U}{2} \sum_{j=1}^{M} a_j^\dagger a_j^\dagger a_j a_j - \frac{K}{2} \sum_{j=1}^{M} \left( a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1} a_j + a_j^\dagger a_{j+1} a_j^\dagger + a_j^\dagger a_j a_{j+1}^\dagger + a_j^\dagger a_j a_{j+1} + a_{j+1}^\dagger a_j a_j^\dagger + a_{j+1}^\dagger a_{j+1} a_j + a_{j+1} a_j a_j^\dagger + a_{j+1} a_j a_j a_{j+1}^\dagger \right)$$

$$u \equiv \frac{NU}{K} \quad \text{[classical, stability, superfluidity, self-trapping]}$$

$$\gamma \equiv \frac{Mu}{N^2} \quad \text{[quantum, Mott-regime]}$$

Dimer ($M=2$): Minimal BHH; Bosonic Josephson junction; Pendulum physics [1].
Driven dimer: Landau-Zener dynamics [2], Kapitza effect [3], Zeno effect [4], Standard-map physics [5].
Trimer ($M=3$): Minimal model for low-dimensional chaos; Many body STIRAP [6].
Triangular trimer ($M=3$): Minimal model with topology, Superfluidity [7], Stirring [7].
Larger rings ($M>3$) High-dimensional chaos; web of non-linear resonances [7].
Coupled subsystems ($M>3$): Minimal model for Thermalization [8,9].

Quantum Chaos perspective on Metastability and Ergodicity

Stability of flow-states (I):
- Landau stability of flow-states ("Landau criterion")
- Bogoliubov perspective of dynamical stability
- KAM perspective of dynamical stability

Stability of flow-states (II):
- Considering high dimensional chaos ($M > 3$).
- Web of non-linear resonances.
- Irrelevance of the familiar Beliaev and Landau damping terms.
- Analysis of the quench scenario.

Coherent Rabi oscillations:
- The hallmark of coherence is Rabi oscillation between flow-states.
- Ohmic-bath perspective $\eta = (\pi/\sqrt{\gamma})$
- Feasibility of Rabi oscillation for $M < 6$ devices.
- Feasibility of chaos-assisted Rabi oscillation.

Thermalization:
- Spreading in phase space is similar to Percolation.
- Resistor-Network calculation of the diffusion coefficient.
- Observing regions with Semiclassical Localization.
- Observing regions with Dynamical Localization.
Bose Hubbard Ring

In the rotating reference frame we have a Coriolis force, which is like magnetic field $B = 2m\Omega$. Hence is is like having flux

$$\mathcal{H} = \sum_{j=1}^{M} \left[ \frac{U}{2} a_j^\dagger a_j^\dagger a_j a_j - \frac{K}{2} \left( e^{i(\Phi/L)} a_{j+1}^\dagger a_j + e^{-i(\Phi/L)} a_j^\dagger a_{j+1} \right) \right]$$

$$\mathcal{H} = \sum_{k} \epsilon_k(\Phi) b_k^\dagger b_k + \frac{U}{2L} \sum b_{k_4}^\dagger b_{k_3}^\dagger b_{k_2} b_{k_1}$$

For $L=3$ sites, using $b_k = \sqrt{n_k} e^{i\varphi_k}$, and $M = (n_1-n_2)/2$, and $n = (n_1+n_2)/2$

$$\mathcal{H}(\varphi, n; \phi, M) = \mathcal{H}^{(0)}(\varphi, n; M) + \left[ \mathcal{H}^{(+)} + \mathcal{H}^{(-)} \right]$$

$$\mathcal{H}^{(0)}(\varphi, n; M) = \mathcal{E}n + \mathcal{E}_\perp M - \frac{U}{3} M^2 + \frac{2U}{3} (N-2n) \left[ \frac{3}{4} n + \sqrt{n^2 - M^2} \cos(\varphi) \right]$$

$$\mathcal{H}^{(\pm)} = \frac{2U}{3} \sqrt{(N-2n)(n\pm M)(n\mp M)} \cos \left( \frac{3\varphi \mp \varphi}{2} \right)$$
Flow-state stability regime diagram

The $I$ of the maximum current state is imaged as a function of $(\Phi, u)$

- solid lines = energetic stability borders (Landau)
- dashed lines = dynamical stability borders (Bogoliubov)

The traditional paradigm associates flow-states with stationary fixed-points in phase space. Consequently the Landau criterion, and more generally the Bogoliubov linear-stability-analysis, are used to determine the viability of superfluidity.

Arwas, Vardi, DC [Scientific Reports 2015]
Monodromy, Chaos, and Metastability of superflow

The swap transition

$M = \frac{(n_1 - n_2)}{2}$, constant of motion.

$n = \frac{(n_1 + n_2)}{2}$, conjugate phase $\varphi$.

Section energy $E = E[SP]$  
Section coordinates $(\varphi, n)$  

Regular separatrix $E = E_x(M)$

Arwas, DC [arXiv 2018]
SQUID-like geometry (weak link or barrier)

Coherent Rabi oscillations ($\gamma \ll 1$):
- The hallmark of coherence is Rabi oscillation between flow-states.
- Ohmic-bath perspective $\eta = (\pi/\sqrt{\gamma})$
- Feasibility of Rabi oscillation for $M < 6$ devices.
- Feasibility of of chaos-assisted Rabi oscillation.

Arwas, DC [New Journal of Physics 2016]

Resonant persistent currents ($\gamma \sim 1$):
- Without optical lattice - non monotonic dependence on $u$
- With optical lattice - Mott transition; Quantum resonances

$$\text{max} [I(\Phi)] = 2J \frac{N}{M} \alpha(w, u)$$

Arwas, DC, Hekking, Minguzzi [PRA 2017] - Editors’ Suggestion
Strong localization and the exploration of phase space for quantum thermalization


Short version of talk prepared for Celebrating 70th birthday of Rick Heller (Cuernavaca 2017)
The minimal model for thermalization

The FPE description makes sense if the sub-systems are chaotic.

Minimal model for a chaotic sub-system: BHH trimer.

Minimal model for thermalization: BHH trimer + monomer

\[ N = 60 \] = number of particles
\[ x \] = occupation of the trimer
\[ N - x \] = occupation of the monomer
\[ f(x) \] = probability distribution

\[ \frac{\partial f(x)}{\partial t} = \frac{\partial}{\partial x} \left[ g(x) D(x) \frac{\partial}{\partial x} \left( \frac{f(x)}{g(x)} \right) \right] \]
Dynamical localization

Here we start the simulation with $x_0 = 60$, meaning that initially all the particles are in the trimer. We plot the saturation profile $P_\infty(x,\varepsilon)$.
The LDOS of the initial preparation

We observed dynamical localization if we start with $x_0 < 30$ or with $x_0 > 55$.

Let us look on the LDOS of representative preparations:

Localization does not always manifests as sparsity. It depends on the geometry of the $r = (x, \epsilon)$ space.
Localization of the eigenstates

Upper panel: The unperturbed states $|r\rangle = |x, \varepsilon\rangle$, color-coded according to $\mathcal{F}^{qm}$, arranged by $(x, \varepsilon)$.

Lower panel: The perturbed states $|E_\alpha\rangle$, color-coded according to $\text{var}(x)_\alpha$, sorted by $(\langle x \rangle_\alpha, \langle \varepsilon \rangle_\alpha)$. 
Participation, Exploration, and Breaktime

We display $\Omega^{cl}(t)$, and $\Omega^{sc}(t)$, and $\Omega^{qm}(t)$, and $\mathcal{N}(t)$.

The breaktime is determined by the intersection of the scaled $\mathcal{N}^{sc}(t)$ with the scaled $\Omega^{cl}(t)$.

$\mathcal{F}^s \equiv \frac{\Omega^{qm}_\infty}{\Omega^{sc}_\infty}$

Prediction:

$\Omega^{qm}_\infty \approx \mathcal{F}^{qm}_{\text{erg}} \Omega^{sc}_t$
Localization measures

\[ F^{cl} = \frac{\Omega^{cl}_\infty}{\Omega_E} \]

\[ F^{qm} = \frac{N^{qm}_\infty}{N_E} \]

\[ F^s = \frac{\Omega^{qm}_\infty}{\Omega^{sc}_\infty} \]

For ergodic system

\[ \Omega^{cl}_\infty \sim \Omega_E \]

\[ \Omega^{sc}_\infty \sim \sqrt{[N_E(r_0)]^2 + \Omega_E^2} \]
The "classical exploration" notion of random walk

Spreading (semiclassical or quantum):
\[ \Omega_{sc/qm}^t = \left\{ \sum_r [P_t(r|r_0)]^2 \right\}^{-1} \equiv \text{PN \{[r_0], t\}} \]

Classical exploration:
\[ \Omega_{cl}^t = \text{PN \{r_0, [0, t]\}} \]

Which can be written as
\[ \Omega_{cl}^t \equiv \left\{ \text{trace } [\bar{\rho}_{cl}(t)^2] \right\}^{-1} \]

Classical exploration for random walk on a lattice [Montroll and Weiss 1965]:
\[ \Omega_t \sim \sqrt{D_0 t} \quad \text{for } d = 1 \]
\[ \Omega_t \sim \frac{v_0 t}{\log(t)} \quad \text{for } d = 2 \]
\[ \Omega_t \sim v_0 t \quad \text{for } d > 2 \]

Example:
Random walk in 3D
\( t = 100 \) steps
explored volume \( \sim 99 \)
spreading radius \( \sim 10 \)
spreading volume \( \sim 1000 \)
The breaktime concept

- Stationary view of strong localization: interference of trajectories.
- Scaling theory of localization: the importance of dimensionality.
- Dynamical view of strong localization: breakdown of quantum-classical correspondence.

\[ t_H[\Omega] = \frac{2\pi}{\Delta_0} \propto \Omega \]

\[ t \ll t_H[\Omega_t] \sim t^* \]

\[ \Omega_t = \sqrt{D_0t} \quad \text{for } d = 1 \sim \text{always localization} \]

\[ \Omega_t = c_0 + v_0t \quad \text{for } d > 2 \sim \text{mobility edge} \]

For diffusion in 1D we get \( \xi = gD \), where \( g \) is the local DOS.

Dittrich, *Spectral statistics for 1D disordered systems* [Phys Rep 1996];
DC, *Periodic Orbits Breaktime and Localization* [JPA 1998].
Manifestation of localization in thermalization?

\[
\frac{\partial f(\varepsilon)}{\partial t} = \frac{\partial}{\partial \varepsilon} \left(g(\varepsilon)D(\varepsilon) \frac{\partial}{\partial \varepsilon} \left(\frac{1}{g(\varepsilon)} f(\varepsilon)\right)\right)
\]

\(g(\varepsilon)\) - local density of states

Rate of energy transfer [FPE version]:

\[A(\varepsilon) = \partial_\varepsilon D + (\beta_1 - \beta_2)D\]

For canonical preparation:

\[
\langle A(\varepsilon) \rangle = \left(\frac{1}{T_1} - \frac{1}{T_2}\right) \langle D(\varepsilon) \rangle
\]

Here we considered a Bose-Hubbard system where the diffusion is in \(x\)

\(x = \) the occupation of subsystem 1

\(N-x = \) the occupation of subsystem 2

**Question:** Do we have \(\xi = g(\varepsilon)D(\varepsilon)\) ?
Phase space formulation of the QCC condition

We propose a generalized QCC condition for the purpose of breaktime determination:

Rough version: \[ t < \left[ \frac{\Omega_{cl}^t}{\Omega_E} \right] t_H \]

Refined version: \[ N_{sc}^t < F_{erg} \left[ \frac{N_E}{\Omega_E} \right] \Omega_{cl}^t \]

\( N_E \) = total number of states within the energy shell \((r_0\) dependent)

\( F_{erg} \) = filling fraction for a quantum ergodic state, say \( = 1/3 \)

\( \Omega_E \) = number of cells that intersect an energy-surface

\( \Omega_{cl}^t \) = explored phase-space volume during time \( t \) (starting at \( r_0 \))

\( N_{sc}^t \approx t/t_E \) = semiclassical number of participating-states during time \( t \)

\[ |\langle r_j | E_\alpha \rangle|^2 \]

It is unavoidable to use in the semiclassical analysis improper Planck cells. Namely, a chaotic eigenstate is represented by a microcanonical energy-shell of thickness \( \propto \hbar^d \) and radius \( \propto \hbar^0 \). For some preparations it is implied that \( N_E \ll \Omega_E \) rather than \( N_E \sim \Omega_E \).

Cartoon: \( \Omega_E = 8 \), while \( N_E = 5 \).

Proper Planck cell: \( \Delta Q \Delta P > \hbar/2 \) for each coordinate.
The "quantum exploration" notion of Heller

The LDOS: \( \varrho(E) = \sum p_\alpha \delta(E - E_\alpha) \)

\( \Delta_0 \) = The mean level spacing
\( \Delta_E \) = The width of the energy shell
\( N_E \) = States within the energy shell
\( N_\infty \) = Participating states
\( \mathcal{F}^{qm} \) = Localization measure

\[
t_H = \frac{2\pi}{\Delta_0}, \quad t_E = \frac{2\pi}{\Delta_E}, \quad N_E = \frac{\Delta_E}{\Delta_0}, \quad N_\infty = \left[ \sum_\alpha p_\alpha^2 \right]^{-1}, \quad \mathcal{F}^{qm} \equiv \frac{N_\infty}{N_E}
\]

The number of states that participate in the dynamics up to time \( t \) is:

\( N_t \equiv \left\{ \text{trace} \left[ \overline{\rho}(t)^2 \right] \right\}^{-1} = \left[ \frac{2}{t} \int_0^t \left( 1 - \frac{\tau}{t} \right) \mathcal{P}(\tau) d\tau \right]^{-1} \quad \overline{\rho}(t) \equiv \frac{1}{t} \int_0^t \rho(t') dt'

Short times: \( N_t^{qm} \approx N_t^{sc} \approx \frac{t}{t_E} \) (based on the classical envelope)

Long times: \( N_t^{cl} \to N_\infty \) (due to the discreteness of the spectrum)