Quantum decay into a non-flat continuum

James Aisenberg[1], Itamar Sela[2], Tsampikos Kottos[1], Doron Cohen[2], Alex Elgart[3]
[1]*Department of Physics, Wesleyan University, Middletown, CT 06459, USA*
[2]*Department of Physics, Ben-Gurion University, Beer-Sheva 84105, Israel*
[3]*Department of Mathematics, Virginia Tech, Blacksburg, VA 24061, USA*

We study the decay of a prepared state into non-flat continuum. We find that the survival probability $P(t)$ might exhibit either stretched-exponential or power-law decay, depending on non-universal features of the model. Still there is a universal characteristic time t_0 that does not depend on the functional form. It is only for a flat continuum that we get a robust exponential decay that is insensitive to the nature of the intra-continuum couplings. The analysis highlights the co-existence of perturbative and non-perturbative features in the local density of states, and the non-linear dependence of $1/t_0$ on the strength of the coupling.

The time relaxation of a quantum-mechanical prepared state into a continuum due to some residual interaction is of great interest in many fields of physics. Applications can be found in areas as diverse as nuclear [1], atomic and molecular physics [2] to quantum information [3], solid-state physics [4, 5] and quantum chaos [6]. The most fundamental measure characterizing the time relaxation process is the so-called survival probability $P(t)$, defined as the probability not to decay before time t.

The study of $P(t)$ goes back to the work of Weisskopf and Wigner [7] regarding the decay of a bound state into a flat continuum. They have found that $P(t)$ follows an exponential decay $P(t) = \exp(-t/t_0)$, with rate $1/t_0$ given by the Fermi Golden Rule. In a bosonic second quantized language this is the decay rate of the site occupation $\hat{n}$, and can be formulated as a *quantum dissipation* problem with a so called *Ohmic* bath. Then $1/t_0$ can be reinterpreted as the friction coefficient that characterizes the damped motion of the generalized coordinate $\hat{n}$. Optionally $P(t)$ could be related to dephasing, and in this case t_0 is reinterpreted as the coherence time, as in Landau's Fermi liquid theory.

Following Wigner, many of the later studies have adopted Random Matrix Theory (RMT) modeling [8, 9] for the investigation of $P(t)$, highlighting the importance of the statistical properties of the spectrum [10]. Notably in the context of a many-particle system, one should understand the role of the whole hierarchy of states and associated couplings, ranging from the single-particle levels to the exponentially dense spectrum of complicated many-particle excitations [11], e.g., leading to a stretched exponential decay $P(t) \sim \exp(-\sqrt{t})$. Non-uniform couplings also emerge upon quantization of chaotic systems where non-universal (semiclassical) features dictate the band-structure of the perturbation, leading to a highly non-linear energy spreading [12]. Despite all the mounting interest in such circumstances, a theoretical investigation of the time relaxation for prototypical RMT models is still missing, and also the general (not model specific) perspective is lacking.

Scope.– In this Letter, we explore a general class of RMT models where the initial state decays into a non-flat continuum. In the language of quantum dissipation studies, this means that we are dealing with a non-Ohmic model. We show that the survival probability $P(t) = g(t/t_0)$ is characterized by a generalized Wigner decay time t_0 that depends in a non-linear way on the strength of the coupling. We also establish that the scaling function g has distinct universal and non-universal features. It is only for the flat continuum of the traditional Wigner model, that we get a robust exponential decay that is insensitive to the nature of the intra-continuum couplings. In addition to $P(t)$ we investigate other characteristics of the evolving wavepacket like the variance $\Delta E_{\text{sprd}}(t)$ and the 50% probability width $\Delta E_{\text{core}}(t)$ of the energy distribution, that describe universal and non-universal features of its decaying component.

Modeling.– We analyze two models whose dynamics is generated by a RMT Hamiltonian $\mathcal{H} = \mathcal{H}_0 + V$, with $\mathcal{H}_0 = \text{diag}\{E_k\}$ and $k \in \mathbb{Z}$. The first one is the Friedrichs model (FM) [13], where the distinguished energy level E_0 is coupled to the rest of the levels $E_{k\neq 0}$ by a rank two matrix. The second one is the generalized Wigner model (WM) [14], where the perturbation V does not discriminate between the levels, and is given by a banded random matrix. In both cases the system is prepared initially in the eigenstate corresponding to E_0, and the coupling to the other levels is characterized by the spectral function

$$\begin{aligned}\tilde{C}(\omega) &= -\text{Im}\left\langle E_0\middle|V\left(E_0+\omega-\tilde{\mathcal{H}}_0+i0\right)^{-1}V\middle|E_0\right\rangle \\ &= \sum_{n\neq 0}|V_{n,0}|^2 2\pi\delta(\omega-(E_n-E_0)) \qquad (1)\\ &= 2\pi\epsilon^2|\omega|^{s-1}\mathrm{e}^{-|\omega|/\omega_c} \qquad (2)\end{aligned}$$

where $\tilde{\mathcal{H}}_0$ is obtained from $\mathcal{H}_0$ by removing the 0^{th} row and column. An RMT averaging over realizations is implicit in the WM case. By integrating Eq.(2) over ω we see that the perturbation V is bounded for the FM provided $s > 0$. The $s = 1$ case is what we refer to as the flat continuum, for which it is well known that both models leads to the same exponential decay for the survival probability. For $s > 2$ the effect of the continuum can be handled using 1^{st} order perturbation theory. Our focus below is in the $0 < s < 2$ regime. We consider the non-Ohmic case ($s \neq 1$) for which a non-linear version of the Wigner decay problem is encountered.

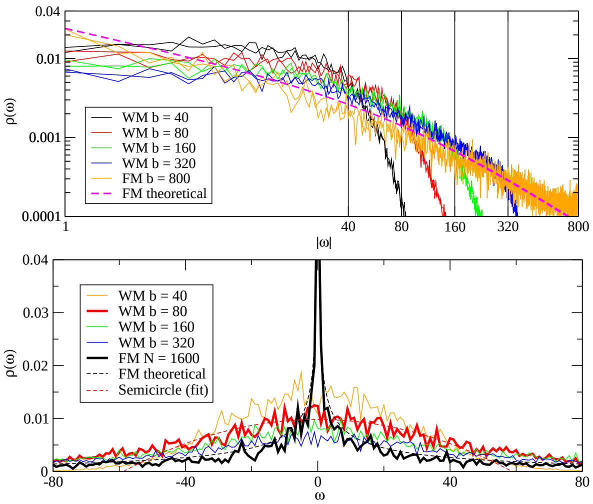

FIG. 1: LDoS for the FM and for the WM via direct diagonalization of 1600×1600 matrices with $s = 1.5$ and $\epsilon = 1.44$. In the FM case $b{=}N/2$. *Upper panel:* The log-log scale emphasizes the universality of the tails up to the cutoff ω_c. *Lower panel:* The log-linear scale emphasizes the difference in the non-universal core component.

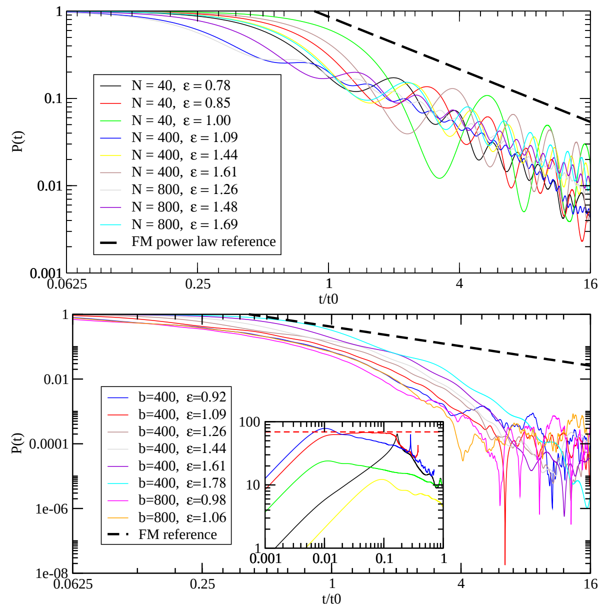

FIG. 2: The survival probability $P(t)$ for the FM (*top*) and for the WM (*bottom*). The time is scaled with respect to t_0. For all curves in the main panels $\varrho = 1$ and $s = 1.5$. The WM simulations are presented in log-log scale in order to contrast it with the FM results. *Inset:* further analysis displaying $Y = -\ln[P(t)]/t$ vs $X = t$ in a log-log plot for representative runs with $(s, \epsilon) =$ black$(0.30, 4.43)$, red$(1.00, 3.24)$, green$(1.25, 1.14)$, blue$(1.50, 1.09)$, yellow$(1.75, 0.50)$, showing that the decay in the WM case is described by a stretched exponential. The red bold dashed line has zero slope, corresponding to simple exponential decay for $s{=}1$.

In the numerical simulations we integrate the Schrödinger equation for the amplitudes $c_n(t) = \langle n|\psi(t)\rangle$ starting with the initial condition $c_n = \delta_{n,0}$ at $t{=}0$. We use units such that $\hbar{=}1$, the density of states is $\varrho{=}1$, and $E_0{=}0$, and we assume a sharp bandwidth $b = \varrho\omega_c$. The integration is done using the self-expanding algorithm of [16] to eliminate finite-size effects, adding $10b$ sites to each edge of the energy lattice whenever the probability of finding the 'particle' at the edge sites exceeds 10^{-12}. The spreading profile is described by the distribution $P_t(n) = \overline{|c_n(t)|^2}$, where the averaging is over realizations of the Hamiltonian. The survival probability is $P(t) = P_t(0)$. The energy spreading is characterized by the standard deviation $\Delta E_{\rm sprd}(t) = [\sum_n (E_n{-}E_0)^2 P_t(n)]^{1/2}$, by the median $E_{50\%} = E_0$, and also by the $E_{25\%}$ and $E_{75\%}$ percentiles. The width of the core component is defined as $\Delta E_{\rm core}(t) = E_{75\%} - E_{25\%}$.

Time Scales.– A dimensional analysis predicts the existence of 3 relevant time scales: The Heisenberg time $t_{\rm H}$ which is related to the density of states ϱ; the semiclassical (correlation) time which is related to the bandwidth ω_c; and the generalized Wigner times t_0 which is related to the perturbation strength:

$$t_{\rm H} \;=\; 2\pi\varrho, \qquad t_c = 2\pi/\omega_c \tag{3}$$

$$t_0 \;=\; \left(\frac{2\pi\epsilon^2}{\boldsymbol{\Gamma}(3{-}s)\sin(s\pi/2)}\right)^{-1/(2-s)} \equiv \frac{1}{\gamma_0} \tag{4}$$

where $\boldsymbol{\Gamma}$ is the Gamma function, and the numerical prefactor will be derived later. We shall refer to ϱ^{-1} and to ω_c as the infrared and ultraviolate cutoffs of the theory. Our main interest is in the *continuum* limit. Assuming further that ω_c is irrelevant, one expects a decay that is determined by the generalized Wigner time t_0.

The LDoS.– Before analyzing the dynamics, it is important to understand the behavior of the Local Density of States (LDoS) [14], which is defined as follows:

$$\rho(\omega) = \sum_n |\langle n|E_0\rangle|^2 \delta(\omega - (E_n{-}E_0)) \tag{5}$$

where $|n\rangle$ are the eigenstates of the full Hamiltonian $\mathcal{H}$. An RMT averaging over realizations is implied in the WM case. Once the LDoS is computed, we can use it to calculate the survival probability:

$$P(t) \;\equiv\; \left|\langle 0|\mathrm{e}^{-i\mathcal{H}t}|0\rangle\right|^2 = \left|\mathrm{FT}\Big[2\pi\rho(\omega)\Big]\right|^2 \tag{6}$$

where FT denotes the Fourier transform. For flat bandprofile $(s = 1)$, the LDoS $\rho(\omega) = (1/\gamma_0)f(\omega/\gamma_0)$ is a Lorentzian $f(x) = (1/\pi)/(1+x^2)$ [14], leading to a Wigner exponential decay for $P(t)$. For $(s \neq 1)$ the ensuing analysis shows that $\rho(\omega)$ has a core-tail structure [12, 15, 16]. Namely, it consists of two distinct regions $x \gg 1$ and $x < 1$ that reflect universal and non-universal features of the problem respectively. The tails $x \gg 1$ can be calculated using 1st order perturbation theory leading to $f(x) \propto 1/x^{3-s}$. This component we regard as universal. The core $(x < 1)$ reflects the non-perturbative mixing of the levels, and it is non-universal. In the WM case we argue that for $x \ll 1$ it is semicircle-like, while for FM we have a singular behaviour $f(x) \sim x^{1-s}$. These findings are supported by the numerical calculations of Fig.1, and are reflected in the behavior of $P(t)$ as confirmed by the numerical simulations of Fig.2.

Friedrichs model.– Using the Schur complement technique, we can calculate analytically the LDoS for the FM. The Green function is $G_{00}(\omega) = \{[\omega - \Delta(\omega)] + i(\Gamma(\omega)/2)\}^{-1}$ with the standard notations $\Gamma(\omega) = \tilde{C}(\omega)$,

$$\begin{aligned} \Delta(\omega) &= -\mathrm{Re}\left\langle E_0 \middle| V \left(E_0 + \omega - \tilde{\mathcal{H}}_0 + i0\right)^{-1} V \middle| E_0 \right\rangle \\ &= \int_{-\infty}^{+\infty} \frac{\widetilde{C}(\omega')}{\omega - \omega'} \mathrm{d}\omega' \qquad (7) \\ &= \epsilon^2 \pi \cot(s\pi/2) \, |\omega|^{s-1} \mathrm{sgn}(\omega) \qquad (8) \end{aligned}$$

In the last line we performed the limit $\omega_c \to \infty$ (with the limiting expression converging in distribution). The LDoS of Eq.(5) is $-(1/\pi)\mathrm{Im}\,[G_{00}(\omega)]$ leading to

$$\rho(\omega) = \frac{1}{\pi} \frac{\Gamma(\omega)/2}{(\omega - \Delta(\omega))^2 + (\Gamma(\omega)/2)^2} \qquad (9)$$

Wigner Model. – The analysis of the LDOS for the WM can be carried out approximately using a combination of heuristic and formal methods. Our numerical results reported in Fig. 1 confirm that the LDoS has 1st order tails $|V_{n,0}/(E_n - E_0)|^2$ that co-exist with the core (non-perturbative) component. We can determine the border γ_0 between the core and the tail simply from the requirement $p_0 \sim 1$ where $p_0 = \int_{\gamma_0}^{\infty} \frac{\tilde{C}(\omega)}{\omega^2} d\omega$. For $s > 2$ we would have for sufficiently small coupling $p_0 \ll 1$ even if we took the limit $\gamma_0 \to 0$. This means that 1st order perturbation theory is valid as a global approximation. But for $s < 2$ the above equation implies breakdown of 1st order perturbation theory at $\gamma_0 \sim \epsilon^{2/(2-s)}$. In the tails $\mathcal{H}_0$ dominates over V, while in the core V dominates. Therefore, as far as the core in concerned, it makes sense to diagonalize V with an effective cutoff γ_0. Following [17], the result for the LDoS lineshape should be semicircle-like, with width given by the expression $\Delta E_{\text{sc}} = \left[\int_0^{\gamma_0} \tilde{C}(\omega) d\omega\right]^{1/2}$, where here we suggest to use the *effective* bandwidth γ_0 instead of the actual bandwidth ω_c. The outcome of the integral is $\Delta E_{\text{sc}} \sim \gamma_0$, demonstrating that our procedure is *self-consistent*: the core has the same width as implied by the breakdown of 1st order perturbation theory. We note that within this perspective the $s = 1$ Lorentzian is regarded as composed of a semicircle-like core and 1st order tails.

The survival probability.– In the WM case the function $\rho(\omega)$ is *smooth* with power law tails $\sim 1/\omega^{1+\alpha}$ where $\alpha = 2-s$. Thanks to the smoothness the FT does not have power law tails but is exponential-like. The similarity with the α-stable Levy distribution suggests that $P(t)$ would be similar to a stretched exponential,

$$P(t) \approx \exp[-(t/t_0)^{2-s}] \qquad (10)$$

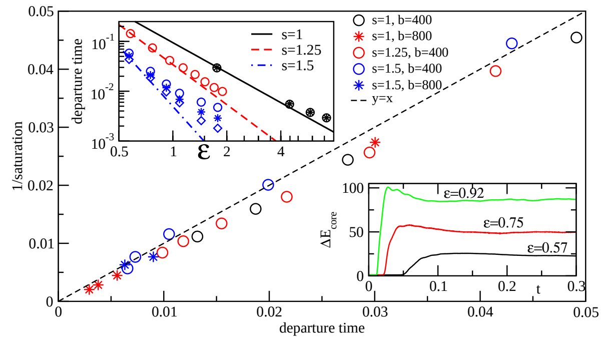

FIG. 3: *Lower Inset:* Examples for the time evolution of ΔE_{core} for s=1.5 and b=800 in the WM case. *Main panel:* The extracted departure time versus the extracted inverse saturation value. This scatter diagram demonstrates the validity of one parameter scaling. *Upper Inset:* The extracted departure time versus the perturbation strength ϵ. The theoretical (dashed) lines are based on the t_0 estimate of Eq.(4). The deviations of the departure time from the theoretical expectation diminish in the limit $\omega_c \to \infty$. The $\circ$ corresponds to $b = 400$, the $\star$ to $b = 800$, and the $\diamond$ to $b = 1600$.

The expression for t_0 in Eq.(4) is implied by the observation that $1/|\omega|^{1+\alpha}$ tails are FT associated with a discontinuity $-C|t|^{\alpha}$, where $C = [2\mathbf{\Gamma}(1+\alpha)\sin(\alpha\pi/2)]^{-1}$.

In the FM case we observe that the function $\rho(\omega)$ in (9) features a crossover from ω^{1-s} for $|\omega| \ll \gamma_0$ to $\Gamma(\omega)/\omega^2$ for $|\omega| \gg \gamma_0$. Thus, compared with the WM case, the FT has an additional contribution from the singularity at ω=0, and consequently by the Tauberian theorem [18], the survival amplitude has a non-exponential decay, that for sufficiently long time is described by a power law:

$$P(t) \;\; = \;\; \left| \frac{2\sin((s-1)\pi)}{(2-s)\pi \;(t/t_0)^{2-s}} \right|^2 \qquad (11)$$

The long time behavior is dominated by the non-smooth feature of the core, and not by the tails. Comparing the exponential and the power-law we can find the expression for the crossover time t_0' that becomes $t_0' \sim [\log|s-1|]^{1/(2-s)} t_0 \gg t_0$ close to the Ohmic limit (s∼1). For $s = 1$ only the exponential decay survives. We emphasize that the cutoff independent behavior appears only after a short transient, i.e. for $t > t_c$. For completeness we note that for the FM with s=2 we get $P(t) \approx |1 + \log(t/t_0)|^2$, that holds for $t_c < t < t_c \mathrm{e}^{1/(2\epsilon^2)}$, while for s>2 there is an immediate but only partial decay that saturates at the value $P(t) = |1-p_0|^2$ for $t > t_c$.

Spreading.– The distinction between core and tail components becomes physically transparent once we analyze the time dependent energy spreading of the wavepacket. Using the same time dependent analysis as in the $s = 1$ case of Ref.[16], it is straightforward to show that the rise of $\Delta E_{\text{core}}(t)$ is at $t \sim t_0$, and its saturation value is $\sim \gamma_0$. Thus ΔE_{core} should exhibit one parameter scaling with respect to t_0. In Fig.3 we present the results of the numerical analysis. Our data, indicate that the

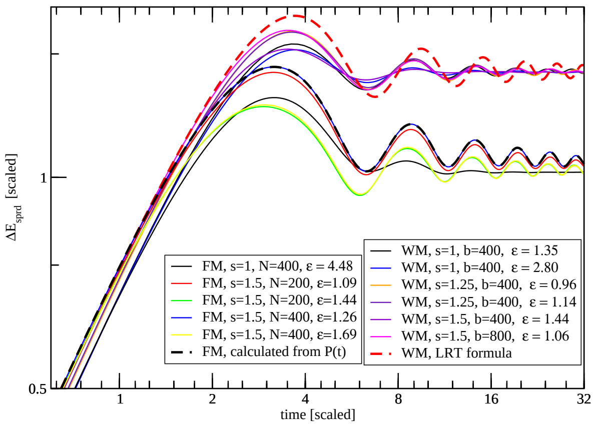

FIG. 4: Scaled spread $\Delta E_{\rm sprd}/(\epsilon^2\omega_c^s/s)$ versus scaled time $\omega_c t$ for the FM and the WM. The theoretical predictions Eq.(12) and Eq.(13) are plotted for comparison.

expected one-parameter scaling is obeyed. We have verified that the deviations from the expected ϵ dependence (for large ϵ) are an artifact due to having finite (rather then infinite) bandwidth.

The physics of $\Delta E_{\rm sprd}$ is quite different and not necessarily universal, because the second moment is dominated by the tails, and hence likely to depend on the cutoff ω_c and diverge in the limit $\omega_c \to \infty$. Indeed in the WM case we can use the Linear response result of [12, 16]

$$\Delta E_{\rm sprd}(t) = \left[2\Big(C(0) - C(t)\Big)\right]^{1/2} \qquad (12)$$

where $C(t)$ is the inverse FT of $\tilde{C}(\omega)$. This gives the saturated value $\omega_c^s\epsilon^2$ as soon as $t > t_c$. We now turn to the FM case. The solution of the Schrödinger equation for $c_n(t)$ is well known [2], and can be expressed using the real amplitude $c(t)\equiv c_0(t)$. In particular $P(t) = |c(t)|^2$ and also the energy spreading can be computed in a closed form, with the end result

$$\Delta E_{\rm sprd}(t) = \Big[(1+c^2(t))C(0) - \dot{c}(t)^2 + 2c(t)\ddot{c}(t)\Big]^{1/2} \qquad (13)$$

For $t < t_0$ we can use the estimates $c(t) \approx 1$ and $\dot{c}(t) \approx 0$ and $\ddot{c}(t) \approx -C(t)$ to conclude that $\Delta E_{\rm sprd}(t)$ behaves as in Eq.(12). But for $t > t_0$ we get $\Delta E_{\rm sprd}(t) \approx [(1+P(t))C(0)]^{1/2}$, leading to a saturation value smaller by factor $\sqrt{2}$, reflecting the non-stationary decay of the fluctuations as a function of time. More interestingly Eq.(13) contains a cutoff independent term that reflects the universal time scale t_0. The numerical results in Fig.4 confirm the validity of the above expressions. We note that in the FM case the effect of recurrences is more pronounced, because they are better synchronized: all the out-in-out traffic goes exclusively through the initial state.

Summary.– In this work we have compared two models that have the same spectral properties, but still different underlying dynamics. One of them has integrable dynamics (FM) while the other is RMT type (WM). This is complementary to our past work [19] where we have contrasted a physical model with its RMT counterpart. In the non-Ohmic decay problem that we have considered in this Letter a universal generalized Wigner time scale has emerged. It is not this time scale but rather the functional form of the decay that reflects the non-universality. Namely, the survival probability $P(t)$ exhibits a stretched-exponential decay in the WM case, but a power-law decay in the FM case. Only the standard case of flat (Ohmic) bandprofile is fully universal.

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