Non-equilibrium steady state (NESS) of sparse systems

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$\mathcal{H}_{\text{total}} = \{E_n\} - f(t)\{V_{nm}\} + \{W_{nm}\} \cdot \text{Bath}$
Driven system + Bath

\[ H_{\text{total}} = \{E_n\} - f(t)\{V_{nm}\} + \{W_{nm}\} \cdot Bath \]

\[ \varepsilon^2 \equiv \text{Driving intensity} \quad \quad T_B \equiv \text{Bath temperature} \]
Steady state temperature

\[ \frac{d\rho}{dt} = -i[H, \rho] - \frac{\varepsilon^2}{2} [V, [V, \rho]] + \mathcal{W}_\beta \rho \]

\[ \frac{p_n}{p_m} = \exp \left( - \frac{E_n - E_m}{T_{nm}} \right) \]

\[ T_{\text{system}} = \text{average}[T_{nm}] \]
Quantum NESS for toy model with n.n. transitions

\[
\frac{d\rho}{dt} = -i[H, \rho] - \frac{\varepsilon^2}{2}[V, [V, \rho]] + \mathcal{W}^\beta \rho
\]

For very strong driving, the NESS is a mixture of \( V \) eigenstates:

\[
p_r \sim \exp\left(-\langle E \rangle_r/T_B\right)
\]

leading to:

\[
p_n \sim \exp(-E_n/T_\infty)
\]

\( T_B < T_\infty < \infty \) [depends on the sparsity]
How the temperature is defined

\[ \mathcal{H}_{\text{total}} = \{E_n\} - f(t)\{V_{nm}\} + \{W_{nm}\} \cdot \text{Bath} \]

The sources temperature: \( T_A = \infty \)
\[ \tilde{S}_A(\omega) \equiv \text{FT}\left\langle \dot{f}(t)\dot{f}(0) \right\rangle \]

The bath temperature: \( T_B \)
\[ \tilde{S}_B(\omega)/\tilde{S}_B(-\omega) = \exp(-\omega/T_B) \]

Temperature of the system?
\[ \dot{W} = \text{rate of heating} = \frac{D(\varepsilon)}{T_{\text{system}}} \]
\[ \dot{Q} = \text{rate of cooling} = \frac{D_B}{T_B} - \frac{D_B}{T_{\text{system}}} \]
\[ \frac{p_n}{p_m} = \exp\left( -\frac{E_n - E_m}{T_{nm}} \right) \]
\[ T_{\text{system}} = \text{average}[T_{nm}] = \left(1 + \frac{D(\varepsilon)}{D_B}\right)T_B \]
Digression - derivation of the cooling rate formula

\[ \dot{Q} = \text{cooling rate} = - \sum_{n,m} (E_n - E_m) w_{nm}^\beta p_m \]

\[ p_n - p_m = \text{occupation imbalance} = \left[ 2 \tanh \left( - \frac{E_n - E_m}{2T_{nm}} \right) \right] \bar{p}_{nm} \]

\[ w_{nm}^\beta - w_{mn}^\beta = \text{up/down transitions imbalance} = \left[ 2 \tanh \left( - \frac{E_n - E_m}{2T_B} \right) \right] w_{nm}^\beta \]

\[ \dot{Q} = \frac{1}{2} \sum_{n,m} (E_n - E_m)^2 \frac{w_{nm}^\beta}{T_B} \bar{p}_{nm} - \frac{1}{2} \sum_{n,m} (E_n - E_m)^2 \frac{w_{nm}^\beta}{T_{nm}} \bar{p}_{nm} = \frac{D_B}{T_B} - \frac{D_B}{T_{\text{system}}} \]

definition of the diffusion coefficient: \( D_B \equiv \left[ \frac{1}{2} \sum_n (E_n - E_m)^2 w_{nm}^\beta \right] \]

definition of effective system temperature: \( \frac{1}{T_{\text{system}}} \equiv \left[ \frac{1}{T_{nm}} \right] \)
Stochastic NESS for toy model with n.n. transitions

\[ D_B = w_\beta \Delta_0^2 \]  

[for use in the rate of cooling formula]

\[ T_{\text{system}} = \left( \frac{1}{T_n} \right)^{-1} = \left( \frac{w_\beta}{w_\beta + w_n} \right)^{-1} T_B \]

[w_n \propto \text{strength of the driving}]

\[ \dot{W} = \frac{D(\varepsilon)}{T_{\text{system}}} \]

\[ D(\varepsilon) = \left( \frac{w_n}{w_\beta + w_n} \right) \left( \frac{1}{w_\beta + w_n} \right)^{-1} \Delta_0^2 \]

\[ D_{[\text{LRT}]} = w_n \Delta_0^2 \]  

[weak driving]

\[ D_{[\text{SLRT}]} = \left( \frac{1}{w_n} \right)^{-1} \Delta_0^2 \]  

[strong driving]
Digression: random walk and the calculation of the diffusion coefficient

\( w_{nm} = \) probability to hop from \( m \) to \( n \) per step.

\[
\text{Var}(n) = \sum_n [w_{nm} t] (n - m)^2 \equiv 2Dt
\]

For n.n. hopping with rate \( w \) we get \( D = w \).

The continuity equation:

\[
\frac{\partial p_n}{\partial t} = -\frac{\partial}{\partial n} J_n
\]

Fick’s law:

\[
J_n = -D \frac{\partial}{\partial n} p_n
\]

The diffusion equation:

\[
\frac{\partial p_n}{\partial t} = D \frac{\partial^2}{\partial n^2} p_n
\]

If we have a sample of length \( N \) then

\[
J = -\frac{D}{N} \times [p_N - p_0]
\]

\[
D/N = \text{inverse resistance of the chain}
\]

If the \( w \) are not the same:

\[
D = \left[ \sum_{n=1}^{N} \frac{1}{w_{n,n-1}} \right]^{-1}
\]

Hence, for n.n. hopping

\[
D = \langle \langle w \rangle \rangle_{\text{harmonic}}
\]

FGR: \( w_{nm} \sim |V_{nm}|^2 \)

\[
D = \langle \langle |V_{nm}|^2 \rangle \rangle
\]
The master equation:
\[
\frac{dp_n}{dt} = - \sum_m w_{nm} (p_n - p_m)
\]

The Hamiltonian in the standard representation:
\[
\mathcal{H} = \{E_n\} - f(t)\{V_{nm}\}
\]

The transformed Hamiltonian:
\[
\tilde{\mathcal{H}} = \{E_n\} - \dot{f}(t) \left\{ \frac{iV_{nm}}{E_n - E_m} \right\}
\]
\[
\tilde{S}(\omega) \equiv \text{FT} \langle \dot{f}(t) \dot{f}(0) \rangle = 2\pi |\dot{f}|^2 \delta_0 (E_n - E_m)
\]

The FGR transition rate due to the low frequency noisy driving:
\[
w_{nm} = \left| \frac{V_{nm}}{E_n - E_m} \right|^2 \tilde{S}(E_n - E_m) \equiv \pi \varrho^3 \ g_{nm} \ \bar{f}^2
\]

The LRT / SLRT formula
\[
D = \text{average} \left[ \frac{1}{2} \sum_n (E_n - E_m)^2 w_{nm} \right] = \pi \varrho \langle \langle |V_{nm}|^2 \rangle \rangle \times \bar{f}^2 \equiv G \ \bar{f}^2
\]
The resistor network calculation

\[ \mathcal{H} = \{E_n\} - f(t)\{V_{nm}\} \]

\[ D = G \overline{f^2} \]

\[ G_{LRT} = \pi \varrho \left\langle \left\langle |V_{nm}|^2 \right\rangle \right\rangle_a \]

\[ G_{SLRT} = \pi \varrho \left\langle \left\langle |V_{nm}|^2 \right\rangle \right\rangle_s \]

\[ g_{nm} = 2 \varrho^{-3} \frac{|V_{nm}|^2}{(E_n - E_m)^2} \delta_0(E_n - E_m) \]

\[ \left\langle \left\langle |V_{nm}|^2 \right\rangle \right\rangle_s \equiv \text{inverse resistivity} \]

\[ g_s \equiv \frac{\left\langle \left\langle |V_{nm}|^2 \right\rangle \right\rangle_s}{\left\langle \left\langle |V_{nm}|^2 \right\rangle \right\rangle_a} \]

LRT applies if the driven transitions are slower than the environmental relaxation.
Bandprofile, sparsity and texture

\[ |V_{nm}|^2 \approx (2\pi \varrho)^{-1} \tilde{C}_c l (E_n - E_m) \]

Hard Qchaos

Weak Qchaos

[median \ll mean]
\{ |V_{nm}|^2 \} as a random matrix \( X = \{ x \} \)

For a random sparse matrix:
\( s, g_s \ll 1 \)

For a uniform (along diagonals):
\( s = g_s = 1 \)

For a Gaussian matrix:
\( s = \frac{1}{3}, \ g_s \sim 1 \)

Histogram of \( x \):
\( x \sim \text{LogNormal} \)
RMT modeling, generalized VRH approx scheme

- log-normal distribution $q$
- finite bandwidth $b$

$$s = q^2 = \left(\text{median/mean}\right)^2$$

$$g_s \approx q \exp\left[2\sqrt{-\ln q \ln(b)}\right]$$
**Digression: Generalized VRH**

Definition of the typical matrix element for a range $\omega$ transition:

$$\left(\frac{\omega}{\Delta}\right) \text{Prob}\left(x > x_\omega\right) \sim 1$$

In the standard-like case (ring with strong disorder):

$$x_\omega \approx v_F^2 \exp\left(\frac{\Delta_l}{|\omega|}\right) \quad [\text{corresponding to a log-box distribution}]$$

An example for the power spectrum of the driving:

$$\tilde{S}(\omega) \propto \exp\left(-\frac{|\omega|}{T}\right) \quad [\text{here the temperature } T \leftrightarrow \omega_c]$$

Generalized VRH estimate:

$$D_{SLRT} \approx \int x_\omega \tilde{S}(\omega) d\omega \quad [\text{should be contrasted with}] \quad D_{LRT} = \int \tilde{C}(\omega) \tilde{S}(\omega) d\omega$$

In the standard-like case (ring with strong disorder):

$$D_{SLRT} \approx \int \exp\left(\frac{\Delta_l}{|\omega|}\right) \exp\left(-\frac{|\omega|}{T}\right) d\omega$$
The rate of heating: LRT and SLRT predictions

\[ \mathcal{H}_{\text{total}} = \{E_n\} - f(t)\{V_{nm}\} \]

\[ f(t) = \text{low freq noisy driving} \]

\[ \sim \text{diffusion in energy space:} \]

\[ D_0 = \frac{4}{3\pi} \frac{M^2 v_E^3}{L_x} f^2 \]

\[ \sim \text{energy absorption:} \]

\[ \dot{E} = (\text{particles/energy}) \times D \]

Beyond the “Wall Formula”

[Beyond the “Drude Formula”]

\[ D_{\text{LRT}} = g_c D_0 \quad \text{[“classical”]} \]

\[ D_{\text{SLRT}} = g_s D_{\text{LRT}} \quad \text{[“quantum”]} \]

LRT applies if the driven transitions are slower than the environmental relaxation, else SLRT applies.
SLRT vs LRT

\[ \mathcal{H}_{\text{total}} = \mathcal{H} + f(t)V \]

\[ \tilde{C}(\omega) = \text{FT} \left\langle V(t)V(0) \right\rangle \]

\[ \tilde{S}(\omega) = \text{FT} \left\langle \dot{f}(t)\dot{f}(0) \right\rangle \]

Kubo formula:

\[ D = \int \tilde{C}(\omega)\tilde{S}(\omega)d\omega \]

SLRT example:

\[ D = \left[ \int R(\omega)\left[ \tilde{S}(\omega) \right]^{-1}d\omega \right]^{-1} \]

Linear response implies

\[ \tilde{S}(\omega) \mapsto \lambda \tilde{S}(\omega) \implies D \mapsto \lambda D \]

\[ \tilde{S}(\omega) \mapsto \sum_i \tilde{S}_i(\omega) \implies D \mapsto \sum_i D_i \]
Perspective and references

The classical LRT approach: Ott, Brown, Grebogi, Wilkinson, Jarzynski, Robbins, Berry, Cohen

The Wall formula (I): Blocki, Boneh, Nix, Randrup, Robel, Sierk, Swiatecki, Koonin

The Wall formula (II): Barnett, Cohen, Heller [1] - regarding $g_c$

Semi Linear response theory: Cohen, Kottos, Schanz... [2-6]

Billiards with vibrating walls: Stotland, Cohen, Davidson, Pecora [7,8] - regarding $g_s$

Sparsity: Austin, Wilkinson, Prosen, Robnik, Alhassid, Levine, Fyodorov, Chubykalo, Izrailev, Casati

\[ u = \left( \frac{t_R}{t_L} \right)^{-1} = \left( \frac{R}{L} \right)^{-1} = \text{deformation} \]

\[ \hbar = \frac{\lambda_E}{L} = \frac{2\pi}{(k_E L)} = \text{function of } E \]

Conclusions

(*) Wigner (\(\sim 1955\)): The perturbation is represented by a random matrix whose elements are taken from a **Gaussian** distribution.

Not always...

1. “weak quantum chaos” \(\implies\) log-wide distribution, “sparsity” and “texture”
2. The heating process \(\sim\) a percolation problem.
3. Resistors network calculation to get \(G_{\text{SLRT}}\).
4. Generalization of the **VRH** estimate.
5. **SLRT** applies if the driving is stronger then the background relaxation.
6. The **stochastic NESS** resembles a **glassy phase** (wide distribution of microscopic temperatures).
7. Definition of effective **NESS temperature**, and extension of the **FDT** phenomenology.
8. For very strong driving - **quantum saturation** of the NESS temperature (\(T \to T_\infty\)).
9. Applications: beyond the “**Drude formula**” and beyond the “**Wall formula**”.