Non-equilibrium steady state (NESS) of sparse systems

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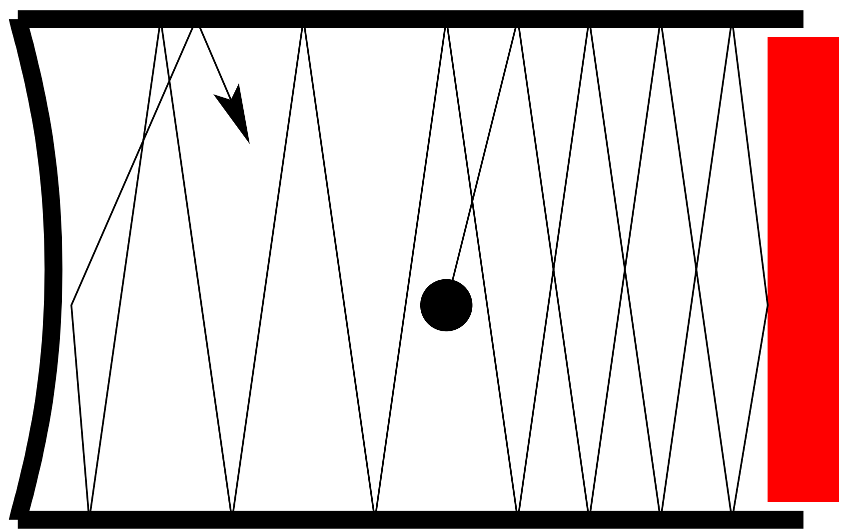
$$\mathcal{H}_{\text{total}} = \{E_n\} - f(t)\{V_{nm}\}$$
$$+\{W_{nm}\} \cdot \mathit{Bath}$$

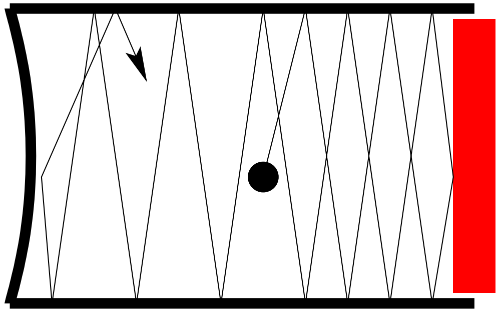


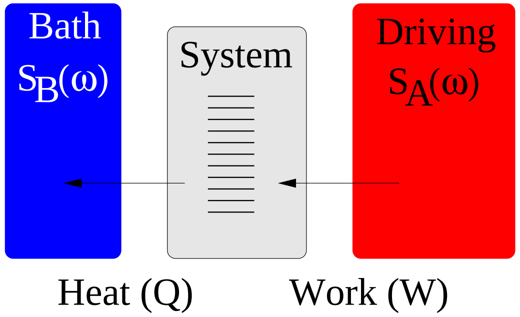
$DIP, $BSF

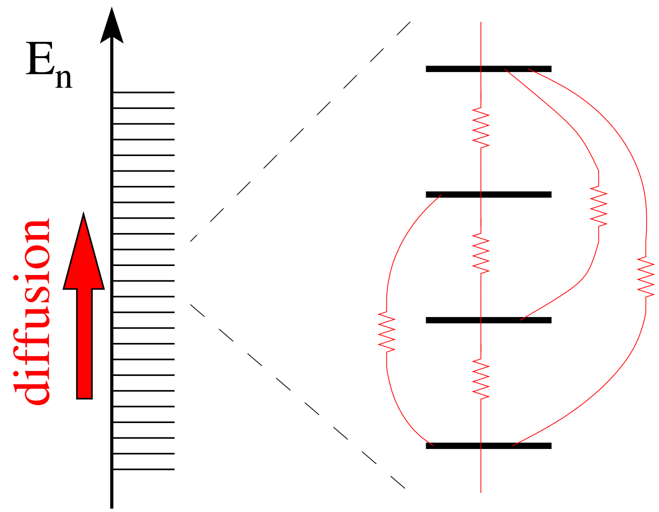
Driven system + Bath

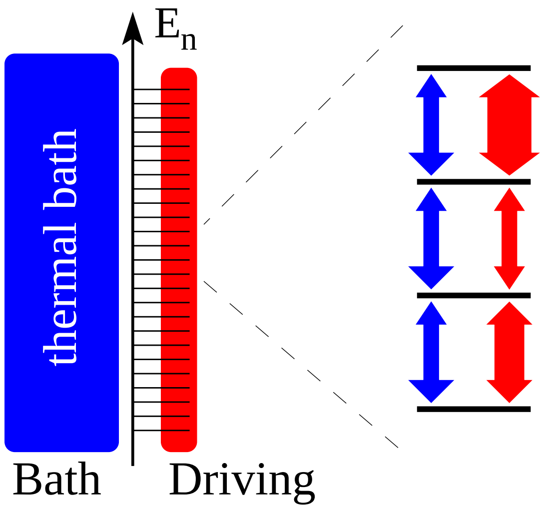
$$\mathcal{H}_{\text{total}} = \{E_n\} - f(t)\{V_{nm}\} + \{W_{nm}\} \cdot Bath$$

$\varepsilon^2 \equiv$ Driving intensity $\qquad T_B \equiv$ Bath temperature







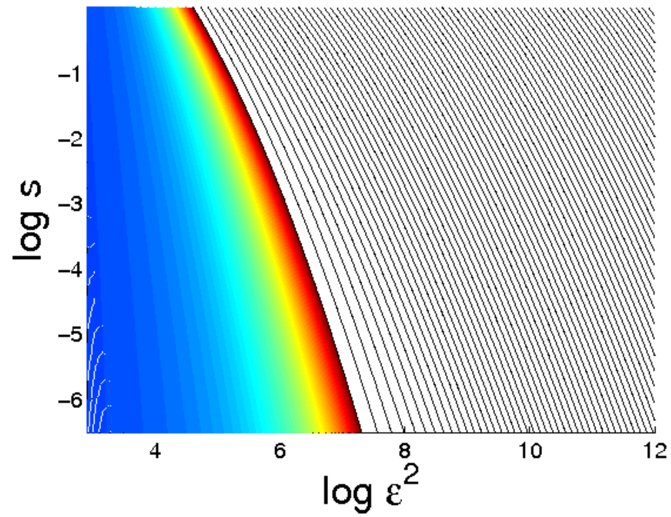


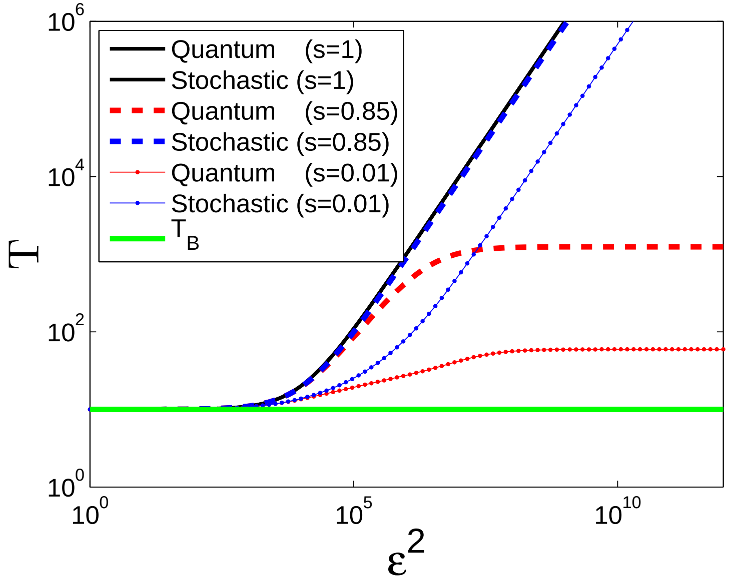
Steady state temperature

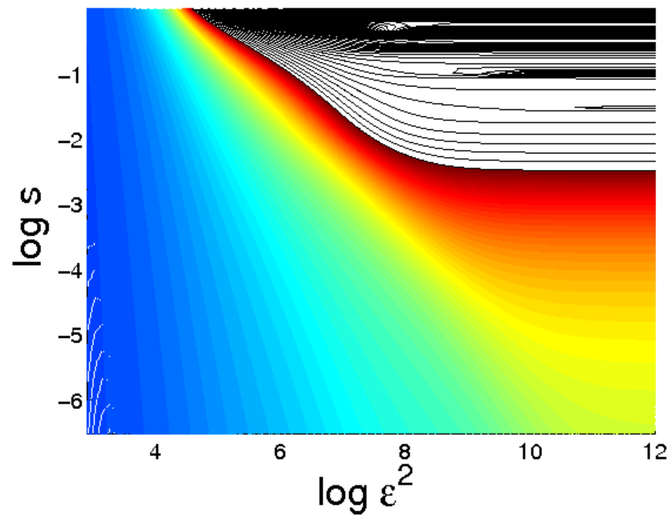
$$\frac{d\rho}{dt} = -i[\mathcal{H}, \rho] - \frac{\varepsilon^2}{2}[V, [V, \rho]] + \mathcal{W}^{\beta}\rho$$

$$\frac{p_n}{p_m} = \exp\left(-\frac{E_n - E_m}{T_{nm}}\right)$$

$$T_{\text{system}} = \text{average}[T_{nm}]$$







Quantum NESS for toy model with n.n. transitions

$$\frac{d\rho}{dt} = -i[\mathcal{H}, \rho] - \frac{\varepsilon^2}{2}[V, [V, \rho]] + \mathcal{W}^{\beta}\rho$$

For very strong driving,

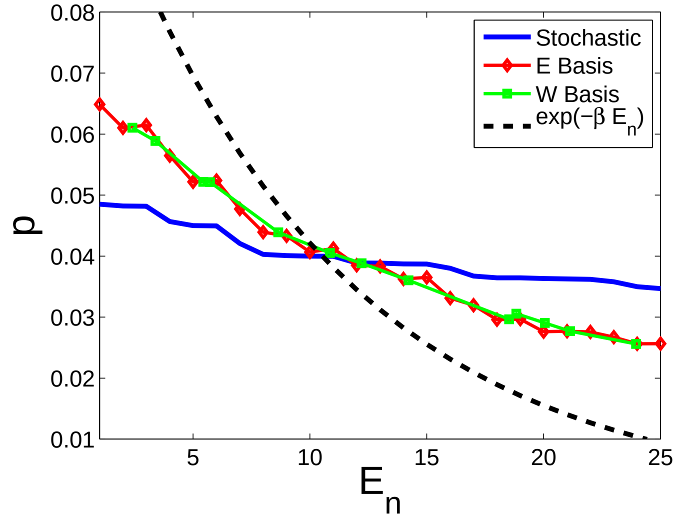
the NESS is a mixture of V eigenstates:

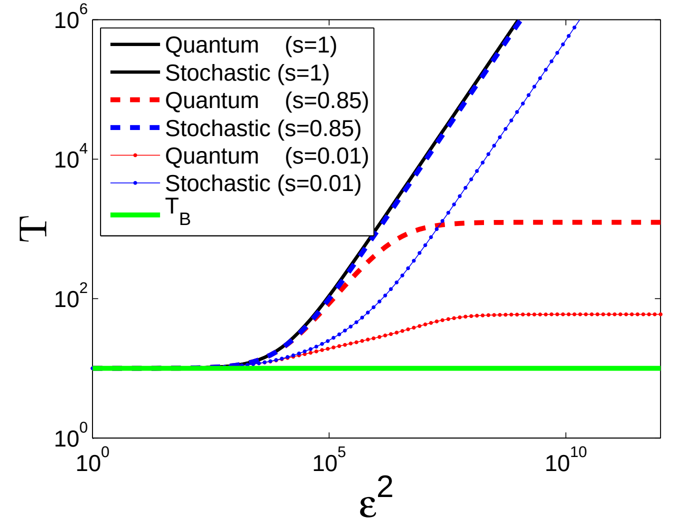
$$p_r \sim \exp(-\langle E \rangle_r / T_B)$$

leading to:

$$p_n \sim \exp(-E_n / T_\infty)$$

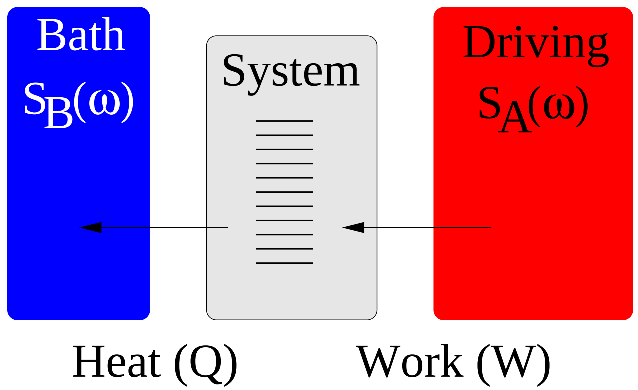
$T_B < T_\infty < \infty$ [depends on the sparsity]





How the temperature is defined

$$\mathcal{H}_{\text{total}} = \{E_n\} - f(t)\{V_{nm}\} + \{W_{nm}\} \cdot Bath$$



The sources temperature: $T_A = \infty$

$$\tilde{S}_A(\omega) \equiv \text{FT} \langle \dot{f}(t)\dot{f}(0)\rangle$$

The bath temperature: T_B

$$\tilde{S}_B(\omega)/\tilde{S}_B(-\omega) = \exp(-\omega/T_B)$$

$$\frac{p_n}{p_m} = \exp\left(-\frac{E_n - E_m}{T_{nm}}\right)$$

Temperature of the system?

$$\dot{\mathsf{W}} = \text{rate of heating} = \frac{D(\varepsilon)}{T_{\text{system}}}$$

$$\dot{\mathsf{Q}} = \text{rate of cooling} = \frac{D_B}{T_B} - \frac{D_B}{T_{\text{system}}}$$

$$T_{\text{system}} = \text{average}[T_{nm}]$$

$$= \left(1 + \frac{D(\varepsilon)}{D_B}\right) T_B$$

Digression - derivation of the cooling rate formula

$$\dot{\mathsf{Q}} \;=\; \text{cooling rate} \;=\; -\sum_{n,m}(E_n - E_m)\, w^{\beta}_{nm}\, p_m$$

$$p_n - p_m \;=\; \text{occupation imbalance} \;=\; \left[2\tanh\left(-\frac{E_n - E_m}{2T_{nm}}\right)\right] \bar{p}_{nm}$$

$$w^{\beta}_{nm} - w^{\beta}_{mn} \;=\; \text{up/down transitions imbalance} \;=\; \left[2\tanh\left(-\frac{E_n - E_m}{2T_B}\right)\right] w^{\beta}_{nm}$$

$$\dot{\mathsf{Q}} \;=\; \frac{1}{2}\sum_{n,m}(E_n-E_m)^2\, \frac{w^{\beta}_{nm}}{T_B}\bar{p}_{nm} - \frac{1}{2}\sum_{n,m}(E_n-E_m)^2\, \frac{w^{\beta}_{nm}}{T_{nm}}\bar{p}_{nm} \;=\; \frac{D_B}{T_B} - \frac{D_B}{T_{\text{system}}}$$

definition of the diffusion coefficient: $D_B \equiv \overline{\left[\frac{1}{2}\sum_{n}(E_n-E_m)^2\, w^{\beta}_{nm}\right]}$

definition of effective system temperature: $\frac{1}{T_{\text{system}}} \equiv \overline{\left[\frac{1}{T_{nm}}\right]}$

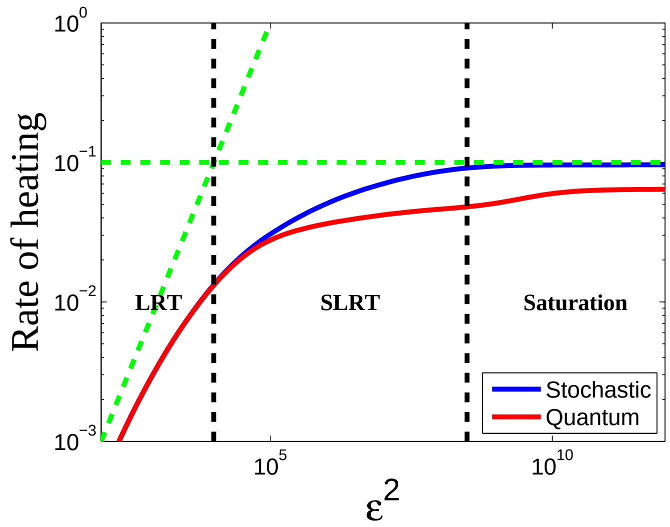
Stochastic NESS for toy model with n.n. transitions

$$D_B = w_\beta \, \Delta_0^2$$ [for use in the rate of cooling formula]

$$T_{\text{system}} = \left[\overline{\left(\frac{1}{T_n}\right)}\right]^{-1} = \left[\overline{\left(\frac{w_\beta}{w_\beta + w_n}\right)}\right]^{-1} T_B$$ [$w_n \propto$ strength of the driving]

$$\dot{\mathsf{W}} = \frac{D(\varepsilon)}{T_{\text{system}}}$$

$$D(\varepsilon) = \left[\overline{\left(\frac{w_n}{w_\beta + w_n}\right)}\right] \left[\overline{\left(\frac{1}{w_\beta + w_n}\right)}\right]^{-1} \Delta_0^2$$



$$D_{[\text{LRT}]} = \overline{w_n} \, \Delta_0^2$$ [weak driving]

$$D_{[\text{SLRT}]} = [\overline{1/w_n}]^{-1} \, \Delta_0^2$$ [strong driving]

Digression: random walk and the calculation of the diffusion coefficient

w_{nm} = probability to hop from m to n per step.

$$\mathrm{Var}(n) = \sum_n [w_{nm} t]\,(n-m)^2 \equiv 2Dt$$

For n.n. hopping with rate w we get $D = w$.

The continuity equation:

$$\frac{\partial p_n}{\partial t} = -\frac{\partial}{\partial n} J_n$$

Fick's law:

$$J_n = -D\frac{\partial}{\partial n} p_n$$

The diffusion equation:

$$\frac{\partial p_n}{\partial t} = D\frac{\partial^2}{\partial n^2} p_n$$

If we have a sample of length N then

$$J = -\frac{D}{N} \times [p_N - p_0]$$

$$D/N = \text{inverse resistance of the chain}$$

If the w are not the same:

$$\frac{D}{N} = \left[\sum_{n=1}^{N} \frac{1}{w_{n,n-1}}\right]^{-1}$$

Hence, for n.n. hopping

$$D = \langle\langle w \rangle\rangle_{\text{harmonic}}$$

FGR: $w_{nm} \sim |V_{nm}|^2$

$$D = \langle\langle |V_{nm}|^2 \rangle\rangle$$

Digression: the Fermi golden rule picture

Master equation:

$$\frac{dp_n}{dt} = -\sum_m w_{nm}(p_n - p_m)$$

The Hamiltonian in the standard representation:

$$\mathcal{H} = \{E_n\} - f(t)\{V_{nm}\}$$

The transformed Hamiltonian:

$$\tilde{\mathcal{H}} = \{E_n\} - \dot{f}(t)\left\{\frac{iV_{nm}}{E_n - E_m}\right\}$$

$$\tilde{S}(\omega) \equiv \text{FT}\ \langle \dot{f}(t)\dot{f}(0)\rangle = 2\pi\overline{|\dot{f}|^2}\ \delta_0(E_n - E_m)$$

The FGR transition rate due to the low frequency noisy driving:

$$w_{nm} = \left|\frac{V_{nm}}{E_n - E_m}\right|^2 \tilde{S}(E_n - E_m) \equiv \pi\varrho^3\ \mathrm{g}_{nm}\ \overline{\dot{f}^2}$$

The LRT / SLRT formula

$$\boldsymbol{D} = \text{average}\left[\frac{1}{2}\sum_n (E_n - E_m)^2\ w_{nm}\right] = \pi\varrho\ \langle\langle |V_{nm}|^2\rangle\rangle \times \overline{\dot{f}^2} \equiv \boldsymbol{G}\ \overline{\dot{f}^2}$$

The resistor network calculation

$$\mathcal{H} = \{E_n\} - f(t)\{V_{nm}\}$$

$$g_s \equiv \frac{\langle\langle |V_{nm}|^2 \rangle\rangle_s}{\langle\langle |V_{nm}|^2 \rangle\rangle_a}$$

$$\boldsymbol{D} = \boldsymbol{G}\ \overline{\dot{f}^2}$$

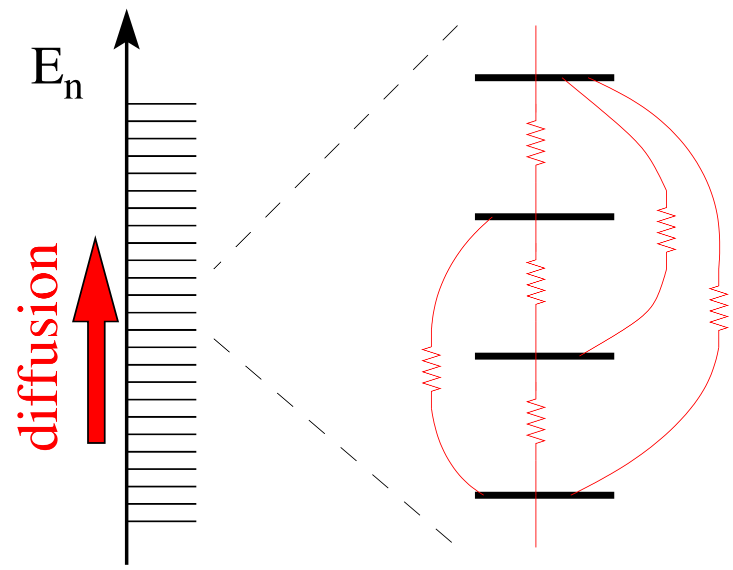
$$\boldsymbol{G}_{\text{LRT}} = \pi\varrho\ \langle\langle |V_{nm}|^2 \rangle\rangle_a$$

$$\boldsymbol{G}_{\text{SLRT}} = \pi\varrho\ \langle\langle |V_{nm}|^2 \rangle\rangle_s$$

LRT applies if the driven transitions are slower than the environmental relaxation

$$\mathrm{g}_{nm} = 2\varrho^{-3}\ \frac{|V_{nm}|^2}{(E_n - E_m)^2}\ \delta_0(E_n - E_m)$$

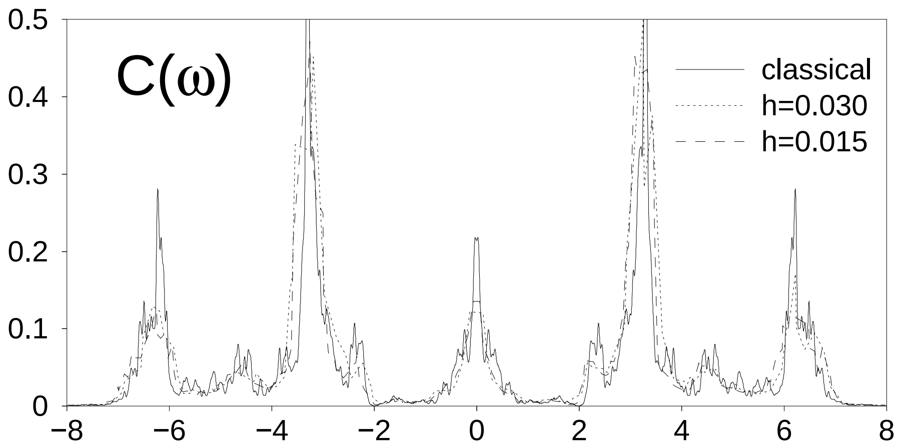
$\langle\langle |V_{nm}|^2 \rangle\rangle_s \equiv$ inverse resistivity

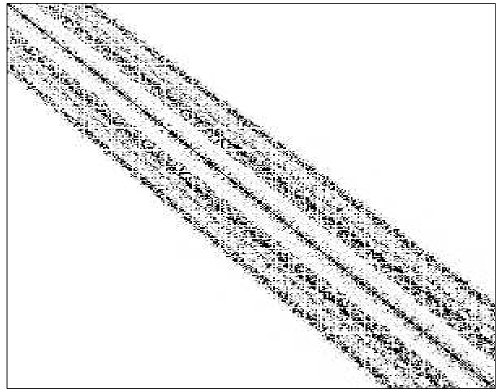


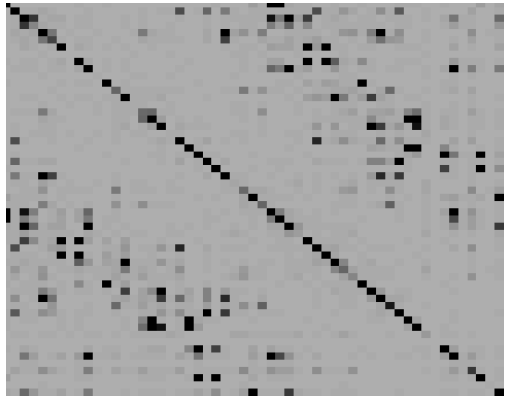
Bandprofile, sparsity and texture

$$|V_{nm}|^2 \approx (2\pi\varrho)^{-1}\tilde{C}_{cl}(E_n - E_m)$$

Hard Qchaos

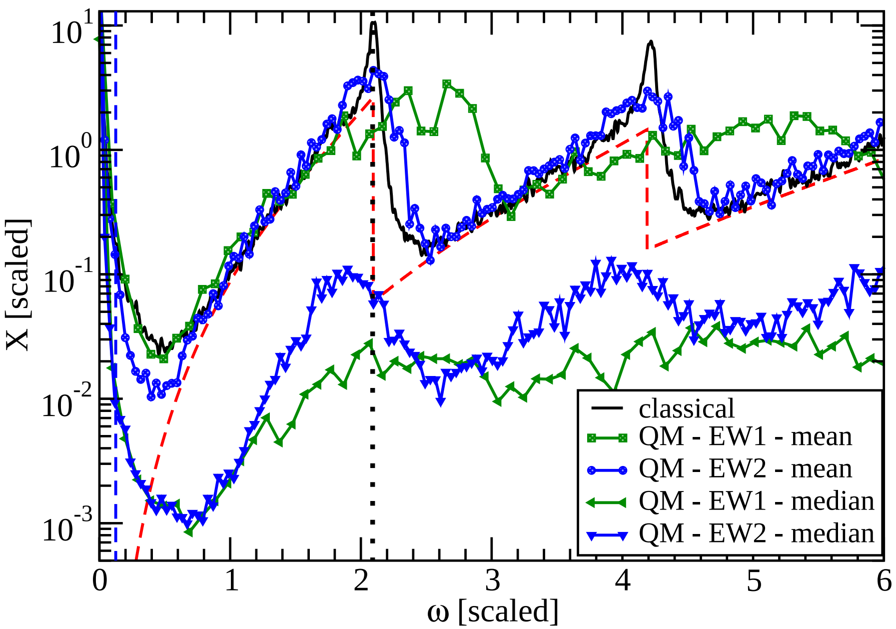


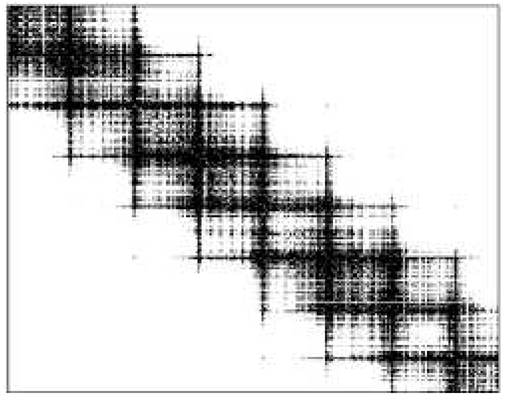




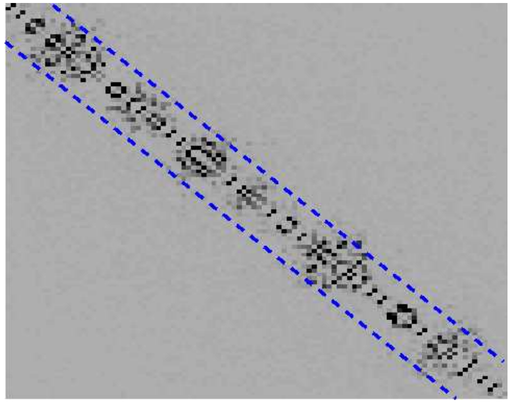
Weak Qchaos

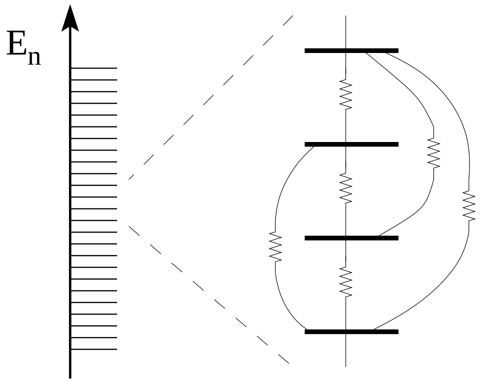
[median ≪ mean]





$\{|V_{nm}|^2\}$ as a random matrix $\boldsymbol{X} = \{x\}$

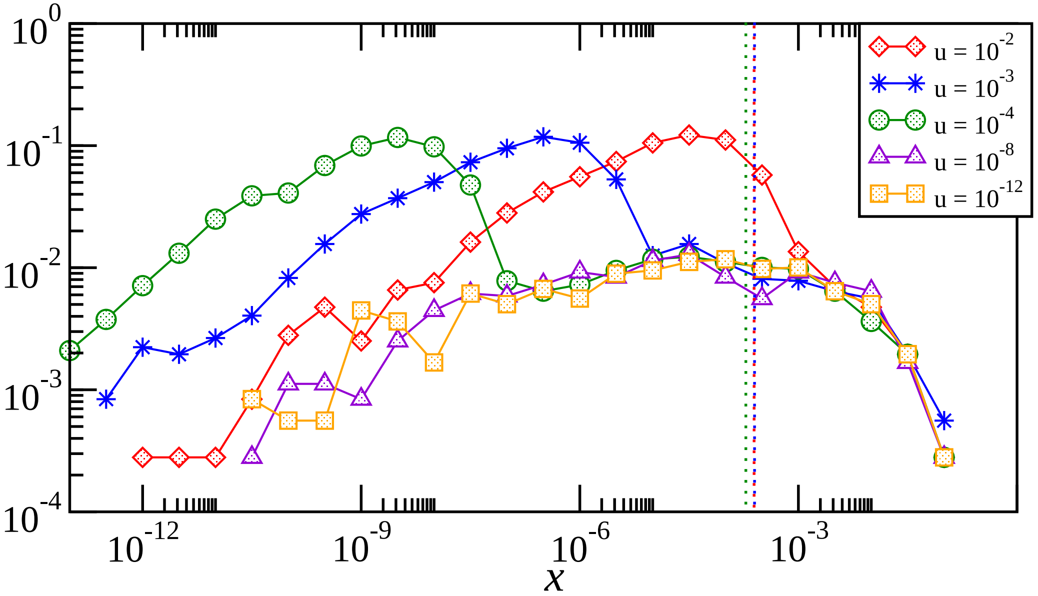




$$s[\boldsymbol{X}] \equiv \frac{\mathrm{PN}\left[\boldsymbol{X}\right]}{\mathrm{PN}\left[\boldsymbol{X}^{\mathrm{unf}}\right]} = \text{sparsity}$$

$$g_s[\boldsymbol{X}] \equiv \frac{\langle\langle \boldsymbol{X} \rangle\rangle_s}{\langle\langle \boldsymbol{X} \rangle\rangle_a} = \text{connectivity}$$

Histogram of x :



$x \sim$ LogNormal

For a random sparse matrix:

$$s,\ g_s \ll 1$$

For a uniform (along diagonals):

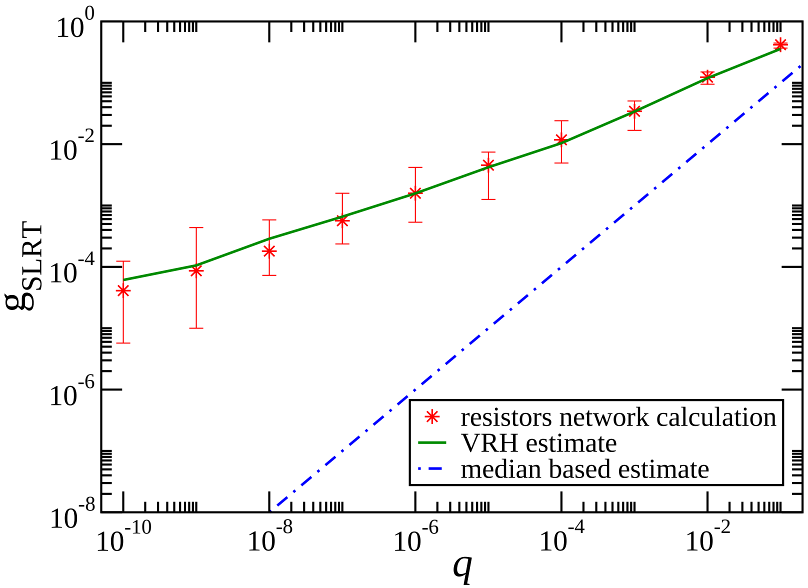
$$s = g_s = 1$$

For a Gaussian matrix:

$$s = 1/3, \quad g_s \sim 1$$

RMT modeling, generalized VRH approx scheme

- log-normal distribution q
- finite bandwidth b



$$s \;=\; q^2 \;=\; (\text{median}/\text{mean})^2$$

$$g_s \;\approx\; q\,\exp\left[2\sqrt{-\ln q\,\ln(b)}\right]$$

Digression: Generalized VRH

Definition of the typical matrix element for a range ω transition:

$$\left(\frac{\omega}{\Delta}\right) \mathrm{Prob}\left(x > x_\omega\right) \sim 1$$

In the standard-like case (ring with strong disorder):

$$x_\omega \approx v_F^2 \exp\left(\frac{\Delta_l}{|\omega|}\right)$$

[corresponding to a log-box distribution]

An example for the power spectrum of the driving:

$$\tilde{S}(\omega) \propto \exp\left(-\frac{|\omega|}{T}\right)$$

[here the temparature $T \Longleftrightarrow \omega_c$]

Generalized VRH estimate:

$$D_{\mathrm{SLRT}} \approx \int x_\omega \tilde{S}(\omega) d\omega$$

[should be contrasted with]

$$D_{\mathrm{LRT}} = \int \tilde{C}(\omega) \tilde{S}(\omega) d\omega$$

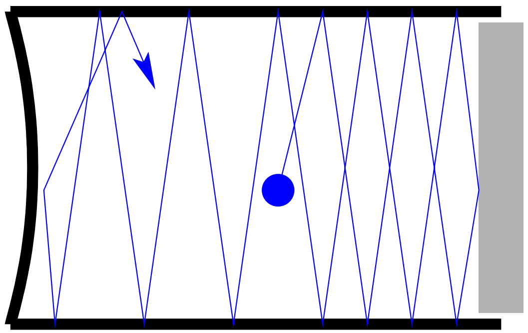
In the standard-like case (ring with strong disorder):

$$D_{\mathrm{SLRT}} \approx \int \exp\left(\frac{\Delta_l}{|\omega|}\right) \exp\left(-\frac{|\omega|}{T}\right) d\omega$$

The rate of heating: LRT and SLRT predictions

$$\mathcal{H}_{\text{total}} = \{E_n\} - f(t)\{V_{nm}\}$$

$f(t)$ = low freq noisy driving



$\rightsquigarrow$ diffusion in energy space:

$$D_0 = \frac{4}{3\pi}\frac{\mathsf{M}^2 v_{\text{E}}^3}{L_x}\overline{\dot{f}^2}$$

$\rightsquigarrow$ energy absorption:

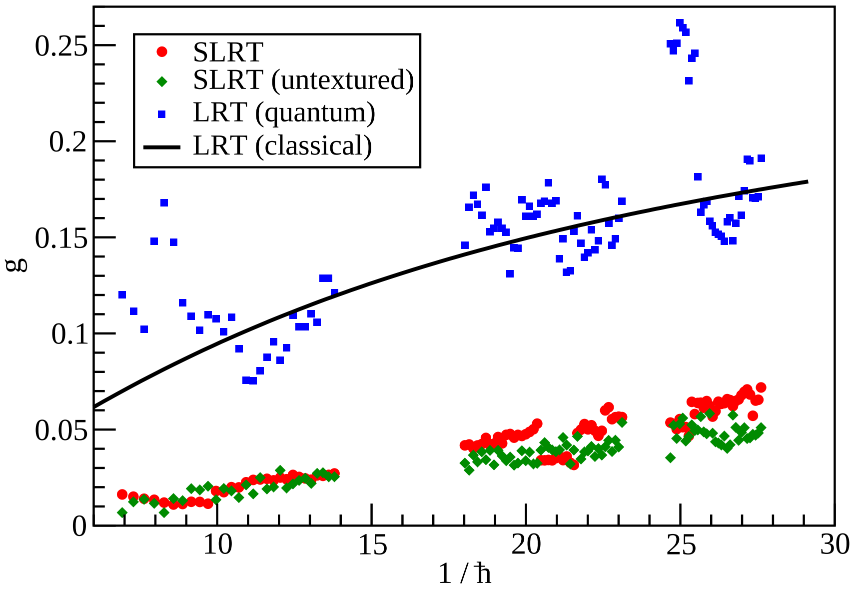
$$\dot{E} = (\text{particles/energy}) \times D$$

Beyond the "Wall Formula"

[Beyond the "Drude Formula"]

$$D_{\text{LRT}} = g_c\, D_0 \qquad [\text{"classical"}]$$

$$D_{\text{SLRT}} = g_s\, D_{\text{LRT}} \qquad [\text{"quantum"}]$$



LRT applies if the driven transitions are slower than the environmental relaxation, else SLRT applies

SLRT *vs* LRT

$$\mathcal{H}_{\text{total}} = \mathcal{H} + f(t)V$$

$$\tilde{C}(\omega) = \text{FT}\ \langle V(t)V(0)\rangle$$
$$\tilde{S}(\omega) = \text{FT}\ \langle \dot{f}(t)\dot{f}(0)\rangle$$

Kubo formula:

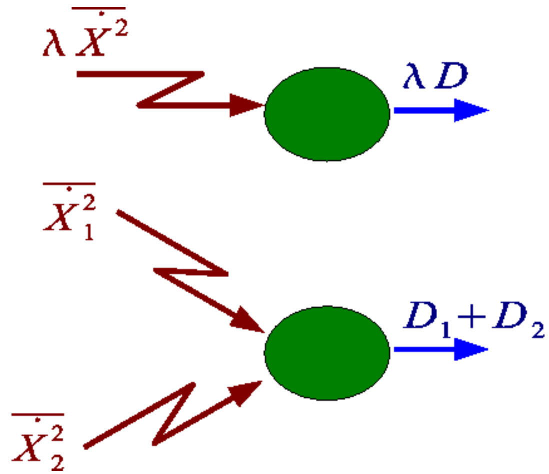
$$\boldsymbol{D} = \int \tilde{C}(\omega)\tilde{S}(\omega)d\omega$$

SLRT example:

$$\boldsymbol{D} = \left[\int R(\omega)\left[\tilde{S}(\omega)\right]^{-1} d\omega\right]^{-1}$$

Linear response implies

$$\tilde{S}(\omega) \mapsto \lambda\tilde{S}(\omega) \quad \Longrightarrow \quad \boldsymbol{D} \mapsto \lambda\boldsymbol{D}$$
$$\tilde{S}(\omega) \mapsto \sum_i \tilde{S}_i(\omega) \quad \Longrightarrow \quad \boldsymbol{D} \mapsto \sum_i \boldsymbol{D}_i$$



Perspective and references

The classical LRT approach: Ott, Brown, Grebogi, Wilkinson, Jarzynski, Robbins, Berry, Cohen

The Wall formula (I): Blocki, Boneh, Nix, Randrup, Robel, Sierk, Swiatecki, Koonin

The Wall formula (II): Barnett, Cohen, Heller [1] - regarding g_c

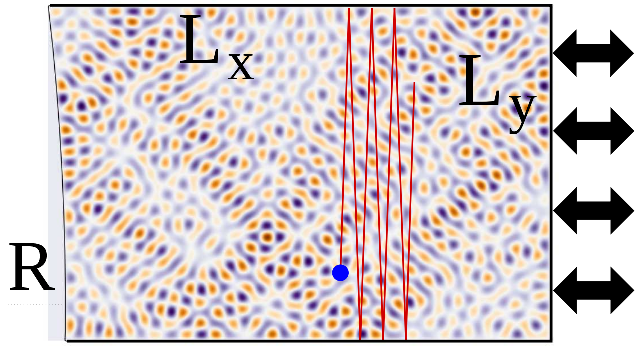
Semi Linear response theory: Cohen, Kottos, Schanz... [2-6]

Billiards with vibrating walls: Stotland, Cohen, Davidson, Pecora [7,8] - regarding g_s

Sparsity: Austin, Wilkinson, Prosen, Robnik, Alhassid, Levine, Fyodorov, Chubykalo, Izrailev, Casati

$$u \;=\; (t_{\rm R}\,/\,t_{\rm L})^{-1} \;=\; (R/L)^{-1} \;=\; \text{deformation}$$

$$\hbar \;=\; \lambda_{\rm E}\,/\,L \;=\; 2\pi/(k_{\rm E}L) \;=\; \text{function of } E$$



[1] A. Barnett, D. Cohen, E.J. Heller (PRL 2000, JPA 2000)

[2] D. Cohen, T. Kottos, H. Schanz (JPA 2006)

[3] S. Bandopadhyay, Y. Etzioni, D. Cohen (EPL 2006)

[4] M. Wilkinson, B. Mehlig, D. Cohen (EPL 2006)

[5] A. Stotland, R. Budoyo, T. Peer, T. Kottos, D. Cohen (JPA/FTC 2008)

[6] A. Stotland, T. Kottos, D. Cohen (PRB 2010)

[7] A. Stotland, D. Cohen, N. Davidson (EPL 2009)

[8] A. Stotland, L.M. Pecora, D. Cohen (arXiv 2010)

Conclusions

(*) Wigner ($\sim$ 1955): The perturbation is represented by a random matrix whose elements are taken from a Gaussian distribution.

Not always...

1. "weak quantum chaos" $\Longrightarrow$ log-wide distribution, "sparsity" and "texture"
2. The heating process $\sim$ a percolation problem.
3. Resistors network calculation to get G_{SLRT}.
4. Generalization of the VRH estimate.
5. SLRT applies if the driving is stronger then the background relaxation.
6. The stochastic NESS resembles a glassy phase (wide distribution of microscopic temperatures).
7. Definition of effective NESS temperature, and extension of the FDT phenomenology.
8. For very strong driving - quantum saturation of the NESS temperature ($T \rightarrow T_\infty$).
9. Applications: beyond the "Drude formula" and beyond the "Wall formula".