Quantum irreversibility, perturbation independent decay, and the parametric theory of the local density of states

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The idea of perturbation independent decay (PID) has appeared in the context of survival-probability studies, and lately has emerged in the context of quantum irreversibility studies. In both cases the PID reflects the Lyapunov instability of the underlying semiclassical dynamics, and it can be distinguished from the Wigner-type decay that holds in the perturbative regime. The theory of the survival probability is manifestly related to the parametric theory of the local density of states (LDOS). In contrast to that the physics of quantum irreversibility requires subtle cross correlations, which are not captured by the LDOS alone, to be taken into account.

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I. INTRODUCTION

The study of quantum irreversibility [1] has become of much interest recently [2–8] due to its potential relevance to quantum computing and to the theory of dephasing [9,10]. Following [1] we define in Sec. II the main object of the present paper, which is the "fidelity," also known as the "Loschmidt echo," that constitutes a measure for quantum irreversibility.

The analysis of fidelity necessitates a generalization of the theory regarding survival probability [11]. In Sec. III we remind the reader that the latter reduces to the analysis of the local density of states (LDOS) [12]. Is it possible to make a similar reduction in the case of the fidelity? At first sight such reduction looks feasible because the general physical picture looks very similar (Secs. V and VI).

In the present paper we claim (Sec. VI), and prove by a numerical example (Secs. VII and VIII), that the study of fidelity cannot be reduced to analysis of LDOS functions. Rather, it is essential to take into account subtle cross correlations which are not captured by the LDOS alone.

The object of the present study is common to almost all the quantum chaos studies, namely, to figure out what is the role of semiclassical mechanics in quantum mechanics. Whenever we find such (semiclassical) "fingerprints," we call them "nonuniversal" effects. Most of the studies in the quantum chaos literature during the last 20 years have been devoted to figuring out the nonuniversal features of the energy spectrum. The main tool in singling out such features is a comparison with the predictions of random matrix theory (RMT).

In the present paper we use the same philosophy. Namely, we identify nonuniversal effects by making a comparison with a corresponding random matrix model. On the other hand, we discuss (Secs. X and XI) a unifying theoretical picture that put the study of quantum irreversibility in the larger context of phase-space-based semiclassical approach.

An important ingredient in the understanding of nonuniversal features follows from studies of the clash between perturbation theory, semiclassical theory, and RMT [10,12,13]. A major realization is that semiclassical theory and RMT lead to *different* nonperturbative limits. Hence the resolution of the clash between the different theories involves the identification of different *regimes* of behavior. This applies in general to the analysis of time-dependent dynamics [13], and in particular to the analysis of wave-packet dynamics, decay of the survival probability, structure of the LDOS [12], and naturally also to quantum irreversibility studies.

Specifically, we distinguish in the present paper between regimes of perturbative and nonuniversal behavior, and we define and study a billiard related model, where we have full control over the "borders" between these regimes. The conclusions are summarized in Sec. XII.

II. THE FIDELITY

Consider a system whose evolution is governed by the chaotic Hamiltonian

$$\mathcal{H}=\mathcal{H}(Q,P;x), \tag{1}$$

where (Q,P) is a set of canonical coordinates, and x is a parameter. Later (Sec. VII) we are going to consider, as an example, a billiard system, where (Q,P) are the position and the momentum of a particle, while x is used in order to parametrize the shape of the billiard. Specifically, for a stadium we define x as the length of the straight edge, and adjust the radius parameter such that the total area is kept constant.

Consider some $\mathcal{H}_0=\mathcal{H}(Q,P;x_0)$, and define $\delta x=x-x_0$. Assume that δx is classically small, so that both $\mathcal{H}_0$ and $\mathcal{H}$ generate classically chaotic dynamics of similar nature. Physically, going from $\mathcal{H}_0$ to $\mathcal{H}$ may signify a small change of an external field. In the case of the billiard system, δx parametrizes the displacement of the walls. Given a preparation Ψ_0, the fidelity is defined as [14]

$$M(t;\delta x)=|m(t;\delta x)|^2, \tag{2}$$

$$m(t;\delta x)\equiv\langle\Psi_0|\exp(+i\mathcal{H}t)\exp(-i\mathcal{H}_0 t)|\Psi_0\rangle. \tag{3}$$

If Ψ_0 is an eigenstate $|E_0\rangle$ of $\mathcal{H}_0$, then $M(t;\delta x)$ is equal to the survival probability $P(t;\delta x)$, which is defined as

$$P(t;\delta x) = |c(t;\delta x)|^2, \tag{4}$$

$$c(t;\delta x) \equiv \langle E_0|\exp(-i\mathcal{H}t)|E_0\rangle. \tag{5}$$

In the general case the preparation Ψ_0 does *not* have to be an eigenstate of $\mathcal{H}_0$. To be specific one assumes that Ψ_0 is a Gaussian wave packet. It is now possible to define a different type of survival probability as follows:

$$P(t;\text{wpk}) = |c(t;\text{wpk})|^2, \tag{6}$$

$$c(t;\text{wpk}) \equiv \langle \Psi_0|\exp(-i\mathcal{H}t)|\Psi_0\rangle. \tag{7}$$

We assume that δx is small enough so that we do not have to distinguish between $\mathcal{H}$ and $\mathcal{H}_0$ in the latter definition.

One may regard Ψ_0 as an eigenstate of some preparation Hamiltonian $\mathcal{H}_{\text{wpk}}$. Specifically, if Ψ_0 is a Gaussian wave packet, then it is the ground state of a Hamiltonian of the type $(P-P_0)^2+(Q-Q_0)^2$ that differs enormously from $\mathcal{H}$. Thus we have in the general case the following three Hamiltonians:

(i) The preparation Hamiltonian $\mathcal{H}_{\text{wpk}}$.

(ii) The unperturbed evolution Hamiltonian $\mathcal{H}_0$.

(iii) The perturbed evolution Hamiltonian $\mathcal{H}$.

Above we have distinguished between two cases: the relatively simple case where $\mathcal{H}_{\text{wpk}}=\mathcal{H}_0$ and the more general case, where we assume that the difference $\|\mathcal{H}_{\text{wpk}}-\mathcal{H}_0\|$ is in fact much larger compared with the perturbation $\|\mathcal{H}-\mathcal{H}_0\|$. The strength of the perturbation is controlled by the parameter δx.

III. THE LDOS FUNCTIONS

Consider first the special case where $\mathcal{H}_{\text{wpk}}=\mathcal{H}_0$. In such case the fidelity amplitude $m(t;\delta x)$ is just the Fourier transform of the local density of states (LDOS),

$$\rho(\omega;\delta x) = \sum_n |\langle n(x)|E_0\rangle|^2 \delta(\omega - [E_n(x)-E_0]). \tag{8}$$

For technical reasons, we would like to assume that there is an implicit average over the reference state $|E_0\rangle$. This will enable a meaningful comparison with the more general case which is discussed below.

In the general case, where $\mathcal{H}_{\text{wpk}}\neq\mathcal{H}_0$, one should recognize the need in defining an additional LDOS function,

$$\rho(\omega;\text{wpk}) = \sum_n |\langle n|\Psi_0\rangle|^2 \delta(\omega-(E_n-E_0)). \tag{9}$$

In this context E_0 is consistently redefined as the mean energy of the wave packet. Recall again that δx is assumed to be small enough, so that we do not have to distinguish between $\mathcal{H}$ and $\mathcal{H}_0$ in the latter definition.

The Fourier transform of $\rho(\omega;\text{wpk})$ is equal (up to a phase factor) to the survival amplitude $c(t;\text{wpk})$ of the wave packet. The physics of $c(t;\text{wpk})$ is assumed to be of ‘‘semiclassical’’ type. We shall define what we mean by semiclassical later on. The same notion is going to be used regarding $c(t;\delta x)$ if the perturbation (δx) is large enough.

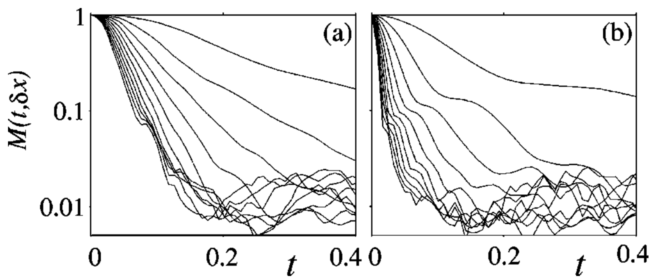

FIG. 1. (a) The decay of $M(t;\delta x)$ in the MBH case. (b) The same for the randomized MBH (RMBH). We use dimensionless units of time that correspond to a stadium billiard with straight edge $x_0=1$, a particle with mass $m=1/2$, the wave number $k\sim 50$, and $\hbar=1$. The values of the perturbation strength are (from the top curve to bottom) $\delta x = 0.0125*i$ with $i=1,\ldots,11$.

IV. DEFINITIONS OF Γ AND γ

In this paper we measure the ‘‘width’’ of the LDOS of Eq. (8) in energy units [14], and denote it by $\Gamma(\delta x)$. A practical numerical definition of $\Gamma(\delta x)$ is as the width of the central region that contains 70% of the probability. This corresponds to the notion of ‘‘core width’’ in [12]. If δx is not too large (see the definition of the ‘‘Wigner regime’’ in the next section), one observes that

$$\Gamma(\delta x) \propto \delta x^{2/(1+g)}. \tag{10}$$

The value $g\sim 0$ applies for strong chaos [12,16], and it is the same as in Wigner's random matrix theory (RMT) [15]. In general (e.g., see the Appendix) we can have $0<g<1$. In fact, the value $g\sim 1$ applies to our numerical model, which will be defined in Sec. VII.

The decay rate of either the fidelity or of the survival probability (depending on the context) is denoted by $\gamma(\delta x)$. The semiclassical value of the decay rate, which is determined via a ‘‘wave-packet dynamics’’ phase-space picture [11], is denoted by γ_{scl}. The Lyapunov exponent is denoted by γ_{cl}.

In order to determine $\gamma(\delta x)$ numerically one should plot $M(t;\delta x)$ against t, for a range of δx values. In Sec. VII we are going to define some model Hamiltonians for which we have done simulations. These are called the linearized billiard Hamiltonian (LBH), the randomized version of LBH (RLBH), the modified billiard Hamiltonian (MBH), and the randomized version of MBH (RMBH).

Figure 1(a) displays the results of the MBH simulations. We see that the MBH decay is well approximated by exponential function [Fig. 1(a)]. The dependence of the decay rate γ_{MBH} on δx is presented in Fig. 2. The RMBH decay [Fig. 1(b)] is badly approximated by exponential function, but in order to make a comparison we still fit it to exponential. This is done in order to have a quantitative measure for the decay time. Thus we have also $\gamma_{\text{RMBH}}(\delta x)$.

In Fig. 2 we also plot the LDOS width $\Gamma(\delta x)$ as a function of δx for the two models. As far as $\Gamma(\delta x)$ is concerned

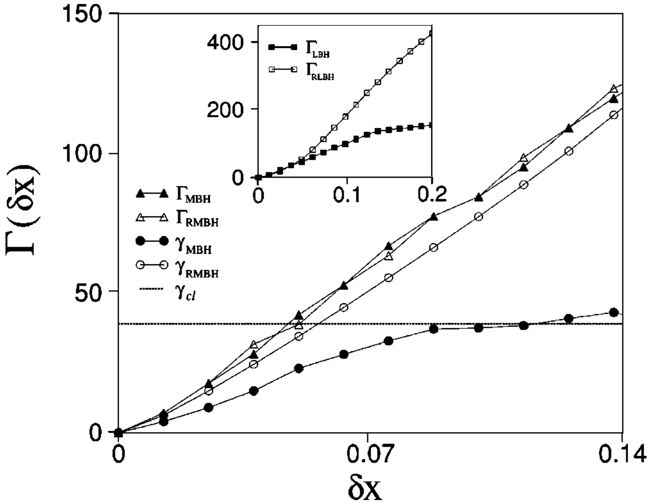

FIG. 2. The LDOS width Γ and the decay constant γ from the MBH/RMBH simulations. The dotted line is the classical Lyapunov exponent. The inset is Γ in the LBH/RLBH case.

the two models are practically indistinguishable. The inset contains plots of $\Gamma(\delta x)$ for the other two models (LBH, RLBH). In later sections we shall discuss the significance of the observed numerical results.

V. THE DECAY OF $P(t;\delta x)$

The theory of the survival amplitude is on firm grounds thanks to the fact that it is the Fourier transform of the LDOS. According to [12] there are three generic δx regimes of behavior:

(i) The standard perturbative regime.

(ii) The Wigner (or Fermi golden rule) regime.

(iii) The nonuniversal (semiclassical) regime.

In the standard perturbative regime ($\delta x \ll \delta x_c$) the LDOS function Eq. (8) is predominantly a Kronecker delta. This characterization constitutes a *definition* of this regime. For an estimate of δx_c in the case of billiards see the Appendix. The survival amplitude is obtained via a Fourier transform of the Kronecker delta dominated LDOS function. This leads to a nonaveraged $m(t;\delta x)$ that does not decay. On the other hand, the E_0 averaged $m(t;\delta x)$ has a Gaussian decay. The latter follows from the observation [1] that the first order correction $E_0(x)-E_0(x_0)$ has typically a Gaussian distribution.

For intermediate values of δx the decay of $P(t;\delta x)$ is typically of exponential type with

$$\gamma=\Gamma(\delta x)/\hbar. \tag{11}$$

This is known as Wigner-type (or as Fermi golden rule) decay. It is a reflection of the Lorentzian-like line shape of the LDOS function. However, for large δx we get into a nonuniversal (semiclassical) regime, where we can apply the "wave-packet dynamics" picture of [11]. Thus we find a semiclassical decay with

$$\gamma=\gamma_{\rm scl}. \tag{12}$$

The Wigner regime, where Eq. (11) holds, is determined [12] by the condition

$$\Gamma(\delta x)\ll\hbar\gamma_{\rm scl}. \tag{13}$$

This inequality can be rewritten as $\delta x<\delta x_{\rm NU}$. The elimination defines a nonuniversal (system specific) parametric scale $\delta x_{\rm NU}$.

In the nonuniversal regime the width of the LDOS is semiclassically determined [12]. In typical cases the width of the LDOS is proportional to the strength of the perturbation, hence

$$\gamma_{\rm scl}\propto\delta x. \tag{14}$$

But in some exceptional cases $\gamma_{\rm scl}$ becomes perturbation independent. Specifically, for billiard systems $1/\gamma_{\rm scl}$ is roughly equal to the mean time between collisions, so we can write

$$\gamma_{\rm scl}\approx\gamma_{\rm cl}. \tag{15}$$

It is important to realize that the perturbation independent decay (PID) of $c(t;\delta x)$ in billiard systems is a reflection of the δx independence of LDOS function $\rho(\omega;\delta x)$ in the nonuniversal regime. See [16] for a numerical study.

VI. THE DECAY OF $M(t;\delta x)$

A mature theory of fidelity is still lacking. However, it has been realized in [3,4] that the same physical picture as in [12] arises: For very small δx we have Gaussian decay (which corresponds to the E_0 averaged decay of the survival amplitude). For intermediate values of δx we have Wigner-type decay with $\gamma=\Gamma(\delta x)/\hbar$. For large δx we enter into the semiclassical regime where one finds "Lyapunov decay" [25] with $\gamma\approx\gamma_{\rm scl}\approx\gamma_{\rm cl}$.

In complete analogy with the case of survival probability studies we can define [via Eq. (13)] an analogous parametric scale [3] that will be denoted by $\delta x_{\rm NUD}$. The semiclassical value ($\gamma_{\rm scl}$) of γ is not necessarily the same for $P(t;\delta x)$ and for $M(t;\delta x)$. Therefore in general $\delta x_{\rm NU}$ and $\delta x_{\rm NUD}$ are not necessarily identical.

For a simple shaped billiard system the γ_{scl} of the survival probability, the γ_{scl} of the fidelity, and the Lyapunov exponent γ_{cl} are all equal to the inverse of the mean collision time. The perturbation parameter δx is defined as the displacement of the billiard wall. In the Appendix we derive the following result:

$$\delta x_{\rm NUD}\sim\delta x_{\rm NU}\sim 2\pi/k, \tag{16}$$

where $2\pi/k$ is the de Broglie wavelength of a particle with mass m, corresponding to the kinetic energy $E_0=(\hbar k)^2/2m$. These results hold for a hard walled billiard.

In the following we want to demonstrate the distinction between $\delta x_{\rm NUD}$ and $\delta x_{\rm NU}$. Therefore we consider a modified billiard Hamiltonian (MBH) for which

$$\delta x_{\rm NUD}\ll\delta x_{\rm NU}. \tag{17}$$

The above inequality reflects the general case, in which the γ_{scl} of $M(t;\delta x)$ is different (smaller) from the γ_{scl} of $P(t;\delta x)$.

The fact that $P(t;\delta x)$ is a special case of $M(t;\delta x)$, and the fact that similar ideas (semiclassical decay versus Wigner-type decay) have emerged in the latter case, naturally suggests that the same physics is concerned. If it were really the "same physics," it would imply that the main features of $M(t;\delta x)$ are determined by a simple-minded theory that involves the LDOS function $\rho(\omega;\delta x)$ in some combination with the LDOS function $\rho(\omega;\text{wpk})$. It is the purpose of the following sections to demonstrate that a simple-minded theory is not enough. The semiclassical PID in the case of $M(t;\delta x)$ necessitates a nontrivial extension of the LDOS parametric theory.

VII. DEFINITION OF THE MODEL

Our model Hamiltonian is the linearized billiard Hamiltonian (LBH) of a stadium system [7]. It can be written as

$$\mathcal{H}=\mathbf{E}+\delta x\mathbf{B}. \tag{18}$$

Here $\mathbf{E}$ is the ordered diagonal matrix $\{E_n(x_0)\}$. The eigenenergies of the quarter stadium billiard, with straight edge $x_0=1$, have been determined numerically. The perturbation due to δx deformation is represented by the matrix $\mathbf{B}$. Also this matrix has been determined numerically as explained in [7].

In the following numerical study we have considered not the LBH, but rather a modified billiard Hamiltonian (MBH), which is obtained from the LBH by the replacement

$$\mathbf{B}_{nm}\mapsto G(n-m)\times\mathbf{B}_{nm}, \tag{19}$$

where $G(n-m)$ is a Gaussian cutoff function. This corresponds physically to having soft walls (for an explanation of this point see Appendix J of [17]). It is important to realize that the "exact" physical interpretation of either the LBH (as an approximation for the billiard Hamiltonian), or the MBH (as a soft wall version of the LBH), is of no importance for the following. The LBH and the MBH are both mathematically "legitimate" Hamiltonians.

In the next section we explain the numerical strategy which we use in order to prove our main point. This incorporates the random matrix theory (RMT) strategy which has been applied in [18] in order to demonstrate that the semiclassical theory and RMT lead to *different nonperturbative limits*. The randomized LBH (RLBH) is obtained by sign randomization of the off-diagonal elements of the $\mathbf{B}$ matrix:

$$\mathbf{B}_{nm}\mapsto\pm\mathbf{B}_{nm} \quad \text{(random sign)}. \tag{20}$$

The randomized MBH (RMBH) is similarly defined. The purpose in making a comparison with a "randomized" Hamiltonian is the ability to distinguish between universal and nonuniversal effects. Making such a distinction is a central theme in the "quantum chaos" literature. Usually such "comparisons" are made in the context of spectral statistics analysis, while here, following [18] we are doing this comparison in the context of quantum dynamics analysis.

VIII. THE NUMERICAL STUDY

The first step of the numerics is to calculate the width $\Gamma(\delta x)$ of the LDOS function $\rho(\omega;\delta x)$. We know from previous studies [12,16] that for a hard-walled billiard system $\Gamma(\delta x)$ shows semiclassical saturation for $\delta x>\delta x_{\text{NU}}$, where δx_{NU} roughly is equal to the de Broglie wavelength [Eq. (16)]. This implies PID for the survival probability. With the LBH we still see (inset of Fig. 2) the reminisces of this saturation. Note that $k\sim50$ and hence $\delta x_{\text{NU}}\sim0.1$. In contrast to that, with the RLBH there is no indication for saturation. This implies that nontrivial correlations of off-diagonal elements play an essential role in the parametric evolution of the LDOS. (See [19] regarding terminology.)

By *modifying* the billiard Hamiltonian we are able to construct an artificial model Hamiltonian (MBH) where the two parametric scales are well separated ($\delta x_{\text{NUD}}\ll\delta x_{\text{NU}}$). Thus within a large intermediate δx range [20] we do not have PID for $P(t;\delta x)$, but we still find PID for $M(t;\delta x)$. See Fig. 2.

In order to prove that the observed PID is not a trivial reflection of $\rho(\omega;\text{wpk})$ we have defined the associated "randomized" Hamiltonian (RMBH). The LDOS functions (8) and (9) are practically *not* affected by the sign-randomization procedure: the sign-randomization procedure has almost no effect on $\Gamma(\delta x)$. In spite of this fact we find that the previously observed PID of $M(t;\delta x)$ goes away: we see (Fig. 2) that for the MBH there is no longer PID in the relevant δx range [20]. This indicates that the PID was of semiclassical "off-diagonal" origin.

We see that both qualitatively and quantitatively the sign-randomization procedure has a big effect on $M(t;\delta x)$. Therefore, we must conclude that the correlations of the off-diagonal terms is still important for the physics of $M(t;\delta x)$. This holds in spite of the fact that the *same* off-diagonal correlations are not important for the LDOS structure. This implies that the theory of $M(t;\delta x)$ necessitates a nontrivial extension of the parametric LDOS theory.

IX. THE SIMPLE-MINDED THEORY

The purpose of the present section is to explain what type of "fidelity physics" can be obtained if we do not take nonuniversal (semiclassical) features of the dynamics into account. Such theory is expected to be valid in case of RMT models. Let $\rho_{eff}(\omega;\delta x)$ be the Fourier transform of $m(t;\delta x)$. It can be written as

$$\rho_{eff}(\omega;\delta x)=\sum_{\omega'} f(\omega')\,\delta(\omega-\omega'), \tag{21}$$

where the summation is over energy differences $\omega'=[E_n(x)-E_m(x_0)]$, and $f(\omega')$ is a product of the overlaps $\langle n(x)|m(x_0)\rangle$, and $\langle m(x_0)|\Psi_0\rangle$ and $\langle\Psi_0|n(x)\rangle$. It is clear that $f(\omega')$ satisfies the sum rule $\Sigma_\omega f(\omega)=1$. On the other hand, if the number of principle components (participation ratio) of the LDOS is N, then the sum over $|f(\omega)|$ gives $N^{1/2}$. Thus we conclude that $f(\omega)$ should have randomlike phase (or randomlike sign) character. Therefore, if we ignore the system specific features, we can regard $f(\omega)$ as the Fourier components of a noisy signal. These Fourier components satisfy

$$\langle f(\omega)\rangle = 0, \tag{22}$$

$$\langle |f(\omega)|^2\rangle = \tilde{\rho}(\omega;\mathrm{wpk})\times\rho(\omega;\delta x), \tag{23}$$

where $\tilde{\rho}$ is, up to normalization, the autoconvolution of $\rho(\omega;\mathrm{wpk})$, and therefore is equal to the Fourier transform of $P(t;\mathrm{wpk})$, and has roughly the same width as $\rho(\omega;\mathrm{wpk})$.

It is worth noticing that for the Lorentzian line shape, which in general is not necessarily the case, Eq. (23) implies that $m(t)$ is characterized by exponential correlations with decay constant $\Gamma/2$. This leads to the decay constant Γ for $M(t)$. The deviation of $\gamma(\delta x)$ from $\Gamma(\delta x)$ in the MBH case cannot be explained by Eq. (23), since the latter does not distinguish between the MBH model and the associated RMBH model. In order to explain the PID in the MBH case it is essential to take into account the nonuniversal (semiclassical) features of the dynamics.

X. THE SEMICLASSICAL THEORY FOR $P(t)$

The semiclassical theory of the survival probability is described within the framework of wave-packet dynamics in Ref. [11]. The short time decay of $c(t;\mathrm{wpk})$ reflects the loss of the overlap between the initial and the evolving wave packets. On the other hand, due to the (inevitable) proximity to periodic orbits, the survival amplitude $c(t;\mathrm{wpk})$ has recurrences. However, because of the (transverse) instability of the classical motion these recurrences are not complete. Consequently the long-time decay may be characterized by the Lyapunov exponent γ_{cl}. Possibly, this "Lyapunov decay" is the simplest example for PID. It is PID because the size of the perturbation ($\|\mathcal{H}-\mathcal{H}_{\mathrm{wpk}}\|$) is not relevant here.

The semiclassical behavior of the survival probability has a reflection in the LDOS structure. A relatively slow "Lyapunov decay" (due to recurrences) implies that the LDOS is "scarred" [11]. Thus the semiclassical LDOS has a "landscape" which is characterized by the energy scale $\hbar\gamma_{cl}$. Note that "scarring," in the mesoscopic physics terminology, is called the weak localization effect.

The above semiclassical picture regarding $c(t;\mathrm{wpk})$ can be extended [12,16,21] to the case of $c(t;\delta x)$, provided δx is large enough. In the other limit, where δx is small, we should be able to use perturbation theory in order to predict the decay rate. Thus we have here a *clash* of two possibilities: having Wigner-type decay with $\gamma=\Gamma(\delta x)/\hbar$, or having nonuniversal decay (NUD) that reflects the semiclassical wave-packet dynamics.

The crossover from the perturbative to the semiclassical regime can be analyzed [12,21] by looking on the parametric evolution of $\rho(\omega;\delta x)$. Depending on δx the LDOS $\rho(\omega;\delta x)$ has (in order of increasing perturbation) standard perturbative structure, core-tail (Lorentzian-like) structure, or purely nonperturbative structure [22]. The width $\Gamma(\delta x)$ of the "core" defines a "window" through which we can view the semiclassical landscape. This landscape is typically characterized by $\hbar\gamma_{cl}$ features, where γ_{cl} is related to the classical dynamics. As δx becomes larger, this window becomes wider, and eventually some of the semiclassical landscape is exposed. Then we say that the LDOS contains a nonuniversal component [22].

XI. THE SEMICLASSICAL THEORY FOR $M(t)$

Whereas Lyapunov decay for $c(t;\delta x)$ is typically a "weak" feature (this is true for generic systems, whereas billiard systems constitute an exception), it is not so for $m(t;\delta x)$. By definition the trajectory of the wave packet is reversed, and therefore the short-time decay due to a loss of wave-packet overlap is avoided. As a result the perturbation independent Lyapunov decay becomes a predominant feature (that does not depend on recurrences). This Lyapunov PID has been discussed in [2].

It is clear, however, that for small δx we can use perturbation theory in order to predict the decay rate of $M(t;\delta x)$. The question that naturally arises, in complete analogy to the $P(t;\delta x)$ case, is how to determine the border δx_{NUD} between the perturbative regime (where we have Wigner-type decay) and the semiclassical regime (where we have NUD).

The natural identification of δx_{NUD} is as the δx for which $\Gamma(\delta x)$ becomes equal to $\hbar\gamma_{scl}$. How is γ_{scl} determined? There are two "mechanisms" that are responsible for the loss of wave-packet overlap. One is indeed related to the instability of the classical motion, while the other is related to the overall energy width of the wave packet.

The survival probability $P(t;\delta x)$ can be regarded as a special case of $M(t;\delta x)$, where the overall energy width of the wave packet is the predominant limiting factor in the decay. The separation between the energy surfaces of $\mathcal{H}$ and of $\mathcal{H}_0$ is proportional to δx. Consequently we typically have $\gamma_{scl}\propto\delta x$.

In the prevailing studies of $M(t;\delta x)$, one assumes *wide* Gaussian wave packets. Therefore the separation between the energy surfaces does not play a major role in the semiclassical analysis. Rather it is the instability of the classical motion that is the predominant limiting factor in the decay. Therefore one typically expects to have $\gamma_{scl}\approx\gamma_{cl}$, which is independent of δx.

XII. CONCLUSIONS

The above discussed criterion for the identification of the nonuniversal regime is in the spirit of spectral statistics studies [23]. In the latter context it is well known that RMT considerations dominate the sub-$\hbar\gamma_{cl}$ energy scale, while nonuniversal corrections dominate the larger energy scales.

In the present paper we have identified the nonuniversal regime for a billiard related model (MBH). The border between the perturbative regime and the nonuniversal regime in the context of $P(t;\delta x)$ is δx_{NU}, while in the context of $M(t;\delta x)$ is δx_{NUD}.

The parametric scales δx_{NU} and δx_{NUD} are similarly defined, but there is an important distinction between them. The first parametric scale marks the exposure of the semiclassical landscape: either that of Eq. (8) or that of Eq. (9).

The second parametric scale, as proved by our numerical strategy, marks the exposure of cross correlations between the corresponding wave amplitudes.

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APPENDIX: THE $\delta X_{\rm NU}$ FOR BILLIARDS

The nonuniversal regime for billiard systems has been identified in [12,16]. Here we would like to complete the missing steps in the generalization of this result. We use the same notations as in [12,16].

In the general case [16] the band profile of the **B** matrix is determined by the semiclassical formula [24]

$$\langle|\mathbf{B}_{nm}|^2\rangle \approx \frac{\Delta}{2\pi\hbar}\tilde{C}\left(\frac{E_n-E_m}{\hbar}\right), \quad \text{(A1)}$$

where $\Delta \propto 1/k^{d-2}$ is the mean level spacing, k is the wave number, and $d=2$ is the dimensionality of the billiard system. The power spectrum of the motion

$$\tilde{C}(\omega) = \text{const} \times k^{3+g}/\omega^g \quad \text{(A2)}$$

is the Fourier transform of a classical correlation function. Here $g=0$ corresponds to strong chaos assumptions, while $0<g<1$ is more appropriate for our type of system due to the bouncing ball effect. The width of the LDOS is determined using a procedure which is explained in [12], leading to Eq. (9) there. Namely,

$$\Gamma(\delta x) = \Delta \times (\delta x/\delta x_c)^{2/(1+g)}, \quad \text{(A3)}$$

where $\delta x_c \propto k^{-[(1-g)+(1+g)d]/2}$ is the generalization of Eq. (8) of [12].

From Eq. (A3) it is clear that δx_c should be interpreted as the deformation which is needed in order to mix neighboring levels. In the standard perturbative regime ($\delta x \ll \delta x_c$) first order perturbation theory is valid as a global approximation. Otherwise, if $\delta x > \delta x_c$, we should distinguish between a nonperturbative "core" of width Γ and perturbative "tails" that lay outside of it.

The expression for Γ can be rewritten as

$$\Gamma(\delta x) \approx \hbar\gamma_{cl} \times (k\,\delta x)^{2/(1+g)} \quad \text{(A4)}$$

where $\gamma_{cl} \propto k$ is roughly the inverse of the ballistic time. In our numerical analysis [7] we find that $\Gamma \approx 0.36k^2 \times \delta x$, corresponding to $g=1$. The nonuniversal scale $\delta x_{\rm NU}$, as well as $\delta x_{\rm NUD}$, is determined by the requirement $\Gamma(\delta x) = \hbar\gamma_{cl}$. Hence we get Eq. (16), which holds *irrespective* of the g value. The latter claim has been stated in [12] without a proof.

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the LDOS has a purely ‘‘nonperturbative’’ structure. For large enough δx the nonperturbative structure becomes purely semiclassical.

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