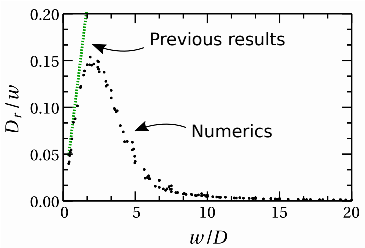
The Lognormal-Like Statistics of a Stochastic Squeeze Process

Dekel Shapira

Ben Gurion University

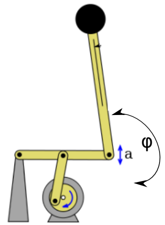


[1] D. Shapira and D. Cohen (Phys. Rev. E, 2017)

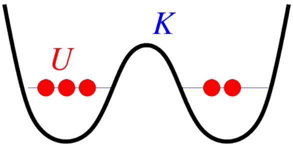
Motivation

Stabilize an inverted pendulum ($\varphi = \pi$):

- Periodic driving (Kapitza)
- Noisy driving



Stabilize condensate in BHH



G. Gordon and G. Kurizki Phys. Rev. Lett. (**2006**).
Y. Khodorkovsky, G. Kurizki, A. Vardi Phys. Rev. Lett. (**2008**), Phys. Rev. (**2009**).
C. Khripkov, A. Vardi, D. Cohen Phys.Rev. A (**2012**).

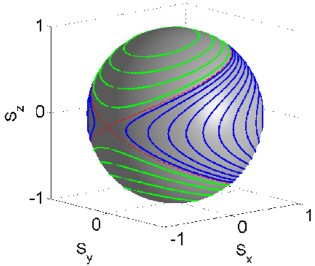
Bose-Hubbard dimer is like a pendulum

$$\mathcal{H} = \frac{U}{2}\sum_{i=1,2}\hat{n}_i(\hat{n}_i-1) - \frac{K}{2}(\hat{a}_2^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_2)$$

N particles in a double well
is like spin $j = N/2$ system

$$\mathcal{H} = U\hat{J}_z^2 - K\hat{J}_x$$

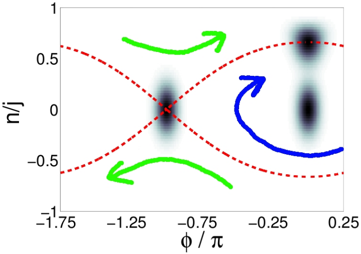
$\hat{J}_z =$ occupation difference



Analogous to Josephson junction
if the occupation difference $\ll N/2$

$$\mathcal{H}(n,\varphi) = Un^2 - \frac{NK}{2}\cos(\varphi)$$

$\hat{n} =$ occupation difference



- Condense all bosons upper orbital (π state) $\sim (a_1^\dagger - a_2^\dagger)^N|0\rangle$

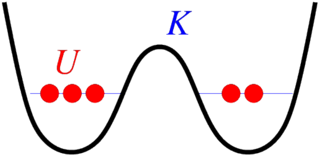
Stabilizing using noisy driving

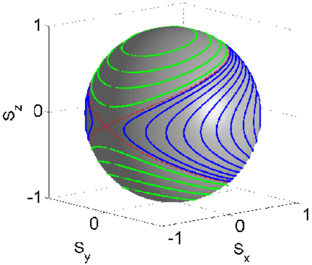
- Condense all bosons upper orbital (π state)
- Stabilizing: noisy driving (QZE):

$$\mathcal{H} \;=\; U\hat{J}_z^2 \;-\; [K + \Omega(t)]\hat{J}_x$$

Dynamics near the π-point

- Gaussian around hyperbolic point
- Equations of motion





The model

- Stochastic Differential Equation in 2D
- 2-parameter* model: w, D
- Langevin equation (Stratonovich):

$$\begin{aligned} \dot{x} &= \ \ wx - \Omega(t)y \\ \dot{y} &= -wy + \Omega(t)x \end{aligned}$$

$$\langle \Omega(t)\Omega(t') \rangle = 2D\delta(t-t')$$

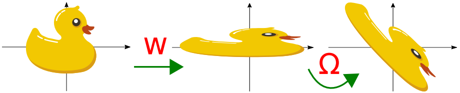
- x, y coordinates $\sim$ major axes of the hyprebolic point

* scaling time: $t \to wt$. One parameter (w/D).

The model

Langevin equation (Stratonovich):

$$\dot{x} = wx - \Omega(t)y$$
$$\dot{y} = -wy + \Omega(t)x$$
$$\langle \Omega(t)\Omega(t') \rangle = 2D\delta(t - t')$$



w generates a squeeze: $x = x_0 e^{wt}$, $y = y_0 e^{-wt}$
$\log(r) \sim wt$

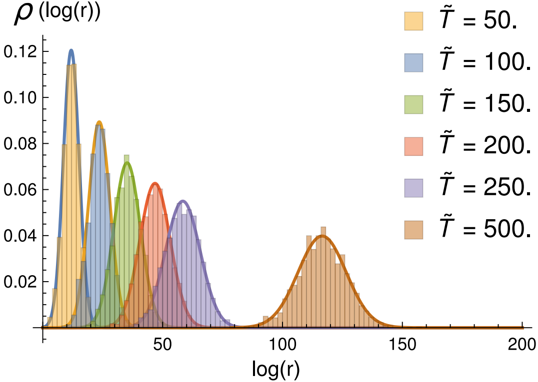
$\Omega(t)$ generates rotations
$\log(r) \sim$ const

Interplay between w and D
$\log(r) \sim$ Diffusive spreading (D_r) with drift (w_r)

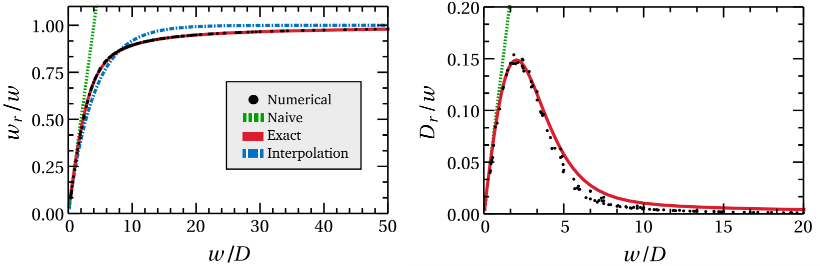
$r^2 = x^2 + y^2$

Drift and Diffusion of $\log(r)$

- Initial condition r_0
- $\log(r) \rightarrow$ Normally distributed
- $\log(r) \rightarrow$ Drifts & diffuses
- $\mu = w_r t \quad \sigma^2 = D_r t$
- w_r and D_r - functions of w and D



$r^2 = x^2 + y^2$



- Naive result: $w_r/w = w/4D$ and $D_r/w = w/8D$
- $w_r \;=\; wX_1 \approx w\left[1 - \exp\left(-\frac{w}{4D}\right)\right]$
- $D_r \;=\; -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \Delta_n X_n w$

$$X_n \;\equiv\; \langle \cos(2n\varphi) \rangle_\infty \;=\; I_n\left(\frac{w}{2D}\right) / I_0\left(\frac{w}{2D}\right)$$

$$\Delta_n \;\equiv\; C_n(0) \;=\; \frac{1}{2}\left(X_{n+1} + X_{n-1}\right) - X_n X_1$$

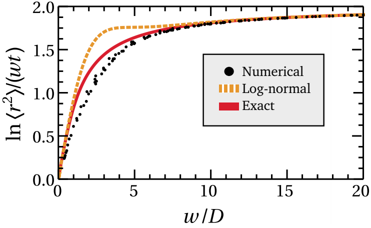
w - Squeeze $\quad D$ - Rotation $\quad I_n(x)$ - Modified Bessel

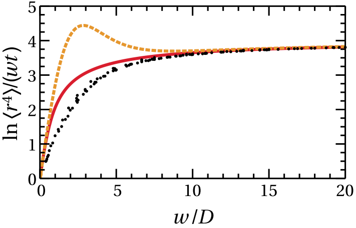
Moments

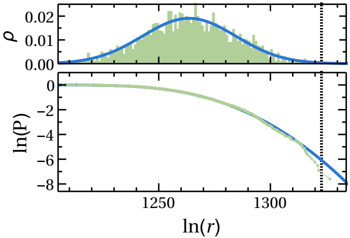
$\log(r) \sim$ Normal: moments of r can be obtained (orange):

$$\log\langle r^n\rangle \;=\; \mu n + \frac{1}{2}\sigma^2 n^2 \;=\; n w_r t + \frac{1}{2} n^2 D_r t$$

Fails when diffusion is big

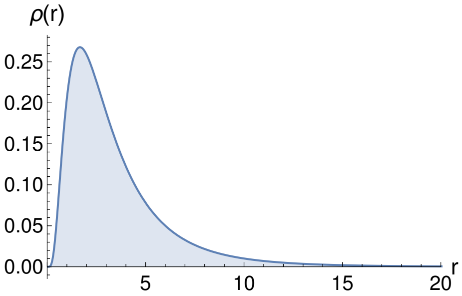


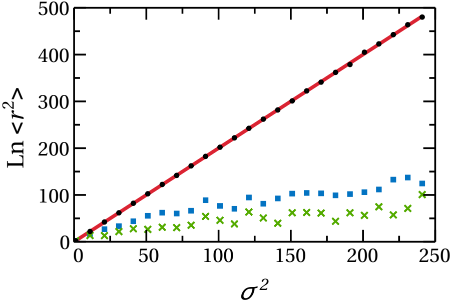




$$\log\langle r^2\rangle/(wt) \sim$$
$$2\left[(1+(D/w)^2)^{1/2} - D/w\right]$$

- $\log(r) \sim N(0, \sigma^2)$
- $\log \langle r^2 \rangle = 2\sigma^2$





- Red line - analytical
- Green and Blue, sample mean: 10^2 and 10^5 samples

Part II - Derivations

Drift and Diffusion for the radial coordinates

In polar coordinates $\mathbf{r}$, φ:

$$\begin{aligned}\dot{\varphi} &= -w\sin(2\varphi) + \Omega(t)\\ \dot{r} &= [w\cos(2\varphi)]\, r\\ \langle\Omega(t)\Omega(t')\rangle &= 2D\delta(t-t')\end{aligned}$$

$\ln(r)$ performs Brownian motion:

$$\frac{d}{dt}\ln(r) = w\cos(2\varphi)$$

After transient time $\ln(r)$ drifts (w_r) and diffuses (D_r)

Need angular distribution $\rho_{(}\varphi, t)$

Angular spreading

Langevin:

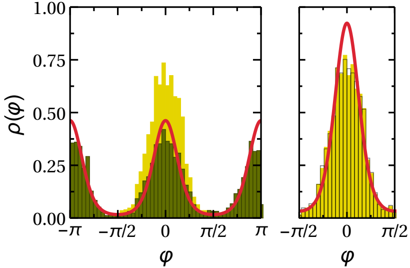
$$\dot{\varphi} \;=\; -w\sin(2\varphi) \;+\; \Omega(t) \qquad\qquad \langle \Omega(t)\Omega(t')\rangle = 2D\delta(t-t')$$

Corresponding Fokker-Planck:

$$\frac{\partial \rho}{\partial t} \;=\; \frac{\partial}{\partial \varphi}\left[\left(D\frac{\partial}{\partial \varphi} + w\sin(2\varphi)\right)\rho\right]$$

Steady state ("detailed balance"):

$$\rho(\varphi) \;\propto\; \exp\left[\frac{w}{2D}\cos(2\varphi)\right]$$



Yellow - $\varphi_0 = 0$
Green - $\varphi_0 = \pi/2$

Drift and Diffusion for the radial coordinates

$\ln(r)$ performs Brownian motion:

$$\frac{d}{dt}\ln(r(t)) \;=\; w\cos(2\varphi)$$

Drift:

$$w_r = \lim_{t\to\infty}\frac{d}{dt}\langle\ln(r)\rangle = \langle w\cos(2\varphi)\rangle_{t=\infty} = w\frac{I_1(w/2D)}{I_0(w/2D)}$$

Diffusion:

$$D_r = \frac{1}{2}\lim_{t\to\infty}\frac{d}{dt}\text{Var}\,\ln(r)$$

$$D_r = \int_0^\infty\left[w^2\left\langle\cos(2\varphi)\cos(2\varphi_t)\right\rangle_\infty - w_r^2\right]dt$$

Diffusion coefficient (sketch)

Reminder: $D_r = w^2 \int_0^\infty \left[\langle \cos(2\varphi) \cos(2\varphi_t) \rangle_\infty - \langle \cos(2\varphi) \rangle_\infty^2 \right] dt$

- Use FPE to obtain differential recursive equations involving the integrand.
- Integrate to obtain non-homogenous recursive equation for D_r.
- Solve non-homogeneous equations
  (Use homogeneous solution)
- Obtain D_r:

$$D_r = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \Delta_n X_n w$$

$$X_n \equiv \langle \cos(2n\varphi) \rangle_\infty = I_n\left(\frac{w}{2D}\right) / I_0\left(\frac{w}{2D}\right)$$

$$\Delta_n \equiv C_n(0) = \frac{1}{2}\left(X_{n+1} + X_{n-1}\right) - X_n X_1$$

w - Squeeze D - Rotation $I_n(x)$ - Modified Bessel

(even) moments of r

- Find $\langle r^2 \rangle$, $\langle r^4 \rangle$
- Use **cartesian** coordinates: x, y
- Use FPE to obtain equation of motion for the the moments of x, y

Langevin equation (Stratonovich):

$$\dot{x_j} = v_j + g_j \Omega(t) \qquad \langle \Omega(t)\Omega(t') \rangle = 2D\delta(t - t')$$

with $v_j = (wx, -wy)$ and $g_j = (-y, x)$.

Associated FPE:

$$\frac{d\rho}{dt} = -\frac{\partial}{\partial x_j}\left[v_j\rho - g_j D \frac{\partial}{\partial x_i}(g_i\rho) \right]$$

Moments equation ($X = X(x_i)$):

$$\frac{d}{dt}\langle X \rangle = \left\langle \frac{\partial X}{\partial x_j}\left(v_j + \frac{\partial g_j}{\partial x_i} D g_i \right) \right\rangle$$

(even) moments of r

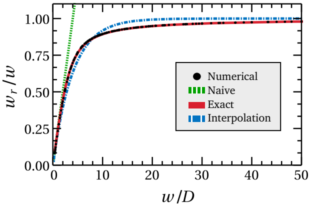
- $X = x^2,\ y^2,\ xy$
- Equations of motion from FPE:

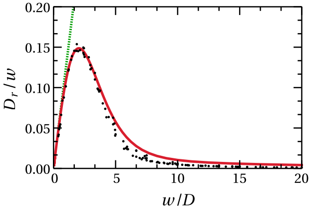
$$\frac{d}{dt}\begin{pmatrix} \langle x^2 \rangle \\ \langle y^2 \rangle \end{pmatrix} = \Big[-2D + 2D\boldsymbol{\sigma}_1 + 2w\boldsymbol{\sigma}_3 \Big] \begin{pmatrix} \langle x^2 \rangle \\ \langle y^2 \rangle \end{pmatrix}$$

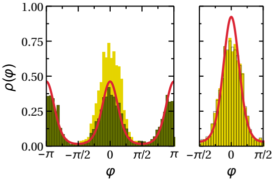
$$\frac{d}{dt}\langle xy \rangle = -4D\langle xy \rangle$$

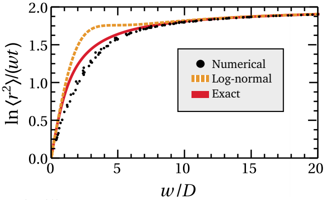
- Initial isotropic distribution, $\langle x^2 \rangle_0 = \langle y^2 \rangle_0 = r_0^2$:

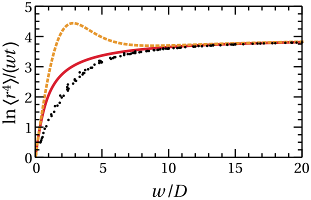
$$\langle r^2 \rangle_t \approx \frac{r_0^2}{2}\left(1 + \frac{D}{\sqrt{w^2+D^2}}\right) \times \exp\left[2\left((w^2 + D^2)^{1/2} - D\right)t\right]$$

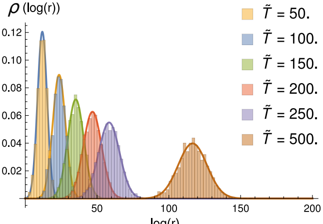

Numerical
Naive
Exact
Interpolation
w_r / w
w / D

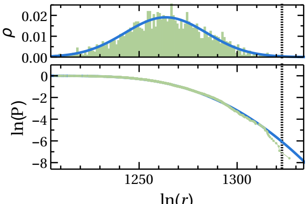

D_r / w
w / D

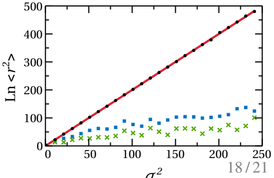

ρ(φ)
φ


Numerical
Log-normal
Exact
ln⟨r²⟩/(wt)
w / D


ln⟨r⁴⟩/(wt)
w / D


ρ (log(r))
T̃ = 50.
T̃ = 100.
T̃ = 150.
T̃ = 200.
T̃ = 250.
T̃ = 500.
log(r)


ρ
ln(P)
ln(r)


Ln <r²>
σ²

Extra

Diffusion coefficient - recursive equation

$$D_r = w^2 \int_0^\infty \left[\langle \cos(2\varphi) \cos(2\varphi_t) \rangle_\infty - \langle \cos(2\varphi) \rangle_\infty^2 \right] dt \qquad \text{(Reminder)}$$

Initial delta distribution $\rho_0(\varphi) = \delta(\varphi - \varphi_0)$. Moments:

$$x_n(t;\varphi_0) \equiv \langle \cos(2n\,\varphi_t) \rangle_0 = \langle \cos(2n\,\varphi) \rangle_t = \int \cos(2n\,\varphi)\, \rho(\varphi, t|\varphi_0) d\varphi$$

$$\frac{d}{dt} x_n = -4Dn^2\, x_n + wn\,(x_{n-1} - x_{n+1}) \qquad \text{(FPE and integration by parts)}$$

Steady state $X_n \equiv x_n(\infty)$

$$C_n(t) = \langle x_n(t;\varphi_0) \cos(2\varphi_0) \rangle_\infty - X_n X_1 \qquad \text{(Conditional probability)}$$

$C_1(t)$ is the integrand

$C_n(t)$ obeys the same recusrive equation as x_n

$$c_n \equiv \int_0^\infty C_n(t) dt \qquad \text{(Obtain Recursive equation with } D_r\text{)}$$

$$4Dn^2\, c_n - wn\,(c_{n-1} - c_{n+1}) = \Delta_n \qquad (\mathbf{c_0 = c_\infty = 0})$$

Solve and obtain $D_r = w^2 c_1$

$$D_r = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \Delta_n X_n w \qquad (I_n(x) - \text{Modified Bessel})$$

$$X_n = I_n\left(\frac{w}{2D}\right) / I_0\left(\frac{w}{2D}\right) \qquad \Delta_n \equiv C_n(0) = \frac{1}{2}(X_{n+1} + X_{n-1}) - X_n X_1$$

Solving the recursive equation

Solve:

$$-W_n^+ c_{n+1} + \Lambda_n c_n - W_n^- c_{n-1} \; = \; \Delta_n \qquad (\mathbf{c_0 = c_\infty = 0})$$

We have $W_n^+ = -wn \qquad W_n^- = wn \qquad \Lambda_n = 4Dn^2$

First solve homogenous equation. Denote solution X_n:

$$-W_n^+ X_{n+1} + \Lambda_n X_n - W_n^- X_{n-1} \; = \; 0 \qquad (X_n \text{ is steady state solution})$$

Otherwise use $R_n = X_n / X_{n-1}$ to reduce to first-order

Rewrite using: $c_n \; = \; X_n \tilde{c}_n$

$$-W_n^+ X_{n+1} \tilde{c}_{n+1} + \Lambda_n X_n \tilde{c}_n - W_n^- X_{n-1} \tilde{c}_{n-1} \; = \; \Delta_n$$

$$-W_n^+ X_{n+1} (\tilde{c}_{n+1} - \tilde{c}_n) + W_n^- X_{n-1} (\tilde{c}_n - \tilde{c}_{n-1}) \; = \; \Delta_n \qquad \text{(Use homogenous solution)}$$

Reduce to first order equation:

$$-W_n^+ X_{n+1} \tilde{a}_{n+1} + W_n^- X_{n-1} \tilde{a}_n \; = \; \Delta_n \qquad (\tilde{a}_n \; = \; \tilde{c}_n - \tilde{c}_{n-1})$$

$$a_n = \tilde{R}_n \left[\tilde{\Delta}_n + a_{n+1} \right] \qquad \left(\tilde{R}_n = \frac{W_n^+}{W_n^-} R_n, \;\; \tilde{\Delta}_n = \frac{\Delta_n}{W_n^+}, \;\; a_n = X_n \tilde{a}_n \right)$$

Solve a_n with $a_\infty = 0$

Finally:

$$c_n = R_n c_{n-1} + a_n \qquad (c_0 = 0)$$

$$c_1 = a_1 = \tilde{R}_1 \tilde{\Delta}_1 + \tilde{R}_1 \tilde{R}_2 \tilde{\Delta}_2 + \ldots = \sum_{n=1}^{\infty} (-1)^n X_n \qquad (\tilde{R}_1 \cdots \tilde{R}_n = (-1)^n X_n)$$