BEC dynamics in a few site systems

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The Bose-Hubbard Hamiltonian (BHH) for a dimer

$\mathcal{H} = \sum_{i=1,2} \left[ \mathcal{E}_i \hat{n}_i + \frac{U}{2} \hat{n}_i(\hat{n}_i - 1) \right] - \frac{K}{2} (\hat{a}_2^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2)$

$N$ particles in a double well is like spin $j = N/2$ system

$\mathcal{H} = -\mathcal{E} \hat{J}_z + U \hat{J}_z^2 - K \hat{J}_x + \text{const}$

Classical phase space

$\mathcal{H}(\theta, \varphi) = \frac{NK}{2} \left[ \frac{1}{2} u (\cos \theta)^2 - \varepsilon \cos \theta - \sin \theta \cos \varphi \right]$

$\mathcal{H}(\hat{n}, \varphi) = \text{(similar to Josephson/pendulum Hamiltonian)}$

$\hat{J}_z = (N/2) \cos(\theta) = \hat{n} = \text{occupation difference}$

$\hat{J}_x \approx (N/2) \sin(\theta) \cos(\varphi), \quad \varphi = \text{relative phase}$

Rabi regime: $u < 1$ (no islands)

Josephson regime: $1 < u < N^2$ (sea, islands, separatrix)

Fock regime: $u > N^2$ (empty sea)

$K = \text{hopping}$

$U = \text{interaction}$

$\mathcal{E} = \mathcal{E}_2 - \mathcal{E}_1 = \text{bias}$

$u \equiv \frac{NU}{K}, \quad \varepsilon \equiv \frac{\varepsilon}{K}$

Assuming $u > 1$ and $|\varepsilon| < \varepsilon_c$

Sea, Islands, Separatrix

$\varepsilon_c = \left( u^{2/3} - 1 \right)^{3/2}$

$A_c \approx 4\pi \left( 1 - u^{-2/3} \right)^{3/2}$
WKB quantization (Josephson regime)

\[ h = \text{Planck cell area in steradians} = \frac{4\pi}{N+1} \]

\[ A(E_\nu) = \left(\frac{1}{2} + \nu\right)h \quad \nu = 0, 1, 2, 3, \ldots \]

\[ \omega(E) \equiv \frac{dE}{dv} = \left[\frac{1}{h}A'(E)\right]^{-1} \]

\[ \omega_K \approx K = \text{Rabi Frequency} \]

\[ \omega_J \approx \sqrt{NUK} = \sqrt{u} \omega_K \]

\[ \omega_+ \approx NU = u \omega_K \]

\[ \omega_x \approx \left[\log\left(\frac{N^2}{u}\right)\right]^{-1} \omega_J \]
The preparations

Eigenstates $|E_{\nu}\rangle$ are like strips along contour lines of $\mathcal{H}$.

Coherent state $|\theta \varphi\rangle$ is like a minimal Gaussian wavepacket.

Fock state $|n\rangle$ is like equi-latitude annulus.

Fock $n=0$ preparation - exactly half of the particles in each site
Fock coherent $\theta=0$ preparation - all particles occupy the left site
Coherent $\varphi=0$ preparation - all particles occupy the symmetric orbital
Coherent $\varphi=\pi$ preparation - all particles occupy the antisymmetric orbital
Wavepacket dynamics

MeanField theory (GPE) = classical evolution of a point in phase space
SemiClassical theory = classical evolution of a distribution in phase space
Quantum theory = recurrences, fluctuations (WKB is very good)

Any operator $\hat{A}$ can be presented by the phase-space function $A_W(\Omega)$

$$\langle \hat{A} \rangle = \text{trace}[\hat{\rho} \hat{A}] = \int \frac{d\Omega}{\hbar} \rho_W(\Omega) A_W(\Omega)$$
\[ P(E) = \text{the LDOS of the preparations} \]

\[ \sim \left[ 1 - \left( \frac{2E}{NK} \right)^2 \right]^{-1/2} \]

\[ \sim \text{BesselI} \left[ \frac{E - E_\nu}{NU} \right] \]

\[ \sim \text{BesselK} \left[ \frac{E - E_x}{NU} \right] \]

\[ \sim \exp \left[ -\frac{1}{N} \left( \frac{E - E_x}{\omega J} \right)^2 \right] \]
\[ M = \text{the participation number} \]

\[ M = \left[ \sum_{\nu} P(E_{\nu})^2 \right]^{-1} = \text{number of participating levels in the LDOS} \]

In the semiclassical analysis there is scaling with respect to \((u/N)^{1/2}\) which is \([\text{the width of the wavepacket}] / [\text{the width of the separatix}]\)

\[ M \approx \left[ \log \left( \frac{N}{u} \right) \right] \sqrt{u} \quad [\text{Pi}] \quad \sim (\text{quasi periodic large fluctuations}) \]

\[ M \approx \left[ \log \left( \frac{N}{u} \right) \right] \sqrt{N} \quad [\text{Edge}] \quad \sim (\text{easy to get the classical limit}) \]
Recurrences and fluctuations

\[ \vec{S} = \langle \vec{J} \rangle / (N/2) = (S_x, S_y, S_z) = \text{Bloch vector} \]

OccupationDifference = \( (N/2) \langle S_z \rangle \)

OneBodyPurity = \( (1/2) \left[ 1 + \langle S_x \rangle^2 + \langle S_y \rangle^2 + \langle S_z \rangle^2 \right] \)

FringeVisibility = \( \left[ \langle S_x \rangle^2 + \langle S_y \rangle^2 \right]^{1/2} \)

Spectral analysis of the fluctuations: dependence on \( u \) and on \( N \), various preparations.
Temporal behavior

Zero prep

Pi prep

Edge prep

$S_x (t \text{ scaled})$

$C_x (\omega)$

$P(E) (E \text{ scaled})$

$\omega / \omega_J$
The spectral content of $S_x$

\[ \omega_{osc} \approx 2\omega_J \quad \text{[Zero]} \]

\[ \omega_{osc} \approx 1 \times \left[ \log \left( \frac{N}{u} \right) \right]^{-1} 2\omega_J \quad \text{[Pi]} \]

\[ \omega_{osc} \approx 2 \times \left[ \log \left( \frac{N}{u} \right) \right]^{-1} 2\omega_J \quad \text{[Edge]} \]

\[ \omega_{osc} \approx \left( \frac{u}{N} \right)^{1/2} 2\omega_J \quad \text{[} u \gg N \text{]} \]
**Fluctuations of $S_x$**

Naive expectation: phase spreading diminishes coherence. In the Fock regime $\langle S_x \rangle_\infty \approx 0$ [Leggett’s “phase diffusion”]

In the Josephson regime $\langle S_x \rangle_\infty$ is determined by $u/N$.

- $\overline{S_x} \approx 1/3$ [TwinFock]
- $\overline{S_x} \approx \exp[-(u/N)]$ [Zero]
- $\overline{S_x} \approx -1 - 4/\log\left[\frac{1}{32}(u/N)\right]$ [Pi]

**RMS**

\[
\text{RMS} \left[ \langle A \rangle_t \right] = \left[ \frac{1}{M} \int \tilde{C}_{cl}(\omega) d\omega \right]^{1/2}
\]

- $\text{RMS} [ S_x(t) ] \sim N^{-1/4}$ [Edge]
- $\text{RMS} [ S_x(t) ] \sim (\log(N))^{-1/2}$ [Pi]

**TwinFock:** Self induced coherence leading to $\overline{S_x} \approx 1/3$.

**Zero:** Coherence maintained if $u/N < 1$ (phase locking).

**Pi:** Fluctuations are suppressed by $u$.

**Edge:** Fluctuations are suppressed by $N$ (classical limit).
The many body Landau-Zener transition

Dynamical scenarios:
adiabatic/diabatic/sudden
Adiabtic-diabatic (quantum) crossover
Diabatic-sudden (semiclassical) crossover
Sub-binomial scaling of $\text{Var}(n)$ versus $\langle n \rangle$
Quantum Stirring in a 3 site system

Control parameters:

\[ X_1 = \left( \frac{1}{k_2} - \frac{1}{k_1} \right) \]
\[ X_2 = \mathcal{E}_0 \quad (\mathcal{E}_1=\mathcal{E}_2=0) \]
\[ X = (X_1, X_2) \]

\[ U = \text{the inter-atomic interaction} \]

\[ \hat{\mathcal{H}} = \sum_{i=0}^{2} \mathcal{E}_i \hat{n}_i + \frac{U}{2} \sum_{i=0}^{2} \hat{n}_i(\hat{n}_i - 1) - k_c(\hat{b}_1^\dagger \hat{b}_2 + \hat{b}_2^\dagger \hat{b}_1) - k_1(\hat{b}_0^\dagger \hat{b}_1 + \hat{b}_1^\dagger \hat{b}_0) - k_2(\hat{b}_0^\dagger \hat{b}_2 + \hat{b}_2^\dagger \hat{b}_0) \]

The induced current:
\[ I = -G \dot{\mathcal{E}} \quad (G = G_2) \]

The pumped particles:
\[ Q = \int I dt = \int G \cdot dX \quad \text{(per cycle)} \]
Stirring of BEC

strong attractive interaction: classical ball dynamics
negligible interaction \(|U| \ll \kappa/N\): mega-crossing
weak repulsive interaction: gradual crossing
strong repulsive interaction \(U \gg N\kappa\): sequential crossing
Results for the geometric conductance

\[ G(R) = -N \frac{(k_1^2 - k_2^2)/2}{[(\varepsilon - \varepsilon_-)^2 + 2(k_1 + k_2)^2]^{3/2}} \]

\[ G(J) \approx - \left[ \frac{k_1 - k_2}{k_1 + k_2} \right] \frac{1}{3U} \]

\[ G(F) = - \left( \frac{k_1 - k_2}{k_1 + k_2} \right) \sum_{n=1}^{N} \frac{(\delta \varepsilon_n)^2}{[(\varepsilon - \varepsilon_n)^2 + (2\delta \varepsilon_n)^2]^{3/2}}, \]

where:

\( R = \) Rabi regime \((U \ll \kappa/N)\)

\( J = \) Josephson regime \((\kappa/N \ll U \ll N\kappa)\)

\( F = \) Fock regime \((U \gg N\kappa)\)

Observation:

It is possible to pump \( Q \gg N \) per cycle.
Summary

- Semiclassical and WKB analysis of the dynamics
- Occupation statistics in a time dependent scenario
- The adiabatic / diabatic / sudden crossovers
- Quantum stirring: mega / gradual / sequential crossings