The rate of heating in vibrating billiards

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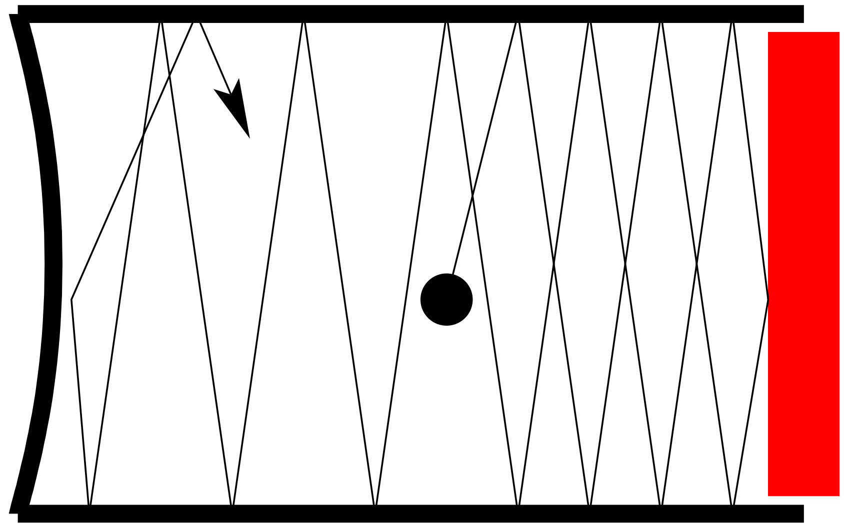
$$\mathcal{H}_{\text{total}} = \{E_n\} - f(t)\{V_{nm}\}$$



$ISF, $GIF, $DIP, $BSF

Dynamics and spectral intensities

$$\mathrm{FT}\Big[\langle\psi(0)|\psi(t)\rangle\Big] \quad\sim\quad \Big|\langle E_n|\psi\rangle\Big|^2$$

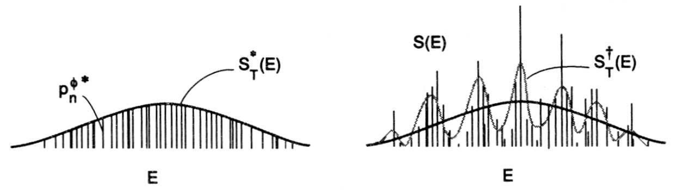


Fig. 38. Ideally ergodic (left) and typically found (right) spectral intensities and envelopes. Both spectra have the same low resolution envelope.

Heller, Les Houches 1989

Analogous relation between correlation function and band-profile:

$$\mathrm{FT}\Big[\langle V(0)V(t)\rangle\Big] \quad\sim\quad \Big|\langle E_n|\hat{V}|E_m\rangle\Big|^2$$

[Feingold-Peres]

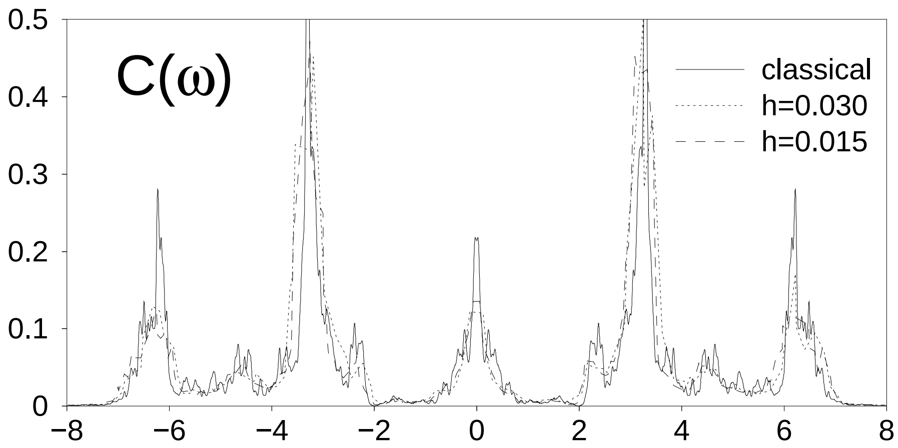
$$\tilde{C}(\omega) = \overline{\sum_n |V_{nm}|^2 \; 2\pi\delta(\omega - (E_n - E_m))}$$

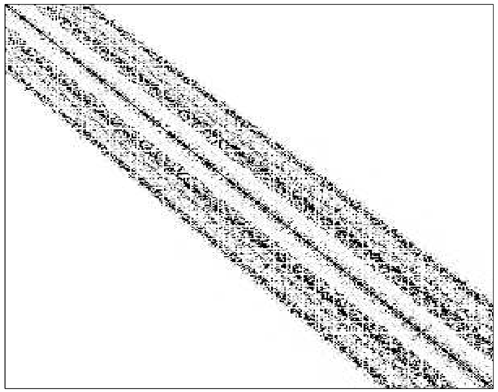
$$\leadsto \leadsto \leadsto \leadsto \leadsto \leadsto \leadsto \leadsto \qquad |V_{nm}|^2 \approx (2\pi\varrho)^{-1}\tilde{C}_{cl}(E_n - E_m)$$

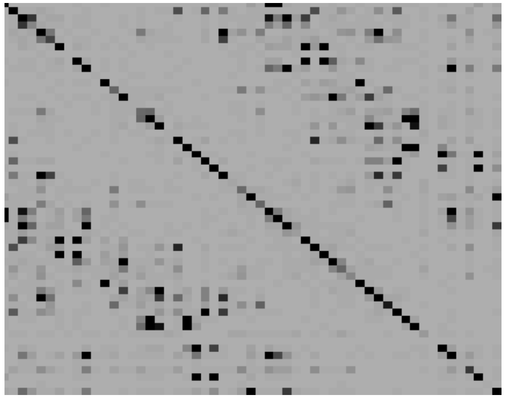
Bandprofile, sparsity and texture

$$|V_{nm}|^2 \approx (2\pi\varrho)^{-1}\tilde{C}_{cl}(E_n - E_m)$$

Hard Qchaos

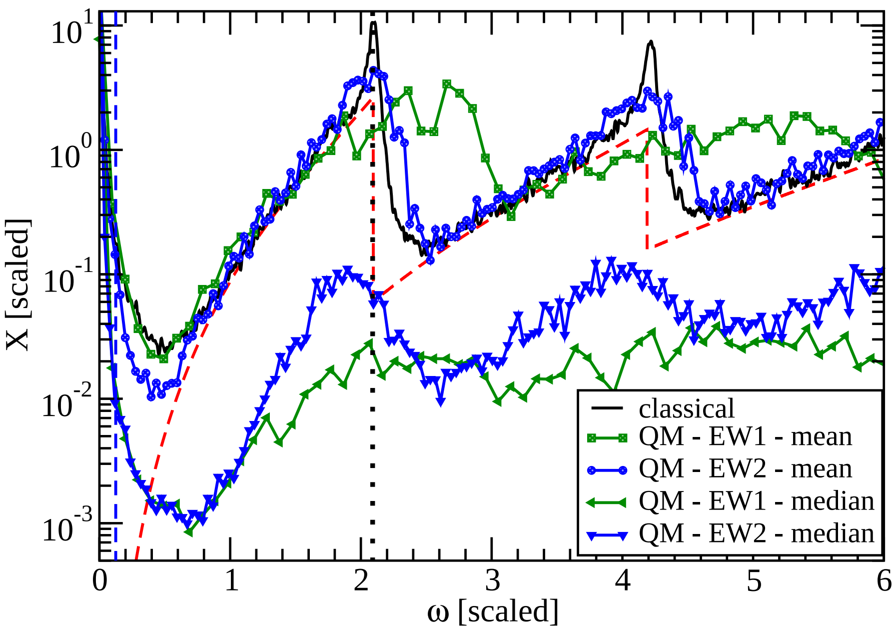


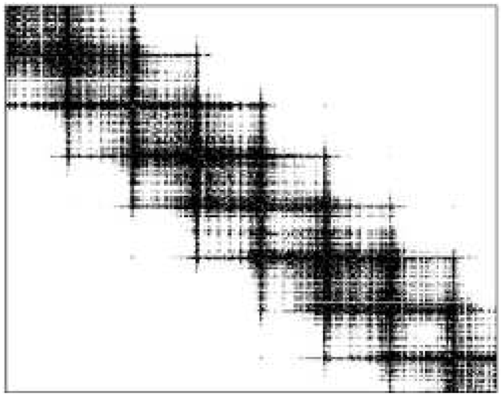




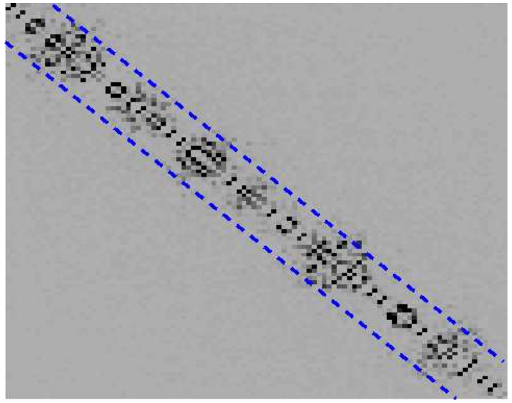
Weak Qchaos

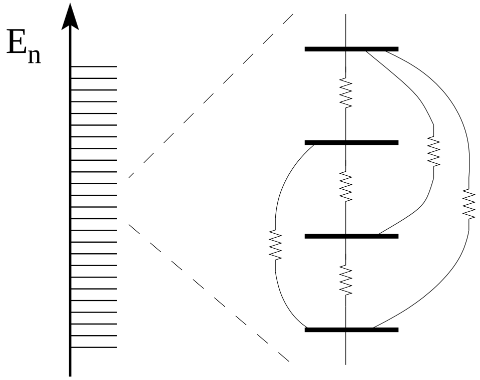
[median ≪ mean]





$\{|V_{nm}|^2\}$ as a random matrix $\boldsymbol{X} = \{x\}$

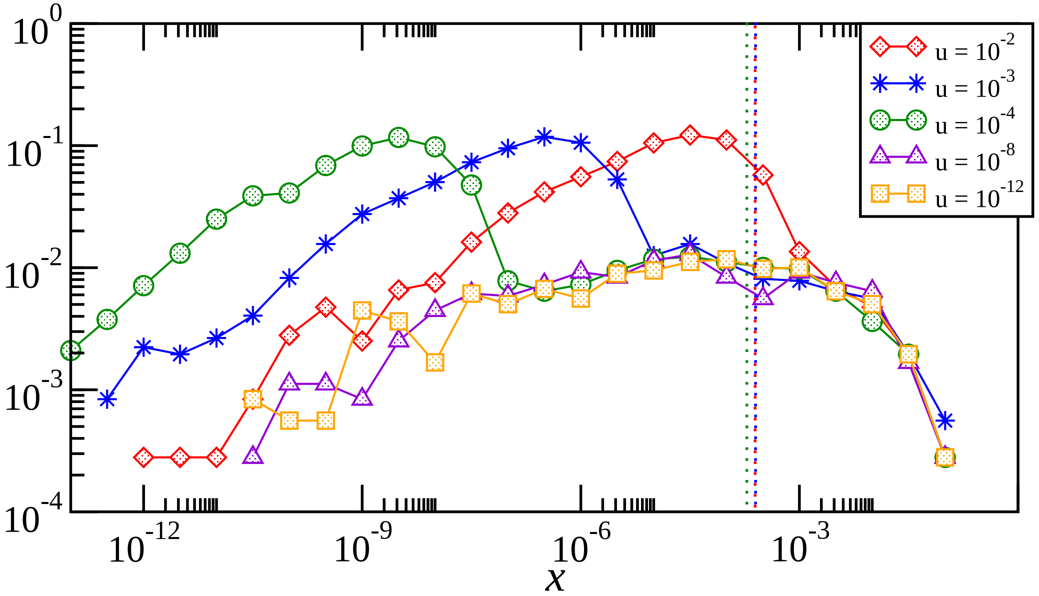




$$s[\boldsymbol{X}] \equiv \frac{\mathrm{PN}[\boldsymbol{X}]}{\mathrm{PN}[\boldsymbol{X}^{\mathrm{unf}}]} = \text{sparsity}$$

$$g_s[\boldsymbol{X}] \equiv \frac{\langle\langle \boldsymbol{X} \rangle\rangle_s}{\langle\langle \boldsymbol{X} \rangle\rangle_a} = \text{connectivity}$$

Histogram of x :



$x \sim \mathrm{LogNormal}$

For a random sparse matrix:

$$s, \; g_s \ll 1$$

For a uniform (along diagonals):

$$s = g_s = 1$$

For a Gaussian matrix:

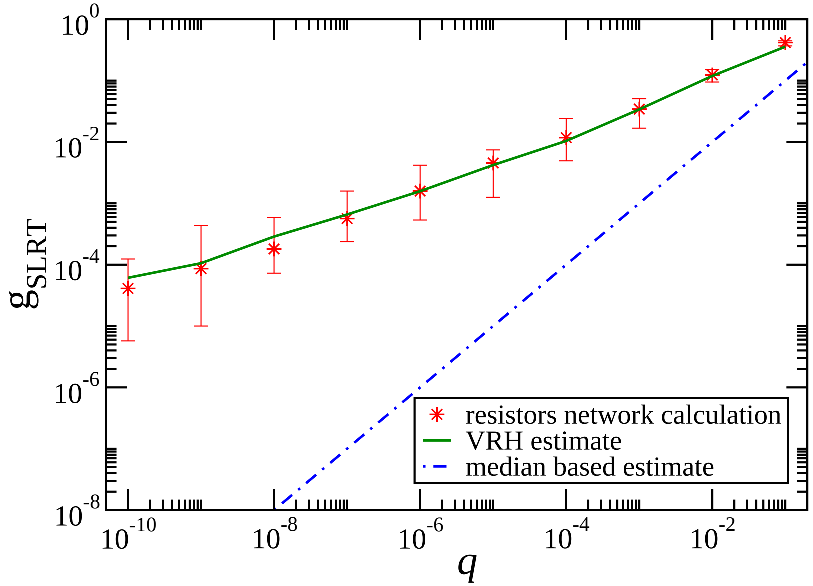
$$s = 1/3, \quad g_s \sim 1$$

RMT modeling, generalized VRH approx scheme

- Log-normal RMT modeling

For the rectangular $\tilde{S}(\omega)$ of width ω_c

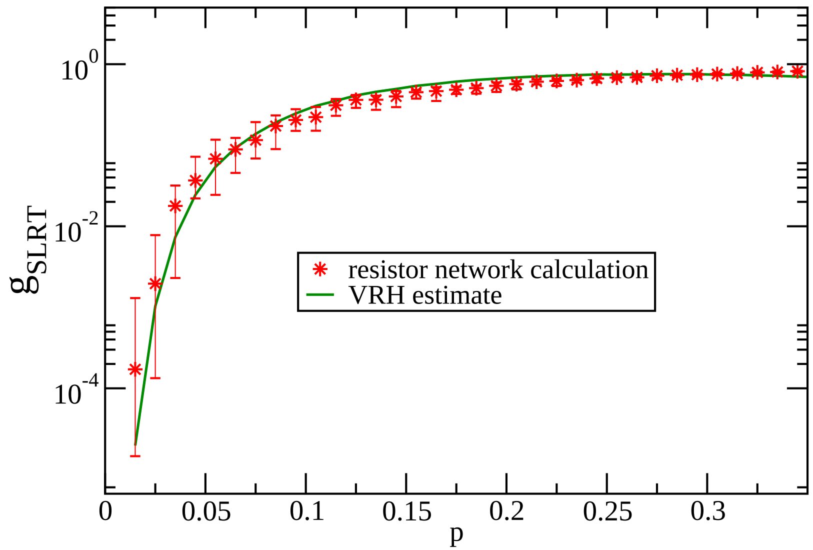
$$g_{\text{SLRT}} \approx q \exp\left[2\sqrt{-\ln q \, \ln(\omega_c/\Delta)}\right]$$



- Log-box RMT modeling

For the exponential $\tilde{S}(\omega)$ of width ω_c

$$g_{\text{SLRT}} \approx \frac{1}{p} \exp\left[-2\sqrt{\frac{\Delta}{p\,\omega_c}}\right]$$



Digression: Generalized VRH

Definition of the typical matrix element for a range ω transition:

$$\left(\frac{\omega}{\Delta}\right) \text{Prob}\Big(x > x_\omega\Big) \;\sim\; 1$$

In the standard-like case (ring with strong disorder):

$$x_\omega \;\approx\; v_{\text{F}}^2 \;\exp\left(\frac{\Delta_l}{|\omega|}\right)$$

[corresponding to a log-box distribution]

An example for the power spectrum of the driving:

$$\tilde{S}(\omega) \;\propto\; \exp\left(-\frac{|\omega|}{T}\right)$$

[here the temparature $T \Longleftrightarrow \omega_c$]

Generalized VRH estimate:

$$D_{\text{SLRT}} \;\approx\; \int x_\omega \; \tilde{S}(\omega) d\omega$$

[should be contrasted with]

$$D_{\text{LRT}} = \int \tilde{C}(\omega) \; \tilde{S}(\omega) d\omega$$

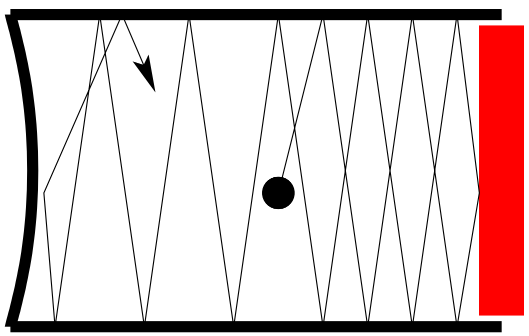
In the standard-like case (ring with strong disorder):

$$D_{\text{SLRT}} \;\approx\; \int \exp\left(\frac{\Delta_l}{|\omega|}\right) \exp\left(-\frac{|\omega|}{T}\right) \; d\omega$$

The rate of heating: LRT and SLRT predictions

$$\mathcal{H}_{\text{total}} = \{E_n\} - f(t)\{V_{nm}\}$$

$f(t) \equiv X(t) - X_0$ = low freq noisy driving



$\rightsquigarrow$ diffusion in energy space:

$$D_0 = \frac{4}{3\pi}\frac{\mathsf{M}^2 v_{\text{E}}^3}{L_x}\overline{\dot{X}^2}$$

$\rightsquigarrow$ energy absorption:

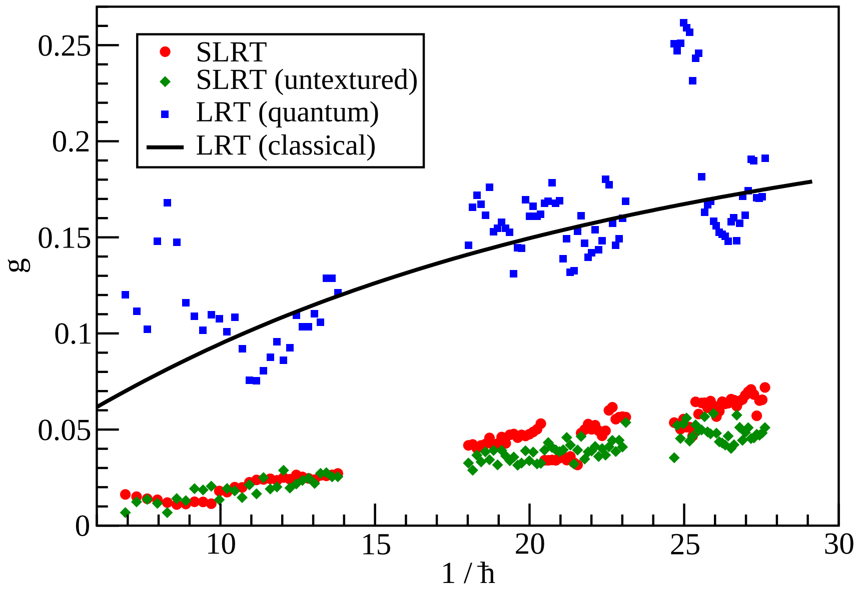
$$\dot{E} = (\text{particles/energy}) \times D$$

Beyond the "Wall Formula"

[Beyond the "Drude Formula"]

$$D_{\text{LRT}} = g_c\, D_0 \qquad [\text{"classical"}]$$

$$D_{\text{SLRT}} = g_s\, D_{\text{LRT}} \qquad [\text{"quantum"}]$$



LRT applies if the driven transitions are slower than the environmental relaxation, else SLRT applies

Perspective and references

The classical LRT approach: Ott, Brown, Grebogi, Wilkinson, Jarzynski, Robbins, Berry, Cohen

The Wall formula (I): Blocki, Boneh, Nix, Randrup, Robel, Sierk, Swiatecki, Koonin

The Wall formula (II): Barnett, Cohen, Heller [1] - regarding g_c

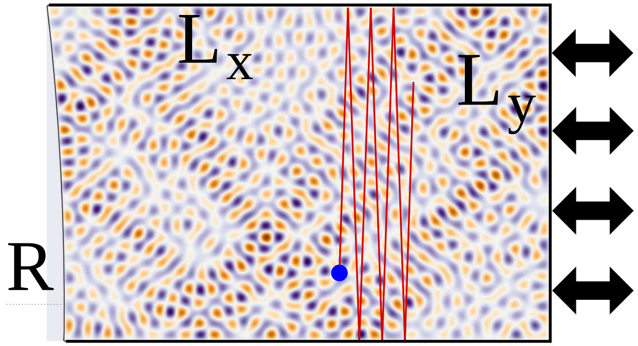
Semi Linear response theory: Cohen, Kottos, Schanz... [2-6]

Billiards with vibrating walls: Stotland, Cohen, Davidson, Pecora [7,8] - regarding g_s

Sparsity: Austin, Wilkinson, Prosen, Robnik, Alhassid, Levine, Fyodorov, Chubykalo, Izrailev, Casati

$$u \quad = \quad (t_{\mathrm{R}} \ / \ t_{\mathrm{L}})^{-1} \quad = \quad (R/L)^{-1} \quad = \quad \text{deformation}$$

$$\hbar \quad = \quad \lambda_{\mathrm{E}} \ / \ L \quad = \quad 2\pi/(k_{\mathrm{E}}L) \quad = \quad \text{function of } E$$



[1] A. Barnett, D. Cohen, E.J. Heller (PRL 2000, JPA 2000)

[2] D. Cohen, T. Kottos, H. Schanz (JPA 2006)

[3] S. Bandopadhyay, Y. Etzioni, D. Cohen (EPL 2006)

[4] M. Wilkinson, B. Mehlig, D. Cohen (EPL 2006)

[5] A. Stotland, R. Budoyo, T. Peer, T. Kottos, D. Cohen (JPA/FTC 2008)

[6] A. Stotland, T. Kottos, D. Cohen (PRB 2010)

[7] A. Stotland, D. Cohen, N. Davidson (EPL 2009)

[8] A. Stotland, L.M. Pecora, D. Cohen (arXiv 2010)

Heating of particles by "shaking" the box

$$\mathcal{H}_{\text{total}} \approx \mathcal{H} + f(t)V$$

$f(t) \equiv X(t) - X_0$ = low freq noisy driving

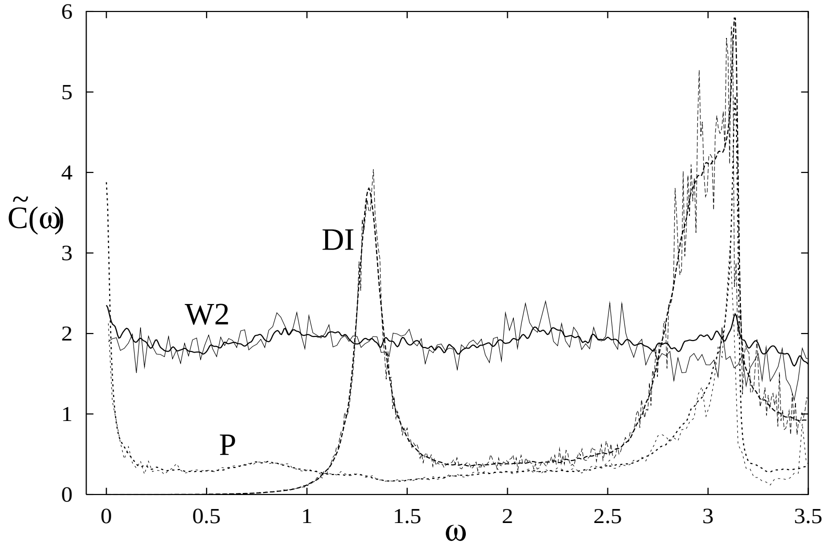
$$\tilde{C}(\omega) = \text{FT}\ \langle V(t)V(0)\rangle$$

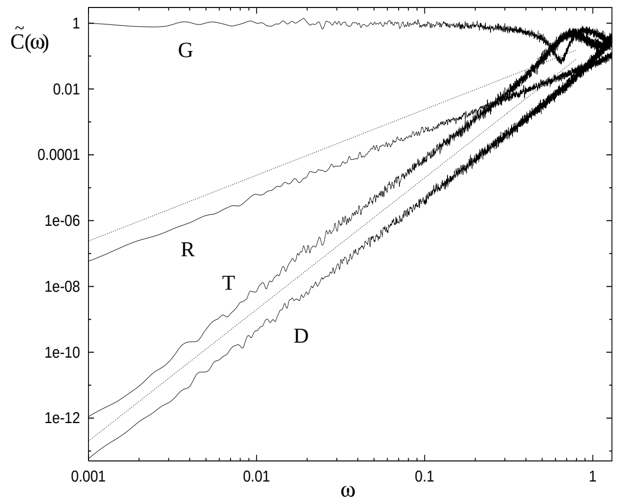
$$\tilde{S}(\omega) = \text{FT}\ \langle \dot{f}(t)\dot{f}(0)\rangle$$

Kubo formula:

$$\boldsymbol{D} = \int_0^\infty \tilde{C}(\omega)\tilde{S}(\omega)d\omega = g_c \frac{4}{3\pi}\frac{\mathsf{M}^2 v_{\text{E}}^3}{L_x}\overline{\dot{X}^2}$$

- $g_c \sim 1$ for "wiggle" deformation.
- $g_c \gg 1$ for "piston" type deformation.
- $g_c \ll 1$ for dilations, translations and rotations.





Barnett, Cohen, Heller (PRL 2000, JPA 2000)

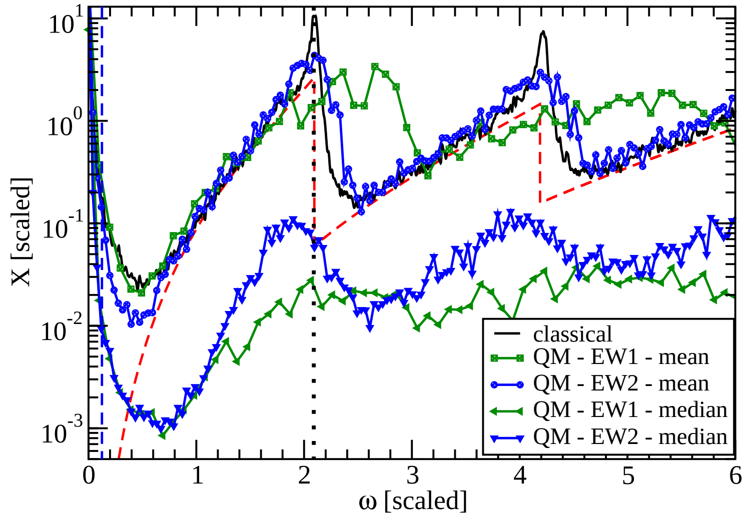
Heating of particles by “vibrating” a piston

$$\mathcal{H}_{\text{total}} \approx \Big[\mathcal{H}_0 + U\Big] + f(t)V$$

$$\mathcal{H} \;=\; \text{rectangular } (L_x \times L_y)$$

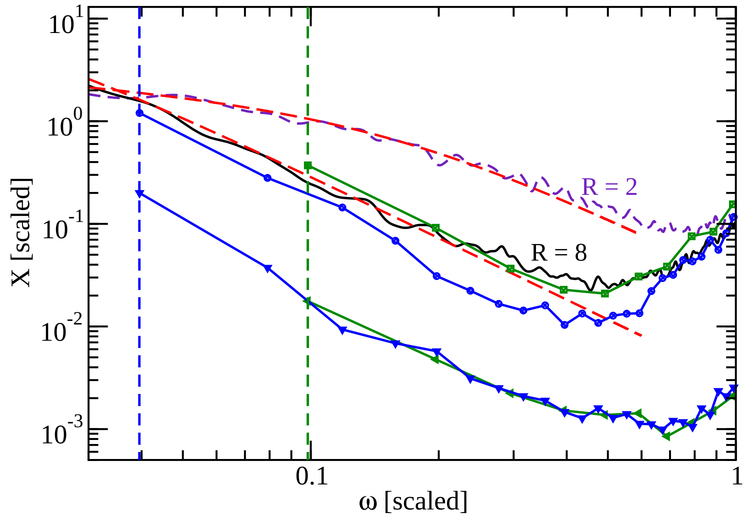
$$U \;=\; \text{deformation } (u = L/R)$$

$$\tilde{C}(\omega) \;=\; \text{FT } \langle V(t)V(0)\rangle$$



$$C(\omega \gg 1/t_{\text{L}}) = \frac{8}{3\pi}\frac{\mathsf{M}^2 v_{\text{E}}^3}{L_x} \;\equiv\; C_\infty$$

$$\tilde{C}(\omega \ll 1/t_{\text{L}}) \approx C_\infty \times \left(\frac{1}{u}\right) \times \ln\left(\frac{2}{\omega t_R}\right)$$



Kubo formula:

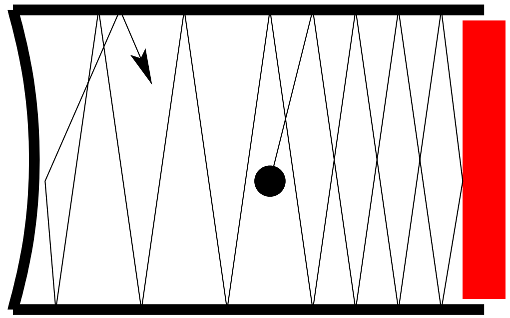
$$\boldsymbol{D} = \int_0^\infty \tilde{C}(\omega)\tilde{S}(\omega)d\omega \;=\; g_c\,\frac{4}{3\pi}\frac{\mathsf{M}^2 v_{\text{E}}^3}{L_x}\,\overline{\dot{X}^2}$$

Stotland, Pecora, Cohen (2010)

The sparsity of the perturbation matrix

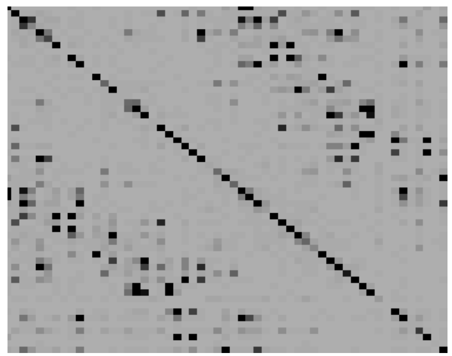
The Hamiltonian in the $\boldsymbol{n} = (n_x, n_y)$ basis:

$$\mathcal{H} = \mathrm{diag}\{E_{\boldsymbol{n}}\} + u\{U_{\boldsymbol{nm}}\} + f(t)\{V_{\boldsymbol{nm}}\}$$



The matrix elements for the wall displacement:

$$V_{\boldsymbol{nm}} \;=\; -\delta_{n_y,m_y} \times \frac{\pi^2}{\mathsf{M} L_x^3} n_x m_x \qquad \text{[sparse]}$$

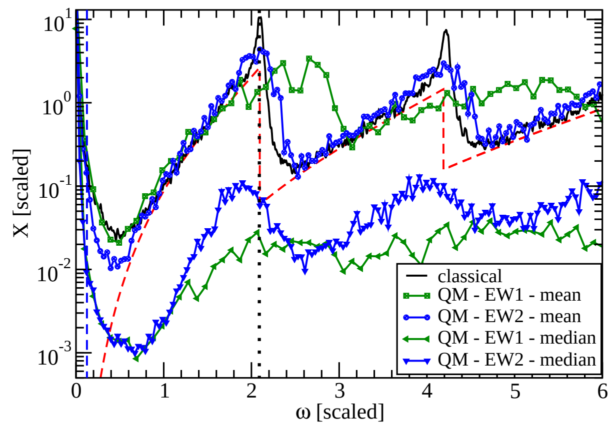


The Hamiltonian in the E_n basis:

$$\mathcal{H} = \mathrm{diag}\{E_n\} + f(t)\{V_{nm}\}$$

The Kubo formula (LRT):

$$\boldsymbol{D} \;=\; \pi \varrho_{\mathrm{E}} \, \langle\langle |V_{mn}|^2 \rangle\rangle_a \, \overline{\dot{X}^2} \;=\; g_c \boldsymbol{D}_0$$



The resistor network calculation

$$\mathcal{H} = \{E_n\} - f(t)\{V_{nm}\}$$

$$\boldsymbol{D} \;=\; \boldsymbol{G}\;\overline{\dot{f}^2}$$

$$\boldsymbol{G}_{\mathrm{LRT}} \;=\; \pi\varrho\;\langle\langle|V_{nm}|^2\rangle\rangle_a$$

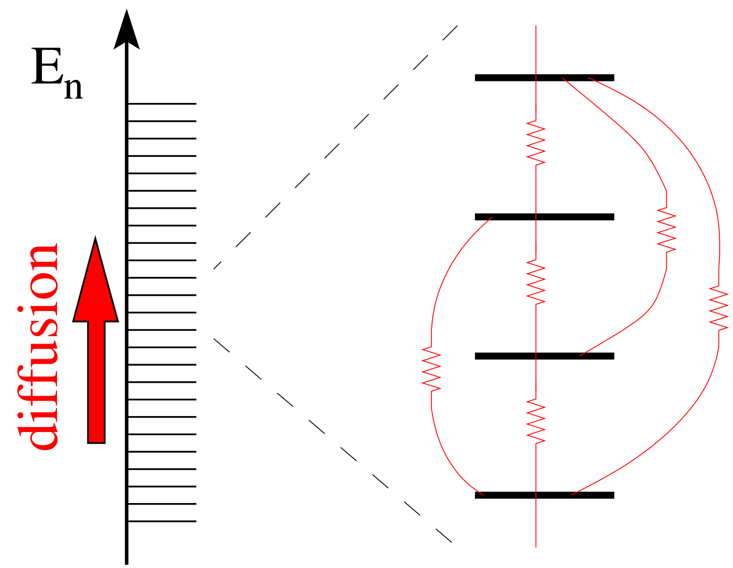
$$\boldsymbol{G}_{\mathrm{SLRT}} \;=\; \pi\varrho\;\langle\langle|V_{nm}|^2\rangle\rangle_s$$

$$g_s \;\equiv\; \frac{\langle\langle|V_{nm}|^2\rangle\rangle_s}{\langle\langle|V_{nm}|^2\rangle\rangle_a}$$

LRT applies if the driven transitions are slower than the environmental relaxation

$$\mathrm{g}_{nm} = 2\varrho^{-3}\;\frac{|V_{nm}|^2}{(E_n-E_m)^2}\;\delta_0(E_n-E_m)$$

$\langle\langle|V_{nm}|^2\rangle\rangle_s \equiv$ inverse resistivity



Digression: the Fermi golden rule picture

Master equation:

$$\frac{dp_n}{dt} = -\sum_m w_{nm}(p_n - p_m)$$

The Hamiltonian in the standard representation:

$$\mathcal{H} = \{E_n\} - f(t)\{V_{nm}\}$$

The transformed Hamiltonian:

$$\tilde{\mathcal{H}} = \{E_n\} - \dot{f}(t)\left\{\frac{iV_{nm}}{E_n - E_m}\right\}$$

$$\tilde{S}(\omega) \equiv \text{FT}\,\langle \dot{f}(t)\dot{f}(0)\rangle = 2\pi\overline{|\dot{f}|^2}\,\delta_0(E_n{-}E_m)$$

The FGR transition rate due to the low frequency noisy driving:

$$w_{nm} = \left|\frac{V_{nm}}{E_n - E_m}\right|^2 \tilde{S}(E_n{-}E_m) \equiv \pi\varrho^3\,\mathrm{g}_{nm}\,\overline{\dot{f}^2}$$

The LRT / SLRT formula

$$\boldsymbol{D} = \text{average}\left[\frac{1}{2}\sum_n (E_n{-}E_m)^2\, w_{nm}\right] = \pi\varrho\,\langle\langle|V_{nm}|^2\rangle\rangle \times \overline{\dot{f}^2} \equiv \boldsymbol{G}\,\overline{\dot{f}^2}$$

Digression: random walk and the calculation of the diffusion coefficient

$w_{nm} =$ probability to hop from m to n per step.

$$\text{Var}(n) = \sum_n [w_{nm}t]\,(n-m)^2 \equiv 2Dt$$

For n.n. hopping with rate w we get $D = w$.

The continuity equation:

$$\frac{\partial p_n}{\partial t} = -\frac{\partial}{\partial n} J_n$$

Fick's law:

$$J_n = -D\frac{\partial}{\partial n} p_n$$

The diffusion equation:

$$\frac{\partial p_n}{\partial t} = D\frac{\partial^2}{\partial n^2} p_n$$

If we have a sample of length N then

$$J = -\frac{D}{N} \times [p_N - p_0]$$

$$D/N = \text{inverse resistance of the chain}$$

If the w are not the same:

$$\frac{D}{N} = \left[\sum_{n=1}^{N} \frac{1}{w_{n,n-1}}\right]^{-1}$$

Hence, for n.n. hopping

$$D = \langle\langle w \rangle\rangle_{\text{harmonic}}$$

FGR: $w_{nm} \sim |V_{nm}|^2$

$$D = \langle\langle |V_{nm}|^2 \rangle\rangle$$

SLRT *vs* LRT

$$\mathcal{H}(X(t)) \approx \mathcal{H}_0 + f(t)V$$
$$f(t) = X(t) - X_0$$

$$\tilde{C}(\omega) = \text{FT } \langle V(t)V(0)\rangle$$
$$\tilde{S}(\omega) = \text{FT } \langle \dot{f}(t)\dot{f}(0)\rangle$$

Kubo formula:

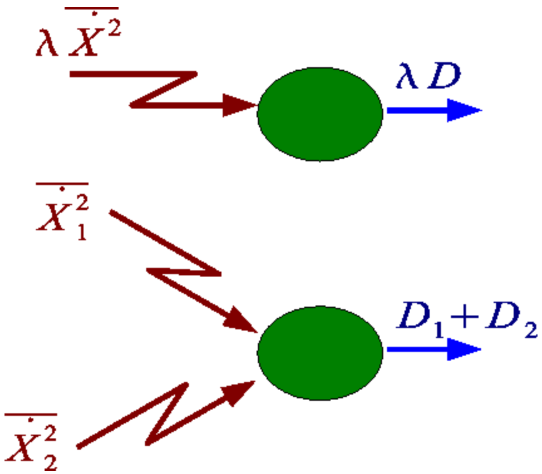
$$\boldsymbol{D} = \int \tilde{C}(\omega)\tilde{S}(\omega)d\omega$$

SLRT example:

$$\boldsymbol{D} = \left[\int R(\omega)\left[\tilde{S}(\omega)\right]^{-1} d\omega\right]^{-1}$$

Linear response implies

$$\tilde{S}(\omega) \mapsto \lambda\tilde{S}(\omega) \implies \boldsymbol{D} \mapsto \lambda\boldsymbol{D}$$
$$\tilde{S}(\omega) \mapsto \sum_i \tilde{S}_i(\omega) \implies \boldsymbol{D} \mapsto \sum_i \boldsymbol{D}_i$$



The weak quantum chaos regime

Consider $\mathcal{H}(u)$, where u is a control parameter.

Generically there are 3 parametric regimes [1]:

- First order perturbation theory regime
- Wigner / Fermi-Golden-Rule / Kubo regime
- Non-perturbative / SC / Lyapunov regime

Deforming a chaotic billiard [2,3]

The Wigner regime: $u_c < u < u_b$

$u_c = \hbar^{3/2}$ mixing of levels starts

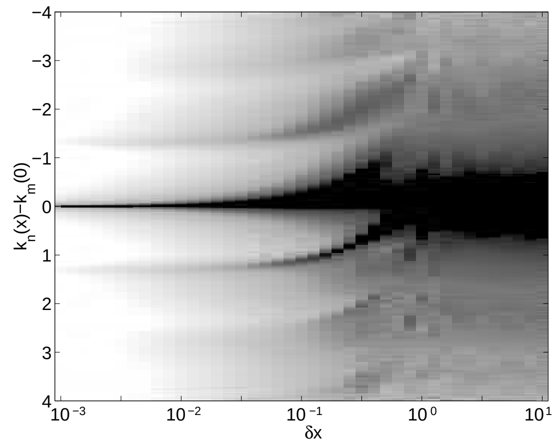
$u_b = \hbar^1$ mixing saturates ($\sim$bandwidth)

Deforming a rectangular billiard [4]

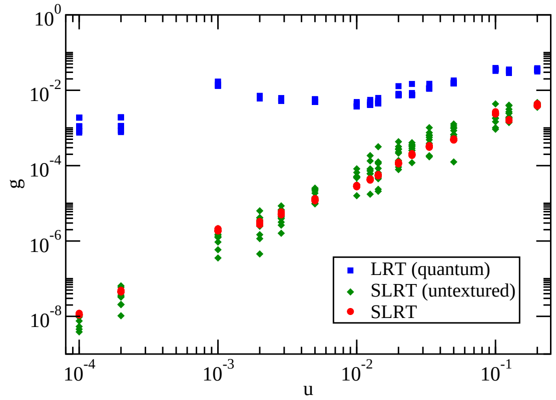
The WQC regime: $u_c < u < u_s$

$u_c = \hbar^2$ mixing of levels starts

$u_s = \hbar^{1/2}$ mixing saturates ($g_s \sim 1$)



mixed levels $\approx (u/u_c)^2$



Mixing is non-uniform

[1] Cohen (PRL 1999, Annals 2000)
[2] Cohen and Heller (PRL 2000)
[3] Cohen, Barnett, Heller (PRE 2001)
[4] Stotland, Pecora, Cohen (2010)

Estimates for an experiment

Consider r^{77}Rb atoms, say ^{77}Rb at $T = 0.1\,\mu$K $\quad\rightsquigarrow\quad$ $\lambda_E = 1\,\mu$m

Linear size of the trap $L = 10\mu$m $\quad\rightsquigarrow\quad$ $h = \lambda_E/L = 0.01$,

SLRT suppression factor for $u \sim 10\%$ deformation is $g_s \sim 0.1$

- Ballistic frequency $\omega_L \approx 220\,$Hz
- Lyapunov frequency $\omega_R \approx 70\,$Hz, $\quad$ Driving $\omega_c \sim \omega_R$
- Level spacing $\omega_0 \approx 7.5\,$Hz

FGR condition: $\quad D/\omega_0^3 < (\omega_c/\omega_0)^{\text{power}}$, $\quad$ power$= 2, 3$

Measurability condition: $\quad D/(T^2\omega_L) > 10^{-3}$

Conclusions

(*) Wigner ($\sim$ 1955):
The perturbation is represented by a random matrix whose elements are taken from a Gaussian distribution.

Not always...

1. "weak quantum chaos" $\Longrightarrow$ (log-wide distribution).
2. The heating process $\sim$ a percolation problem.
3. Resistors network calculation to get G_{SLRT}.
4. Generalization of the VRH estimate
5. SLRT is essential whenever the distribution of matrix elements is wide ("sparsity") or if the matrix has "texture".