

Quantum response of weakly chaotic systems

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Abstract. - Chaotic systems, that have small Lyapunov exponent, do not obey the common random matrix theory predictions within a wide “weak quantum chaos” regime. This leads to a novel prediction for the rate of heating for cold atoms in optical billiards with vibrating walls. The Hamiltonian matrix of the driven system does not look like taken from a Gaussian ensemble, but rather it is very sparse. This sparsity can be characterized by parameters s and g_s that reflect the percentage of large elements, and their connectivity respectively. For g we use a resistor network calculation that has direct relation to the semi-linear response characteristics of the system.

The heating of particles in a box with vibrating walls is a prototype problem for exploring the limitations of linear response theory (LRT) and the quantum-to-classical correspondence (QCC) principle. In the experimental arena this topic arises in the theory of *nuclear friction* [1], and more recently in the studies of cold atoms that are trapped in *optical billiards* [2]. It is also related to the analysis of mesoscopic conductance of ballistic rings [3]. In typical circumstances the classical analysis predicts an absorption coefficient that is determined by the Kubo formula [4–8], leading to the “Wall formula” in the nuclear context, or to the analogous “Drude formula” in the mesoscopic context. The question arises [6–12], are there circumstances in which the quantum theory leads to a novel anomalous result that does not resemble the semiclassical prediction? The established examples for such anomalies assume “microscopic” circumstances where QCC is a priori not expected. In this Letter we demonstrate that a quantum anomaly shows up even in genuine “mesoscopic” circumstances where QCC would be expected by common wisdom.

We consider a weakly chaotic billiard, say Fig. 1, that has linear size L and a convex wall of radius R . The Hamiltonian can be written schematically as

$$\mathcal{H}_{\text{total}} = \mathcal{H} - f(t)F = \mathcal{H}_0 + U - f(t)F \quad (1)$$

where \mathcal{H}_0 describes a non-deformed rectangular box, U describes the deformation of the fixed (left) wall, and F is the perturbation due to the displacement $f(t)$ of the moving (right) wall which can be regarded as a piston. In the classical analysis, if one assumes that the colli-

sions with the walls are uncorrelated, one obtains that the absorption coefficient is given by the “Wall formula”, which is based on Eq.(4) below. But our interest is focused in circumstances in which the Lyapunov (correlation) time $t_R = R/v_E$ is much longer than the ballistic time $t_L = L/v_E$, where $v_E = (2E/m)^{1/2}$ is the velocity of the particle. Consequently, the low frequency response is enhanced by factor t_R/t_L as implied by Eq.(5).

We call the (classical) value of the absorption coefficient G_{LRT} . In the quantum analysis we write the result for the absorption coefficient as $G = g_s G_{\text{LRT}}$. If QCC considerations apply then $g_s \sim 1$, with small \hbar dependent corrections. Otherwise, if QCC does not apply, we call it an anomaly. There are circumstances in which an anomaly is not a big surprise: **(1)** If $f(t)$ is slowly varying, so-called quantum adiabatic parametric driving, then Landau-Zener transitions between neighboring levels might be the dominate mechanism for heating [6], and hence QCC is not expected. **(2)** If $f(t)$ is low frequency noisy driving, that induces Fermi-Golden-Rule (FGR) transitions between neighboring levels only, the result would be determined by the level spacing statistics, and hence QCC is not expected [11]. **(3)** If u is very small, such that levels are barely mixed, then QCC is not expected. In the latter case the suppression of the heating rate ($g_s \ll 1$) can be regarded as arising from the non-random nature of the eigenfunctions. This is the case that had been considered in [12]. In the present work we ask what happens if u is *not* very small, such that the eigenfunctions look like *random waves*. Do we enter the traditional “quantum chaos” regime where LRT and QCC

are commonly expected to be valid?

Turning to the quantum analysis we realize that the minimal model for \mathcal{H} depends on *two* dimensionless parameters which are the relative deformation $u = L/R$, and the dimensionless Planck constant

$$\hbar = \lambda_E/L \quad (2)$$

Here $\lambda_E = 2\pi\hbar_{\text{planck}}/(mv_E)$ is the de Broglie wavelength. For a given deformation (R determines u) and energy window (E determines \hbar) we calculate the eigenvalues and eigenfunctions of \mathcal{H} using the boundary element method [13], find the ordered eigenenergies E_n , and calculate the matrix elements F_{nm} . An image of a representative matrix is displayed in Fig. 2, and its bandprofile is presented in Fig. 3.

The power spectrum of the driving $\hat{f}(t)$ is described by a spectral function $\tilde{S}(\omega)$. As common in the mesoscopic context we assume its spectral support to be narrower than $1/t_R$ but larger compared with the mean level spacing. We assume FGR transitions between levels, whose rate is proportional to $|F_{nm}|^2\tilde{S}(E_n - E_m)$. The calculations of g_s is done within the framework of semi-linear response theory (SLRT) [10–12], using a resistor-network analogy. The results for g_s are presented in Fig. 4.

Conflicting expectations.— The rate of heating due to low frequency driving, and hence g_s , are determined by the couplings $|F_{nm}|^2$ between nearby levels. For a small deformation, first order perturbation theory (FOPT) implies that these couplings are $\propto u^2$. But as u becomes larger the common expectation, based on Wigner theory, is to have Lorentzian mixing, leading to $\propto 1/u^2$. In the formally equivalent problem of a conductance calculation this implies $G \propto 1/u^2$, where u represents the strength of the disordered potential (instead of using the FGR or Wigner picture one can use the equivalent Drude picture where the Born mean free path is $\propto 1/u^2$). On the other hand QCC considerations, based on Eq. (3) below and using Eq. (5), imply that the couplings should be $\propto 1/u$. The purpose of the following paragraphs is to resolve this confusion by adopting a generalized random matrix theory (RMT) perspective and hence to highlight the emergence of a quantum anomaly.

RMT modeling.— So called “quantum chaos” is the study of quantized chaotic systems. Assuming that the classical dynamics is fully chaotic, as in the case of a billiard with convex walls (Fig. 1), one expects the Hamiltonian to be like a random matrix with elements that have a Gaussian distribution. This is of course a sloppy statement, since any Hamiltonian is diagonal in some basis. The more precise statement is following [14]: Assume that \mathcal{H} generates chaotic dynamics, and consider an observable F that has some classical correlation function $C(t)$, with some correlation time t_R . Then the matrix representation F_{nm} in the basis of \mathcal{H} looks like a random banded matrix. The bandwidth is \hbar/t_R . If t_R is small, such that

the bandwidth is large compared with the energy window of interest, then the matrix looks like it is taken from a Gaussian ensemble.

What emerges in our numerical example, we would like to call “weak quantum chaos” (WQC) circumstances, for which the traditional RMT modeling does not apply. Namely, in such circumstances it is not enough to characterize F_{nm} by its semiclassically-determined *bandprofile*. Rather one should further characterize F_{nm} by its quantum-mechanically-determined *sparsity* [15] and by its *texture*.

Bandprofile.— Define a matrix \mathbf{X} whose elements are $X_{nm} = |F_{nm}|^2$. The bandprofile $\tilde{C}_a(r)$ is obtained by averaging the elements X_{nm} along the diagonals $n-m = r$, within the energy window of interest. In the same way we also define a *median* based bandprofile $\tilde{C}_s(r)$. Given that the mean level spacing $\Delta_0 = 2\pi/(mL_xL_y)$ is small compared with the energy range of interest, it is well known [14] that

$$\tilde{C}_a(n-m) = \left(\frac{2\pi}{\Delta_0}\right)^{-1} \tilde{C}(E_n - E_m) \quad (3)$$

where $\tilde{C}(\omega)$ is the classical power spectrum, that can be obtained via the Fourier transform of the classical auto-correlation function $\langle F(0)F(t) \rangle$.

The calculation of the classical $\tilde{C}(\omega)$ and of its quantum mechanical version is carried out as described in [17]. The applicability of Eq. (3) to the analysis of our billiard system is confirmed in Fig. 3, down to very small frequencies. Analytical results for $\tilde{C}(\omega)$ can be obtained. For large frequencies the power spectrum becomes flat and reaches the constant value [12]

$$C(\omega \gg 1/t_L) = \frac{8}{3\pi} \frac{m^2 v_E^3}{L_x} \quad (4)$$

For intermediate frequencies the effect of the deformation is mainly to ergodize the collision angle and one can obtain analytical expression (represented in Fig. 3 by dashed red line). For small frequencies the effect of the deformation is less trivial and we find that the power spectrum is logarithmically divergent:

$$\tilde{C}(\omega \ll 1/t_L) \approx \frac{m^2 v_E^3 R}{16L_x^2} \ln \frac{2v_E}{\omega R} \quad (5)$$

The divergence comes because there are vertically bouncing trajectories with very long horizontal bouncing period, as in the related analysis of [18].

Sparsity and Texture.— For strongly chaotic systems the elements within the band have approximately a Gaussian distribution. But in the WQC regime the matrix becomes *sparse* and *textured* as demonstrated in Fig. 2. Loosely speaking sparsity means that only a small fraction ($s \ll 1$) of elements are large, while the texture refers

to their non-random arrangement. The precise definition of $s[[X]]$ can be found in Section III of [19]. It is related to the size distribution of the in-band elements. [An optional measure for sparsity is the ratio ($q \ll 1$) of the median to the average.] In the WQC regime the size distribution of the in-band elements becomes log-wide (approximately log-normal) as seen in Fig. 5. This is reflected by having $\bar{C}_s(r) \ll \bar{C}_a(r)$ as seen in Fig. 3.

The sparsity and the texture of F_{nm} are important for the analysis of the energy absorption rate [12] as implied by SLRT [10, 11]. Accordingly, we suggest to characterize the sparsity by a resistor network measure

$$g_s = g_s[\mathbf{X}] \equiv \langle\langle \mathbf{X} \rangle\rangle_s / \langle\langle \mathbf{X} \rangle\rangle_a \quad (6)$$

Here $\langle\langle \mathbf{X} \rangle\rangle_a$ is the algebraic average over the in-band elements of the matrix, while $\langle\langle \mathbf{X} \rangle\rangle_s$ is the corresponding resistor network average that takes their connectivity into account. For precise definitions of these averages see Section IV of [19], where it is also explained why g_s is the SLRT suppression factor of the heating rate. For a strictly uniform matrix $g_s = s = 1$, for a Gaussian matrix $s = 1/3$ and $g_s \sim 1$, while for sparse matrix $s, g_s \ll 1$. In the RMT context a realistic estimate for $\langle\langle \mathbf{X} \rangle\rangle_s$ can be obtained using a generalized variable-range-hopping procedure [19]. In the numerics of Fig. 4 we see that g_s reflects better than s the variation in the ‘‘sparsity’’ of the perturbation matrix as we go higher in energy.

The WQC regime.— With the classical t_L and t_R , we can associate the energies $\Delta_L = 2\pi/t_L$ and $\Delta_R = 2\pi/t_R$. Conversely, with the mean levels spacing we can associate the Heisenberg time $t_H = 2\pi/\Delta_0$. Note that $t_H = (1/\hbar)^{d-1}t_L$ where $d=2$. It is also possible to define the Ehrenfest time $t_E = [\log(1/\hbar)]t_R$, which is the time required for the instability to show up in the quantum dynamics. The traditional condition for hard quantum chaos (HQC) is $t_E \ll t_H$, but if we neglect the log factor it is simply $t_R \ll t_H$. This can be rewritten as $\Delta_R \gg \Delta_0$, or in a more illuminating way as $u \gg u_b$, where $u_b = \hbar$. Consequently, the naive expectation is to have the WQC regime roughly within $u < u_b$.

We observe (Fig. 4) that the WQC regime is much larger than naively expected. This can be explained as follows. If a wall of a billiard is deformed, the levels are mixed. FOPT is valid provided $|U_{nm}| < \Delta_0$. This condition determines a parametric scale u_c . If the unperturbed billiard were chaotic, the variation required for level mixing would be [16] $u_c \approx \hbar/(k_E L)^{1/2} = \hbar^{3/2}$. This expression assumes that the eigenstates look like random waves. In the Wigner regime ($u_c < u < u_b$) there is a Lorentzian mixing of the levels and accordingly, the number of mixed levels is $\sim (u/u_c)^2$. But our unperturbed (rectangular) billiard is not chaotic, the unperturbed levels of the non-deformed billiards are not like random waves. Therefore, the mixing of the levels is *non-uniform*.

By inspection of the $U_{n_x n_y, m_x m_y}$ matrix elements one observes that the dominant matrix elements that are re-

sponsible for the mixing are those with large n_x but small $|n_y - m_y|$. Accordingly, within the energy shell $E_{n_x n_y} \sim E$, the levels that are mixed first are those with maximal n_x , while those those with minimal n_x are mixed last. The mixing threshold for the former is

$$u_c \approx \hbar/(k_E L) = \hbar^2 \quad (7)$$

while for the latter one finds $u_c^\infty \sim \hbar^0$, which is much larger than $u_b = \hbar^1$. Straightforward analysis of this mixing (extending that of [12]) leads to the result

$$g_s \approx u^3/\hbar \quad (8)$$

which is confirmed by our numerics. It follows that the WQC-HQC crossover is at

$$u_b = \hbar^{1/3} \quad (9)$$

and not at $u_b = \hbar$. Accordingly, the WQC regime extends well beyond the traditional boundary of the Wigner regime, and in any case it is well beyond the FOPT border u_c .

Comments.— As a side remark we note that we are studying in this work a driven chaotic system, and not a driven integrable system. Remarkable examples for driven integrable system are the kicked rotator [20] and vibrating elliptical billiards [21]. In the absence of driving such systems are integrable, while in the presence of driving a *mixed phase space* emerges. This is not what we call here *weak chaos*. It is also important to note that questions regarding dynamical localization [22] are irrelevant here because we assume low frequency noisy driving and not strictly periodic driving.

Summary.— The discovery of ‘‘anomalies’’, i.e. major deviations from QCC in circumstances where QCC is expected by common wisdom, is a major challenge in quantum-mechanics studies. For example: Anderson’s Localization (wavefunctions were commonly expected to be extended); Heller’s scars (wavefunctions were commonly expected to look like random waves). Here we highlighted an anomaly in the theory of response: the rate of heating is unexpectedly suppressed for a quantized chaotic system. Our analysis has been based on SLRT. This theory applies to circumstances in which the environmental relaxation is weak compared with the $f(t)$ -induced transitions. In such circumstances the connectivity of the transitions from level to level is important, and the LRT result should be multiplied by g_s . We have highlighted that there is a *distinct WQC regime*, where semiclassics and Wigner-type mixing co-exist. This is the regime where an LRT to SLRT crossover is expected as the intensity of the driving is increased.

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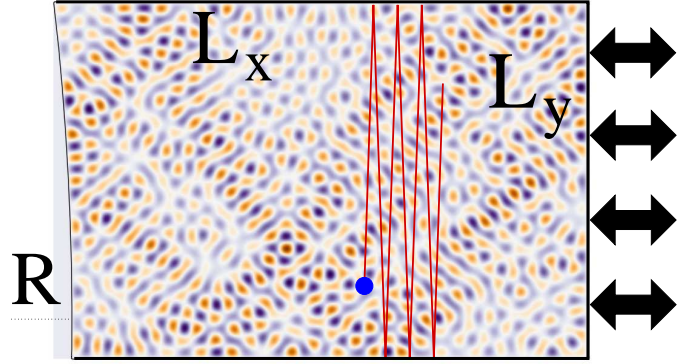


Fig. 1: **The billiard system.** Sketch of the model system of Eq. (1). The unperturbed billiard is a rectangular of size $L_x=1.5$ and $L_y=1.0$. The deformation U , due to the curvature of the left wall (radius $R=8$), is characterized by the parameter $u = L_y/R$. In order to break the mirror symmetry the center of the curved wall is shifted upwards a distance $\epsilon=0.1$. The time dependent perturbation is due to the displacement $f(t)$ of the right wall. In the numerics the units are chosen such that $\hbar_{\text{planck}}=1$ and the mass is $m=1/2$. The image in the background represents the eigenstate $E_n \simeq 13618$.

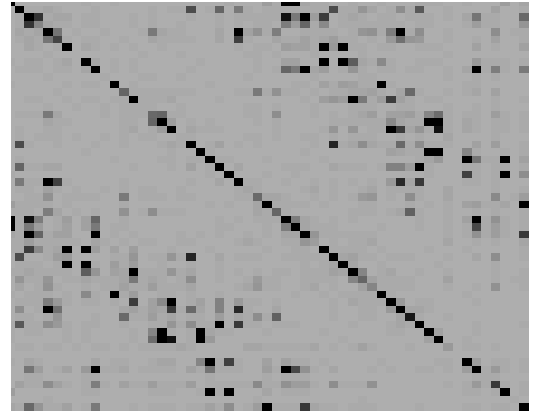


Fig. 2: **Image of the perturbation matrix.** Image of the matrix $\mathbf{X} = \{|F_{nm}|^2\}$ for the billiard of Fig. 1 within the energy window $3500 < E_n < 4000$. This matrix is *sparse*. More generally it might have some *texture*. The latter term applies if the arrangement of the large elements is characterized by some pattern.

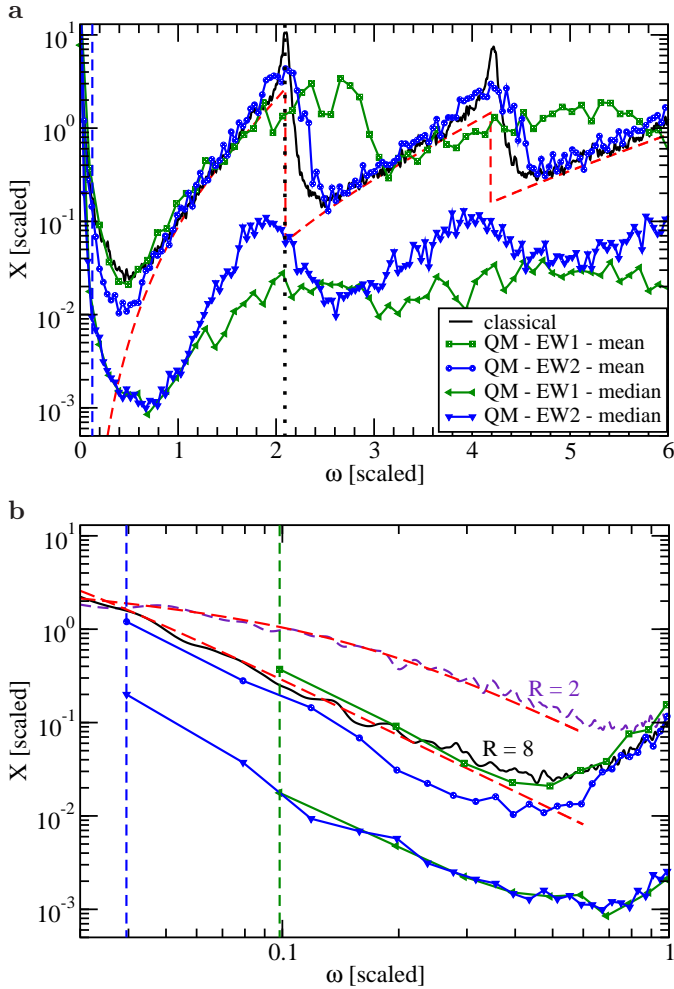


Fig. 3: **The band profile of the matrix.** (a) The algebraic average and median along the diagonals of the X_{nm} matrix versus $\omega \equiv (E_n - E_m)$. The vertical axis is normalized with respect to the prediction of Eq. (4), while the horizontal axis is ω/v_E . The classical power spectrum is presented to demonstrate the applicability of the semiclassical relation Eq. (3). The red line is the analytical expression that applies to zero deformation. The quantum analysis is for $R = 8$ with $100 < E < 4000$ (EW1), and with $10000 < E < 14000$ (EW2). The dotted vertical line is the frequency $1/t_L$ and the dashed one is $1/t_R$. (b) Zoom of the $\omega \ll 1/t_L$ region. For sake of comparison we display results also for $R = 2$. The vertical lines indicate the mean level spacing. The dashed red curves are a refined version of Eq.(5).

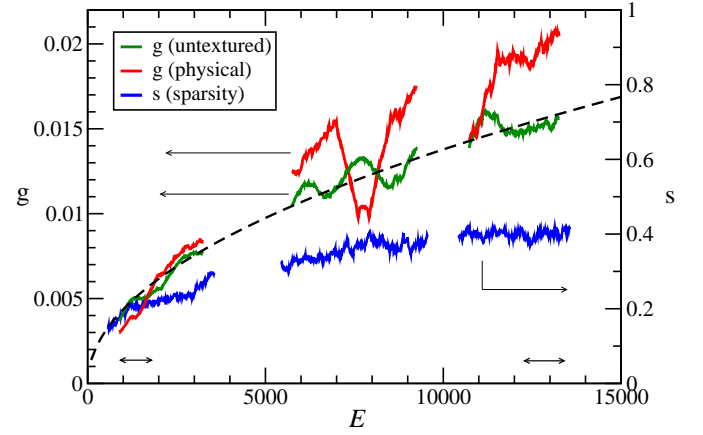


Fig. 4: **Sparsity versus energy.** The sparsity measure s and the resistor network result for g_s versus the energy E for the billiard of Fig. 1. The calculation of each point has been carried out on a 100×100 sub-matrix of \mathbf{X} centered around E . The “untextured” data points are calculated for an artificial random matrix with the same bandprofile and sparsity (but no texture). The dashed line is $g \propto 1/\hbar$, where $\hbar = \lambda_E/L$ with $\lambda_E = 2\pi\hbar_{\text{planck}}/(mv_E)$.

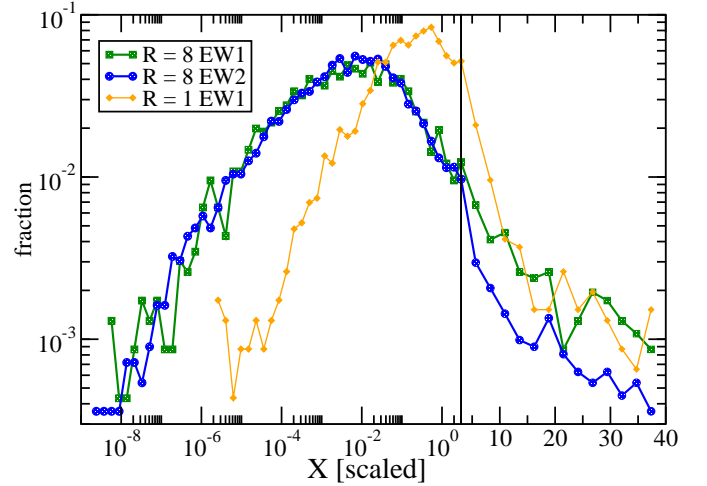


Fig. 5: **The size distribution of the matrix.** Histogram of the values of X_{nm} for the central band of the EW1 and EW2 matrices as defined in Fig. 3. For sake of comparison we display results also for $R = 1$.