Counting statistics in multiple path geometries and quantum stirring

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Refs:

Counting statistics for a coherent transition, MC and DC (PRA 2008)

Counting statistics in multiple path geometries, MC and DC (JPA 2008)

Quantum stirring of electrons in a ring, IS and DC (PRBs 2008)

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$DIP$, $BSF$
The counting operator:

\[ Q = \int_0^t I(t') dt' \]

- Single path coherent transition
- Double path coherent transition
- Quantum stirring (full cycle)
- Condensed particles / interactions

**Outline**

\[ \langle Q \rangle = ??? \]

\[ \text{Var}(Q) = ??? \]

\[ P(Q) = ??? \quad \text{[FCS]} \]

Stirring = inducing DC current by AC driving.
$Q$ is not an observable

The counting statistics can be determined using a continuous measurement scheme: The current induces a translation of a Von-Neumann pointer. At the final time, the position of the pointer is measured.

$$P(Q) = \frac{1}{2\pi} \int \left\langle \left[ \mathcal{T} e^{-i(r/2)Q} \right]^\dagger \left[ \mathcal{T} e^{i(r/2)Q} \right] \right\rangle e^{-iQr} dr$$

$$Q = \langle Q \rangle$$

$$Q^2 = \langle Q^2 \rangle$$

where $Q = \int_0^t \mathcal{I}(t') dt'$

H. Everett, Rev. Mod. Phys. 29, 454 (1957).


MC and DC, PRA (2008)
Single path coherent transition

\[ \langle N \rangle = p = \text{occupation} \]
\[ \langle Q \rangle = p = \text{counting} \]
\[ \text{Var}(Q) = (1 - p)p \]

**classical:**
\[ N = 1 \begin{bmatrix} p \\ 1-p \end{bmatrix}, 0 \begin{bmatrix} 1-p \\ 1-p \end{bmatrix} \]
\[ Q = 1 \begin{bmatrix} p \\ 1-p \end{bmatrix}, 0 \begin{bmatrix} 1-p \\ 1-p \end{bmatrix} \]

**Quantum:**
\[ N = 1 \begin{bmatrix} p \\ 1-p \end{bmatrix}, 0 \begin{bmatrix} 1-p \\ 1-p \end{bmatrix} \]
\[ Q = \pm \sqrt{p} \begin{bmatrix} (1 \pm \sqrt{p})/2 \end{bmatrix} \]

\[ p = 1 - P_{LZ} \]
\[ P_{LZ} = e^{-2\pi \frac{c^2}{u}} \]
Restricted Quantum to Classical Correspondence (QCC)

$\mathcal{N} =$ occupation operator (eigenvalues = 0, 1)

$\mathcal{I} =$ current operator

Heisenberg equation of motion:

$$\frac{d}{dt} \mathcal{N}(t) = \mathcal{I}(t)$$

Counting vs change in Occupation:

$$\mathcal{N}(t) - \mathcal{N}(0) = Q$$

Counting statistics = Occupation statistics:

$$\langle Q^k \rangle = \langle (\mathcal{N}(t) - \mathcal{N}(0))^k \rangle = \langle \mathcal{N}^k \rangle_t = p \quad \text{for } k = 1, 2 \text{ only}$$

$$\langle 0 | (\mathcal{N}(t) - \mathcal{N}(0)) (\mathcal{N}(t) - \mathcal{N}(0)) (\mathcal{N}(t) - \mathcal{N}(0)) | 0 \rangle \neq \langle 0 | \mathcal{N}(t)^3 | 0 \rangle$$

Restricted QCC is robust

Detailed QCC is fragile

**Double path coherent transition**

\[
\langle N \rangle = p, \quad \langle Q \rangle = \lambda p, \quad \text{Var}(Q) = \lambda^2 (1 - p)p
\]

\[
\lambda = \frac{c_1}{c_1 + c_2} = \text{splitting ratio}
\]

Coherent splitting is not like incoherent partitioning:

\[
\text{Var}(Q) \neq (1 - \lambda p)\lambda p
\]

\[
\lambda \neq \frac{|c_1|^2}{|c_1|^2 + |c_2|^2}
\]

\[
p = 1 - P_{LZ}
\]

\[
P_{LZ} = e^{-\pi \frac{(c_1 + c_2)^2}{\ddot{u}}}
\]
Splitting vs Partitioning

\[ |\Psi\rangle = \left( |Q=0\rangle + |Q=1\rangle \right) \otimes |q=0\rangle \quad q = \text{pointer} \]

The Schrodinger-cat paradigm:
The state of the particle becomes mixed;
One measures \( Q = 0, 1 \) with 50%-50% probabilities.

\[ |\Psi\rangle = |Q=0\rangle \otimes |q=0\rangle + |Q=1\rangle \otimes |q=1\rangle \]

The Born-Oppenheimer paradigm:
The state of the particle remains pure;
One measures \( Q = \frac{1}{2} \) with 100% probability.

\[ |\Psi\rangle = \left( |Q=0\rangle + |Q=1\rangle \right) \otimes |q=\frac{1}{2}\rangle \]
Splitting and stirring

The scattering point of view:
The particle has two paths to its destination.

The stirring point of view:
A circulating current is induced due to the driving.
The splitting ratio approach\(^*\) to quantum stirring\(^**\)

\[
\langle \mathcal{N} \rangle \approx \left| \sqrt{P_{\mathcal{L}Z}} - e^{i\varphi} \sqrt{P_{\mathcal{L}Z}} \right|^2
\]

\[
\langle Q \rangle \approx \lambda_{\mathcal{L}} - \lambda_{\mathcal{R}}
\]

\[
\text{Var}(Q) \approx \left| \tilde{\lambda}_{\mathcal{L}} \sqrt{P_{\mathcal{L}Z}} + e^{i\varphi} \lambda_{\mathcal{R}} \sqrt{P_{\mathcal{L}Z}} \right|^2
\]

\(^*\) In the classical context a similar approach has been independently proposed under the name current decomposition formula. S. Rahav, J. Horowitz, and C. Jarzynski \(\text{PRL 2008}\).

\(^**\) The splitting ratio approach allows to bypass the Kubo formula approach to quantum stirring, D. Cohen \(\text{PRB 2003}\), which is based on the adiabatic transport formalism of Thouless (1983), Avron (1988), Berry and Robbins (1993).

\[
\tilde{\lambda}_{\mathcal{L}} = \lambda_{\mathcal{L}} - 2\lambda_{\mathcal{R}}
\]
Derivation, using the adiabatic approximation

\[ U(t) \approx \sum_n \langle n(t) \rangle \exp \left[ -i \int_{t_0}^t E_n(t') dt' \right] \langle n(t_0) \rangle \]

\[ \mathcal{I}(t)_{nm} = \langle n|U(t)^\dagger \mathcal{I} U(t)|m \rangle \approx \langle n(t)|\mathcal{I}|m(t) \rangle \exp \left[ i \int_{t_0}^t E_{nm}(t') dt' \right] \]

\[ Q \equiv \begin{pmatrix} +Q_\parallel & iQ_\perp \\ -iQ_\perp^* & -Q_\parallel \end{pmatrix} \]

\[ \text{Var}(Q) = |Q_\perp|^2 \approx \left| \int_{-\infty}^\infty c e^{i\Phi(t)} dt \right|^2 \]

\[ \Phi(t) \equiv \int_0^t \sqrt{u(t')^2 + (2c)^2} \, dt' \quad \text{[for a single LZ transition]} \]

\[ \text{Var}(Q) \approx \left| \int_{-\infty}^\infty c e^{i\Phi(t)} dt \right|^2 = \left| \frac{2c^2}{\dot{u}} \int_{-\infty}^\infty \cosh(z) e^{i\Phi(z)} dz \right|^2 \sim \left( \frac{2c^2}{\dot{u}} \right)^{2/3} \exp \left[ -\pi \frac{c^2}{\dot{u}} \right] \]

\[ P_{\text{LZ}} \approx \left| \int_{-\infty}^\infty \frac{c\dot{u}}{u^2 + (2c)^2} e^{i\Phi(t)} dt \right|^2 = \left| \frac{1}{2} \int_{-\infty}^\infty \frac{1}{\cosh(z)} e^{i\Phi(z)} dz \right|^2 \sim \left( \frac{\pi}{3} \right)^2 \exp \left[ -\pi \frac{c^2}{\dot{u}} \right] \]
Derivation, using the adiabatic approximation (cont.)

A sequence of two Landau Zener crossings:

$$\langle Q \rangle \approx \lambda_\circ - \lambda_\circ$$

[assume for simplicity that only the splitting ratio is different]

\[
\text{Var}(Q) = \left| \int_{-\infty}^{\infty} \lambda_c e^{i\Phi(t)} dt \right|^2 \approx \left| \lambda_\circ e^{i\varphi_1} + \lambda_\circ e^{i\varphi_2} \right|^2 P_{LZ}
\]

\[
P_{LZ+LZ} = \left| \int_{-\infty}^{\infty} \frac{c\dot{u}}{u^2 + (2c)^2} e^{i\Phi(t)} dt \right|^2 \approx \left| e^{i\varphi_1} - e^{i\varphi_2} \right|^2 P_{LZ}
\]
Derivation, using the splitting ratio approach

\[
\mathcal{H} = \begin{pmatrix}
    u(t) & c_1 & c_2 \\
    c_1 & 0 & 1 \\
    c_2 & 1 & 0
\end{pmatrix}, \quad \mathcal{I} = \begin{pmatrix}
    0 & ic_1 & 0 \\
    -ic_1 & 0 & 0 \\
    0 & 0 & 0
\end{pmatrix}
\]

\[
\mathcal{H} = \begin{pmatrix}
    u(t) & \frac{c_1+c_2}{\sqrt{2}} \\
    \frac{(c_1+c_2)}{\sqrt{2}} & 1
\end{pmatrix}, \quad \mathcal{I} = \frac{c_1}{\sqrt{2}} \begin{pmatrix}
    0 & i \\
    -i & 0
\end{pmatrix}
\]

\[
\mathcal{H} = \begin{pmatrix}
    u(t) & c \\
    c & 0
\end{pmatrix}, \quad \mathcal{I} = \lambda \begin{pmatrix}
    0 & ic \\
    -ic & 0
\end{pmatrix}
\]

\[U_{LZ} \approx \begin{pmatrix}
    \sqrt{P_{LZ}} & -\sqrt{1-P_{LZ}} \\
    \sqrt{1-P_{LZ}} & \sqrt{P_{LZ}}
\end{pmatrix}\]

\[U_{LZ}(\text{cycle}) = \left[ T \ U_{LZ} \circ T \right] e^{-i\phi} \left[ U_{LZ} \right]\]

\[Q = \int \mathcal{I}(t) dt \approx \lambda \circ Q_{LZ}^\circ - [T e^{-i\phi} U_{LZ}^\circ]^\dagger \lambda \circ Q_{LZ}^\circ [T e^{-i\phi} U_{LZ}^\circ]\]
Long time Counting Statistics

Naive expectation:
Probabilistic point of view implies
\[ \delta Q \propto \sqrt{t} \]

Quantum result:
The eigenvalues \( Q_\pm \) of the \( Q \) operator grow linearly with the number of cycles
\[ \delta Q \propto t \]

If we have good control over the preparation we can select it to be a Floque state of the quantum evolution operator. For such preparation the linear growth of \( \delta Q \) is avoided, and it oscillates around a residual value.
Stirring of BEC

strong attractive interaction: classical ball dynamics

negligible interaction (|U| \ll \kappa/N): mega-crossing

weak repulsive interaction: gradual crossing

strong repulsive interaction (U \gg N\kappa): sequential crossing
References and further work

Refs:

Counting statistics for a coherent transition, MC and DC (PRA 2008)

Counting statistics in multiple path geometries, MC and DC (JPA 2008, FQMT proc. 2009)

Quantum stirring of electrons in a ring, IS and DC (PRBs 2008)

Further work:

BEC in 2-sites - Bloch-Josephson oscillations, E.Boukobza, MC, DC and A.Vardi (PRL 2009)


BEC in 3-sites - Quantum stirring, M.Hiller, T.Kottos and DC (EPL 2008 & PRA 2008)

URL:

http://www.bgu.ac.il/~dcohen
Main messages

- FCS for a 2-site coherent transition - the simplest solvable model.
- Counting statistics for a 3-site system - multiple path geometry.
- Restricted QCC fails for multiple path transitions.
- Coherent splitting is not like incoherent partitioning.
- Splitting ratio approach to quantum stirring (vs Kubo).
- Interference in the calculation of $\text{Var}(Q)$.
- Exact vs adiabatic results for $\text{Var}(Q)$.
- Long time counting statistics for multiple cycle stirring process.
- Stirring of BEC - the effect of interactions.