Semi-linear response of energy absorption

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$ISF$, $GIF$, $DIP$, $BSF$
Diffusion and Energy absorption

Driven chaotic system with Hamiltonian $\mathcal{H}(X(t))$

$X = \text{some control parameter}$

$\dot{X} = \text{rate of the (noisy) driving}$

$\sim \text{diffusion in energy space:}$

$D = G_{\text{diffusion}} \overline{\dot{X}^2}$

$\sim \text{energy absorption:}$

$\dot{E} = G_{\text{absorption}} \overline{\dot{X}^2}$

[Ott, Brown, Grebogi, Wilkinson, Jarzynski, D.C.]

There is a dissipation-diffusion relation.

In the canonical case $\dot{E} = D/T$.

Below we use for $G$ scaled units.
Models

\[ \mathcal{H} = \{E_n\} - X(t)\{V_{nm}\} \]

with:
- Stotland
- Davidson
- deformed boundary
- or point scatterer
- or Gaussian bump

with:
- Wilkinson
- Mehlig
- Metallic grain

with:
- Stotland
- Budoyo
- Peer
- Kottos
- Flux
Some results

Cold atoms in vibrating traps:

Metallic rings driven by EMF:
Digression: size distribution

Given a matrix that looks random \{V_{nm}\},

Consider the \textit{size distribution} of the elements.

Histogram of $\log(x)$ where $x = |V_{nm}|^2$

\begin{align*}
\text{Algebraic average:} & \quad \langle \langle x \rangle \rangle_a = \langle x \rangle \\
\text{Harmonic average:} & \quad \langle \langle x \rangle \rangle_h = [\langle 1/x \rangle]^{-1} \\
\text{Geometric average:} & \quad \langle \langle x \rangle \rangle_g = \exp[\langle \log x \rangle]
\end{align*}

$\langle \langle x \rangle \rangle_h \ll \langle \langle x \rangle \rangle_g \ll \langle \langle x \rangle \rangle_a$
Digression: random walk

$w_{nm} =$ probability to hop from $m$ to $n$ per step.

$$\text{Var}(n) = \sum_n [w_{nm} t] (n - m)^2 \equiv 2Dt$$

For n.n. hopping with rate $w$ we get $D = w$.

The diffusion equation:

$$\frac{\partial p_n}{\partial t} = -\frac{\partial}{\partial n} J_n = D \frac{\partial^2}{\partial n^2} p_n$$

Fick’s law:

$$J_n = -D \frac{\partial}{\partial n} p_n$$

If we have a sample of length $N$ then

$$J = -\frac{D}{N} \times [p_N - p_0]$$

$$D/N = \text{inverse resistance of the chain}$$

If the $w$ are not the same:

$$\frac{D}{N} = \left[ \sum_{n=1}^{N} \frac{1}{w_{n,n-1}} \right]^{-1}$$

Hence

$$D = \langle \langle w \rangle \rangle_n \quad \text{for n.n. hopping}$$
**Digression: Fermi Golden rule**

The Hamiltonian in the standard representation:

\[ \mathcal{H} = \{E_n\} - X(t)\{V_{nm}\} \]

The transformed Hamiltonian:

\[ \tilde{\mathcal{H}} = \{E_n\} - \dot{X}(t) \left\{ \frac{iV_{nm}}{E_n - E_m} \right\} \]

The FGR transition rate for \( \omega \sim 0 \) driving:

\[ w_{nm} = 2\pi \left| \frac{V_{nm}}{E_n - E_m} \right|^2 \frac{|\dot{X}|^2}{\Gamma} \delta(\omega - (E_n - E_m)) \]

Note that the spectral content of the driving is

\[ \tilde{S}(\omega) = |\dot{X}|^2 \delta(\omega - (E_n - E_m)) \]
Semi Linear Response Theory (SLRT)

\[ \mathcal{H} = \{E_n\} - X(t)\{V_{nm}\} \]

\[ \frac{dp_n}{dt} = -\sum_m w_{nm}(p_n - p_m) \]

\[ w_{nm} = \text{const} \times g_{nm} \times \dot{X}^2 \]

\[ g_{nm} = 2\varrho^{-3} \frac{|V_{nm}|^2}{(E_n - E_m)^2} \delta \Gamma(E_n - E_m) \]

\[ \langle\langle |V_{mn}|^2 \rangle\rangle \equiv \text{inverse resistivity of the network} \]

\[ D = \pi \varrho \langle\langle |V_{mn}|^2 \rangle\rangle \times \dot{X}^2 \equiv G \dot{X}^2 \]
**Example: cold atoms in vibrating trap**

The Hamiltonian in the $\mathbf{n} = (n_x, n_y)$ basis:

$$\mathcal{H} = \text{diag}\{E_n\} + u\{U_{nm}\} + f(t)\{V_{nm}\}$$

The matrix elements for the wall displacement:

$$V_{nm} = -\delta_{n_y,m_y} \times \frac{\pi^2}{M L_x^3} n_x m_x$$

The Hamiltonian in the $E_n$ basis:

$$\mathcal{H} = \text{diag}\{E_n\} + f(t)\{V_{nm}\}$$

$$\langle\langle |V_{nm}|^2 \rangle\rangle_a \approx \frac{M v_E^3}{2\pi L_x^2 L_y}$$

$$\langle\langle |V_{nm}|^2 \rangle\rangle_g \approx \frac{4M^2 v_E^4}{L_x^3 L_y \omega_c^2} \exp \left[-M^2 v_E^2 (\sigma_x^2 + \sigma_y^2)\right] \times u^2$$

The SLRT result:

$$G_{\text{SLRT}} = q \exp \left[2\sqrt{-\ln q}\right] \times G_{\text{LRT}}$$
SLRT vs LRT

\[ X = \text{some control parameter} \]
\[ \dot{X} = \text{rate of the (noisy) driving} \]

The definition of the "conductance":
\[ D = G \dot{X}^2 \]

LRT implies
\[ D = \int G(\omega) |\dot{X}_\omega|^2 d\omega = \int G(\omega) \tilde{S}(\omega) d\omega \]

Within the framework of LRT
\[ \tilde{S}(\omega) \mapsto \lambda \tilde{S}(\omega) \implies D \mapsto \lambda D \]
\[ \tilde{S}(\omega) \mapsto \sum_i \tilde{S}_i(\omega) \implies D \mapsto \sum_i D_i \]

But there are circumstance such that e.g.
\[ D = \left[ \int R(\omega) \left[ \tilde{S}(\omega) \right]^{-1} d\omega \right]^{-1} \]
Simplest illustration

\[ S(\omega) \]

\[ \omega_1 \]
\[ \omega_2 \]

\[ E \]

\[ D \gg D_1 + D_2 \]
Example: energy absorption by metallic grains

Linear response theory:

\[
D = \sigma^2 \hbar \rho \int_0^\infty \text{d} \omega \omega^2 R_2(\hbar \omega) \tilde{S}(\omega)
\]

Semi-linear response theory:

\[
D = \sigma^2 \hbar \rho \int \frac{\text{d}x}{(2\pi)^{N/2}} e^{-x^2/2} \left[ \int_0^\infty \text{d} \omega \frac{P_2(\rho \hbar \omega)}{\tilde{S}(\omega)} \right]^{-1}
\]

Level spacing statistics:

\[
P_2(s) \approx a_\beta s^\beta \exp(-c_\beta s^2) \quad \text{with } \beta = 1, 2, 4
\]

The LRT result of Gorkov and Eliashberg:

\[
G = C_\beta \sigma^2 (\hbar \rho)^{\beta+1} T^{\beta+2}
\]

Our SLRT result (large \( s \) statistics!):

\[
G = \frac{\sigma^2}{2\hbar \rho (\hbar \rho \omega_0)^{\beta-1}} \exp \left[-\frac{1}{\pi(\hbar \rho T)^2} \right]
\]
The conductance of metallic rings

Non interacting “spinless” electrons in a ring.

\[ \mathcal{H}(r, p; \Phi(t)) \]

\[
-\dot{\Phi} = \text{electro motive force (RMS)} \\
G \dot{\Phi}^2 = \text{rate of energy absorption}
\]

\[
G = \pi \left(\frac{e}{L}\right)^2 \mathrm{DOS}^2 \langle \langle |v_{mn}|^2 \rangle \rangle
\]

\[
\langle \langle |v_{mn}|^2 \rangle \rangle_{\text{harmonic}} \ll \langle \langle |v_{mn}|^2 \rangle \rangle \ll \langle \langle |v_{mn}|^2 \rangle \rangle_{\text{algebraic}}
\]

\[ \mathcal{M} \text{ mode ring of length } L \text{ with disorder } W \]
Numerical Results

Regimes: ballistic; diffusive; localization

- SLRT (Meso)
- LRT (Kubo)
- Drude

- results for the tight binding model
- results for untextured matrices
- results for log-normal RMT ensemble
- results for log-box RMT ensemble
Linear response theory (LRT)

\( \mathcal{H} = \{E_n\} - \frac{e}{L} \Phi(t) \{v_{nm}\} \)

\[
\mathbf{G} = \pi \left( \frac{e}{L} \right)^2 \sum_{n,m} |v_{mn}|^2 \delta_T (E_n - E_F) \delta_\Gamma (E_m - E_n)
\]

\[
\mathbf{G} = \pi \left( \frac{e}{L} \right)^2 \text{DOS}^2 \langle \langle |v_{mn}|^2 \rangle \rangle_{\text{algebraic}}
\]

applies if

EMF driven transitions \( \ll \) relaxation

otherwise

*connected sequences of transitions* are essential.

leading to

Semi Linear Response Theory (SLRT)
Bandprofile, sparsity and texture

\[ G = \pi \left( \frac{e}{L} \right)^2 \text{DOS}^2 \langle \langle |v_{mn}|^2 \rangle \rangle \]

\[ \langle \langle |v_{mn}|^2 \rangle \rangle \equiv \text{inverse resistivity of the network} \]

Bounds:

\[ \langle \langle |v_{mn}|^2 \rangle \rangle_{\text{harmonic}} \ll \langle \langle |v_{mn}|^2 \rangle \rangle \ll \langle \langle |v_{mn}|^2 \rangle \rangle_{\text{algebraic}} \]

Analytical estimates:

- Mixed average scheme
- Variable range hopping scheme
Naive expectation (assuming $\Gamma > \Delta$):

$$G = \frac{e^2}{2\pi\hbar} \mathcal{M} \frac{\ell}{L} + \mathcal{O}\left(\frac{\Delta}{\Gamma}\right)$$

$L$ = perimeter of the ring

$\ell$ = mean free path $\propto W^2$

$\ell_\infty$ = localization length $\approx \mathcal{M}\ell$

Ballistic regime: $L \ll \ell$

Diffusive regime: $\ell \ll L \ll \ell_\infty$

Anderson regime: $\ell_\infty \ll L$
Strategy of analysis

Given $W$ ...

Characterization of the eigenstates:

- participation ratio ($\text{PR}$)

Characterization of $v_{nm}$ and RMT modeling

- bandwidth
- sparsity ($p$)
- texture

Approximation schemes for $G$

- Mixed average
- Variable range hopping estimate
**Ergodicity of the eigenstates**

- **Weak disorder** (ballistic rings): Wavefunctions are localized in mode space.
- **Strong disorder** (Anderson localization): Wavefunctions are localized in real space.

The PR of eigenstates of a ring with a single scatterer. The horizontal axis is the reflection of the scatterer.

The PR of eigenstates of a ring with disorder. The horizontal axis is $W$.

The sparsity ($p$) of the perturbation matrix is related to the ergodicity of the eigenstates.
\{ |v_{nm}|^2 \} as a random matrix \{ X \}

The fraction of "large" elements:

\[ p \equiv F(\langle X \rangle) \]

Sparsity: \( p \ll 1 \).

Histograms of \( X \):

**Ballistic:**

\( X \sim \text{LogNormal} \)

**Localization:**

\( X \sim \text{LogBox} \)
RMT based prediction for $G_{\text{SLRT}}/G_{\text{LRT}}$

RMT implied dependence on $p$

Log-normal distribution:

![Graph showing log-normal distribution](image1)

Log-box distribution:

![Graph showing log-box distribution](image2)
The VRH estimate

\[
G = \pi \hbar \left( \frac{e}{L} \right)^2 \sum_{n,m} |v_{mn}|^2 \delta_T (E_n - E_F) \delta_\Gamma (E_m - E_n)
\]

\[
G = \frac{1}{2} \left( \frac{e}{L} \right)^2 \varrho_F \int \tilde{C}_{qm}(\omega) \delta_\Gamma (\omega) \, d\omega
\]

\[
\tilde{C}_{qm}\text{-LRT}(\omega) \equiv 2\pi \varrho_F \langle X \rangle
\]
\[
\tilde{C}_{qm}\text{-SLRT}(\omega) \equiv 2\pi \varrho_F \overline{X}
\]

where by definition: \( \left( \frac{\omega}{\Delta} \right) \text{Prob}(X > \overline{X}) \sim 1 \)

For strong disorder we get:

\[
\overline{X} \approx v_F^2 \exp \left( -\frac{\Delta}{\omega} \right)
\]

\[
G \propto \int \exp \left( -\frac{\Delta}{|\omega|} \right) \exp \left( -\frac{|\omega|}{T} \right) \, d\omega
\]
LRT, SLRT and beyond

$-\dot{\Phi} = \text{electro motive force (RMS)}$

$G\dot{\Phi}^2 = \text{rate of energy absorption}$

Semi linear response theory


Beyond (semi) linear response theory


Conclusions

(*) Wigner (~ 1955):
The perturbation is represented by a random matrix whose elements are taken from a Gaussian distribution.

Not always...

1. Ballistic ring $\implies$ log-normal distribution.
2. Strong localization $\implies$ log-box distribution.
3. Resistors network calculation to get $G_{\text{SLRT}}$.
4. Generalization of the VRH estimate
5. SLRT is essential whenever the distribution of matrix elements is wide ("sparsity") or if the matrix has "texture".
6. Other applications of SLRT...