Semi-linear response of energy absorption

Doron Cohen, Ben-Gurion University

Collaborations:

Tsampikos Kottos (Wesleyan)
Holger Schanz (Gottingen)
Michael Wilkinson (UK)
Bernhard Mehlig (Goteborg)
Swarnali Bandopadhyay (BGU)
Yoav Etzioni (BGU)
Alex Stotland (BGU)
Rangga Budoyo (Wesleyan)
Tal Peer (BGU)
Nir Davidson (Weizmann)

Discussions:

Yuval Gefen (Weizmann)
Shmuel Fishman (Technion)

$ISF, $GIF, $DIP, $BSF

Diffusion and Energy absorption

Driven chaotic system with Hamiltonian $\mathcal{H}(X(t))$

$$X \quad = \quad \text{some control parameter}$$

$$\dot{X} \quad = \quad \text{rate of the (noisy) driving}$$

$\rightsquigarrow$ diffusion in energy space:

$$\boldsymbol{D} \quad = \quad \boldsymbol{G}_{\text{diffusion}} \; \overline{\dot{X}^2}$$

$\rightsquigarrow$ energy absorption:

$$\dot{\boldsymbol{E}} \quad = \quad \boldsymbol{G}_{\text{absorption}} \; \overline{\dot{X}^2}$$

[Ott, Brown, Grebogi, Wilkinson, Jarzynski, D.C.]

There is a dissipation-diffusion relation.

In the canonical case $\dot{E} = D/T$.

Below we use for G scaled units.

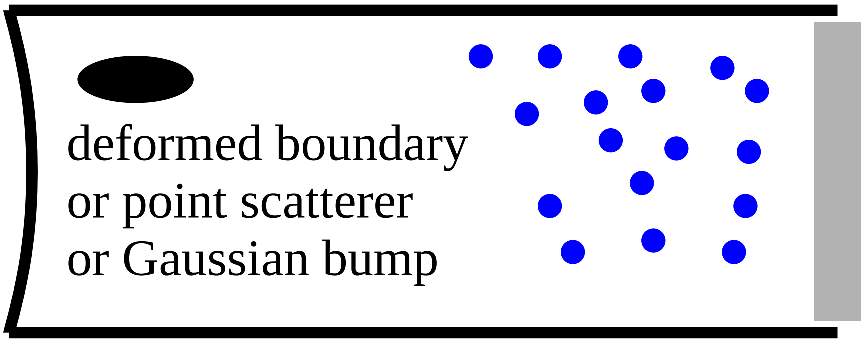
Models

$$\mathcal{H} = \{E_n\} - X(t)\{V_{nm}\}$$

with:

Stotland

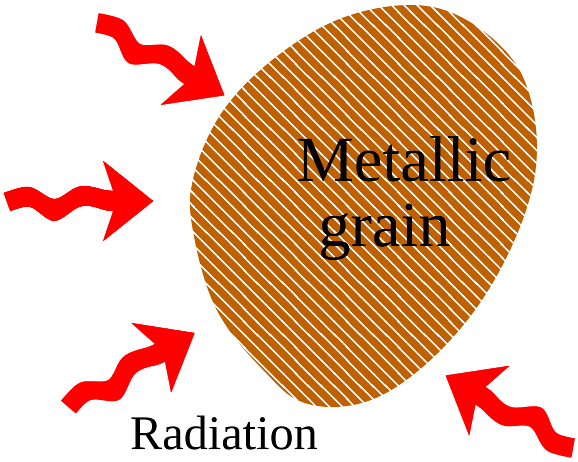
Davidson



with:

Wilkinson

Mehlig



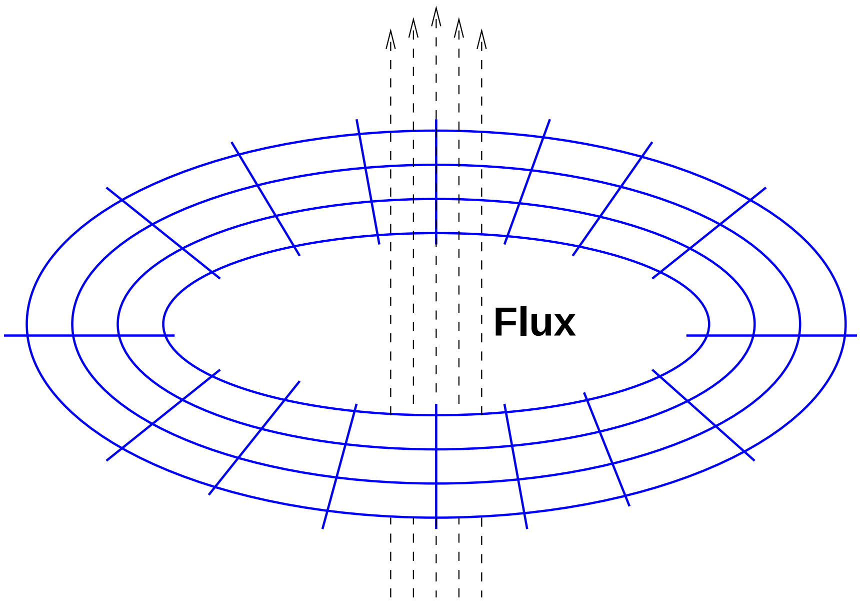
with:

Stotland

Budoyo

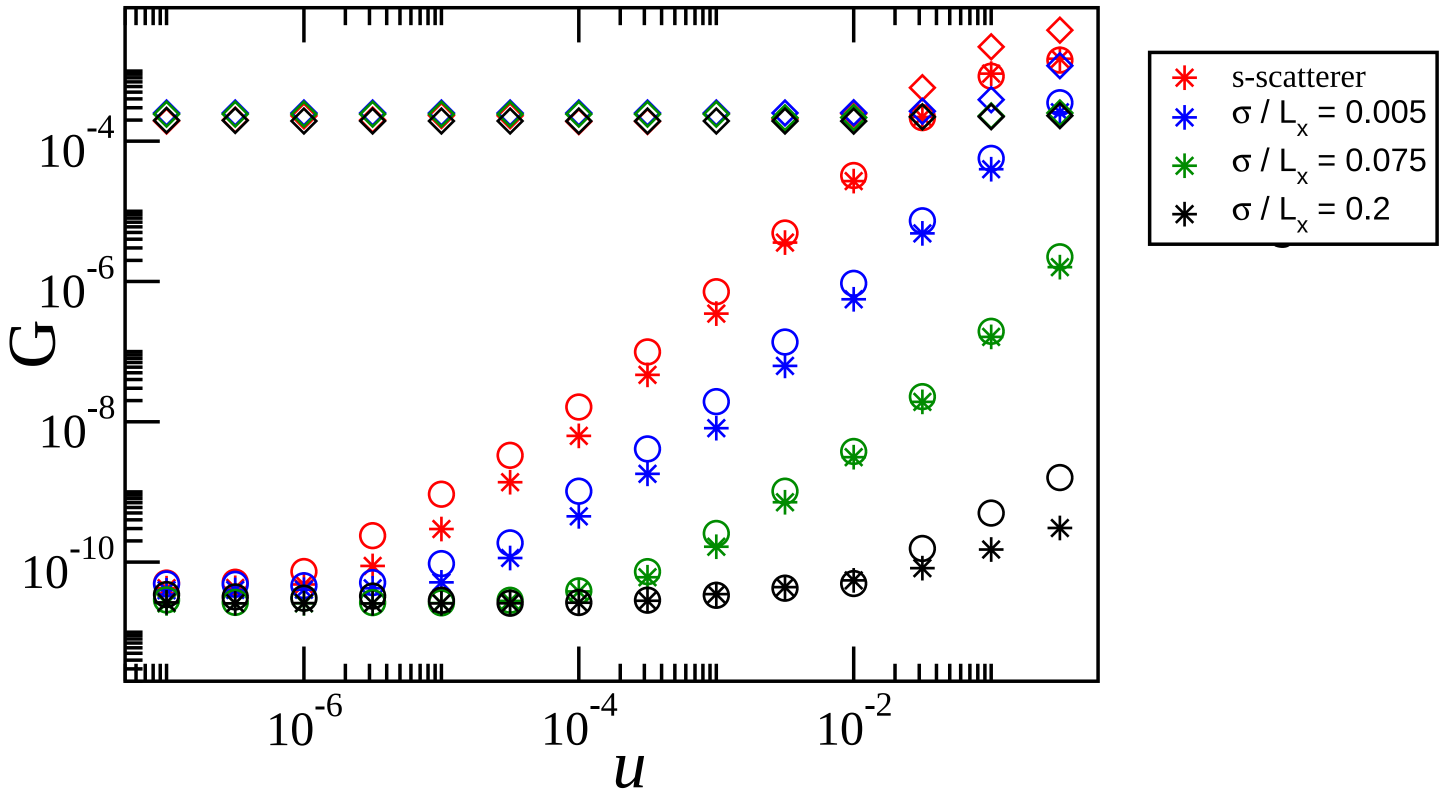
Peer

Kottos

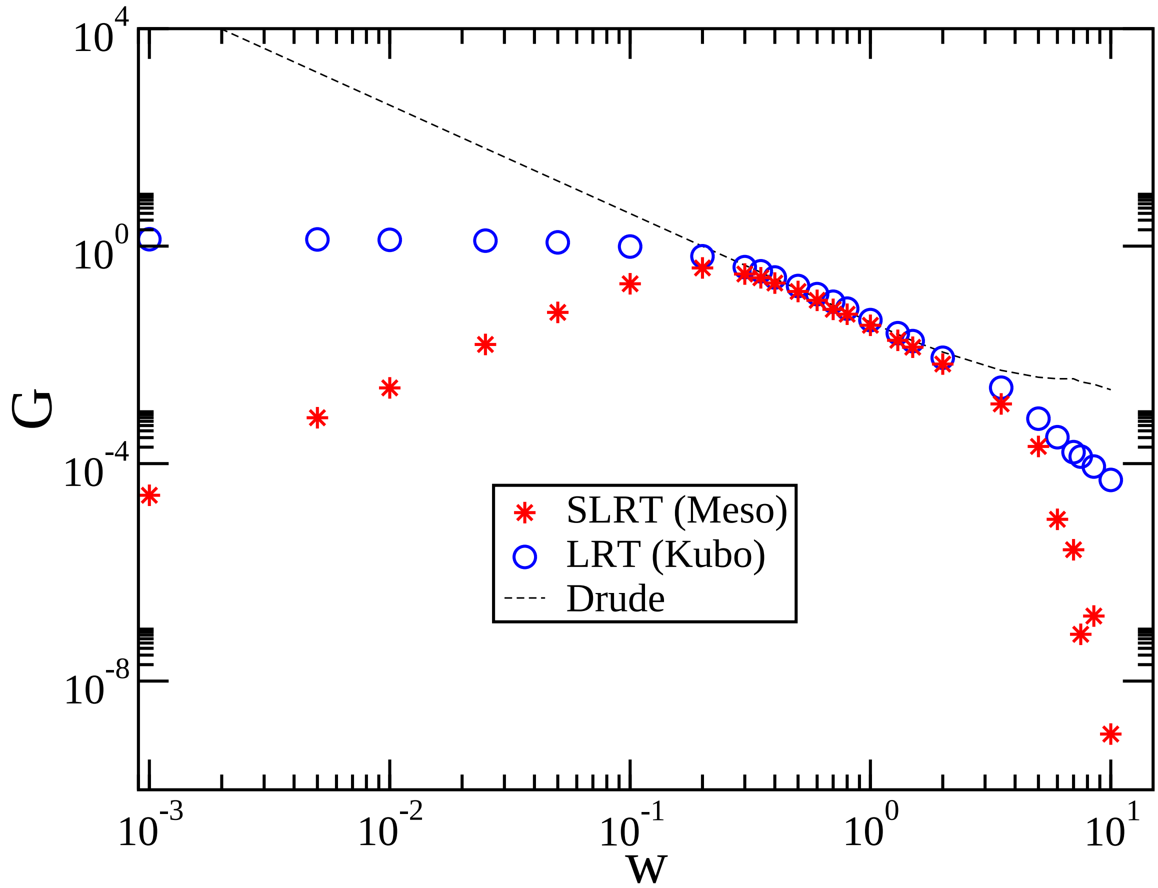


Some results

Cold atoms in vibrating traps:



Metallic rings driven by EMF:

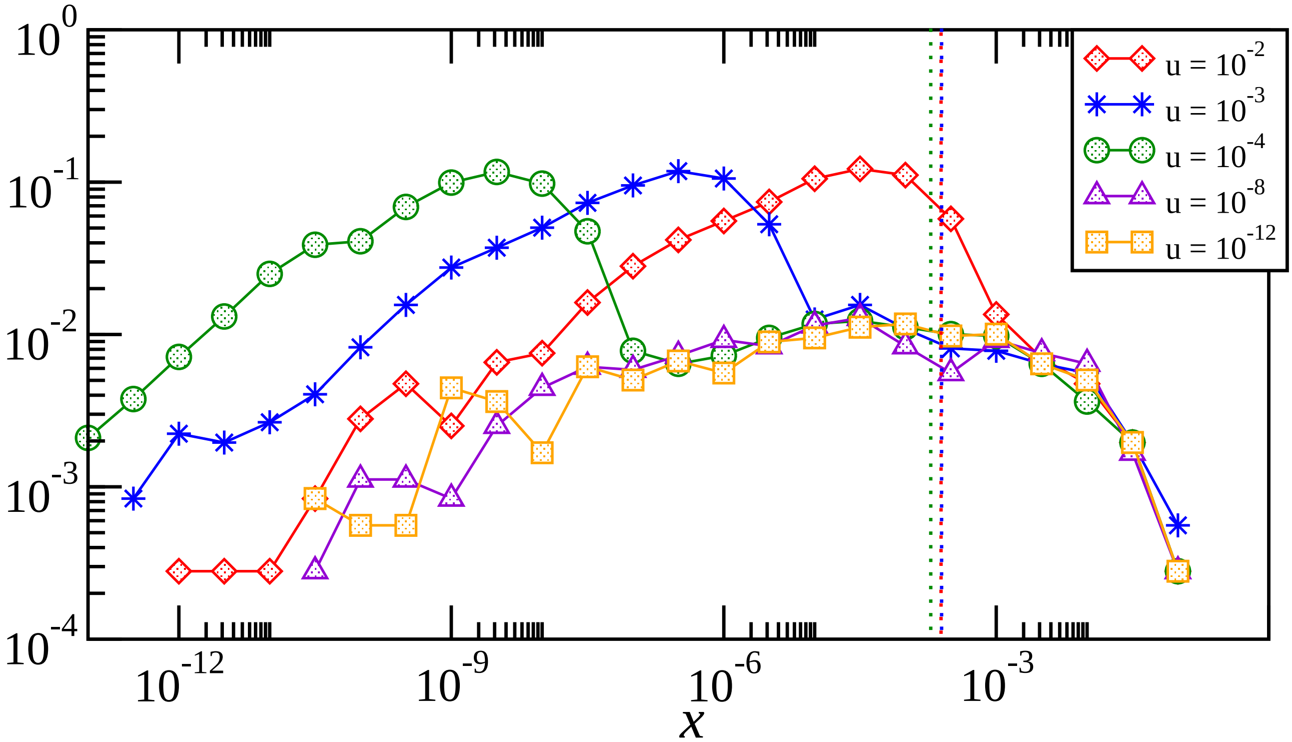


Digression: size distribution

Given a matrix that looks random $\{V_{nm}\}$,

Consider the *size distribution* of the elements.

Histogram of $\log(x)$ where $x = |V_{nm}|^2$



Algebraic average: $\langle\langle x \rangle\rangle_a = \langle x \rangle$

Harmonic average: $\langle\langle x \rangle\rangle_h = [\langle 1/x \rangle]^{-1}$

Geometric average: $\langle\langle x \rangle\rangle_g = \exp[\langle \log x \rangle]$

$$\langle\langle x \rangle\rangle_h \quad \ll \quad \langle\langle x \rangle\rangle_g \quad \ll \quad \langle\langle x \rangle\rangle_a$$

Digression: random walk

$w_{nm} =$ probability to hop from m to n per step.

$$\mathrm{Var}(n) \;=\; \sum_n [w_{nm} t]\,(n-m)^2 \;\equiv\; 2Dt$$

For n.n. hopping with rate w we get $D = w$.

The diffusion equation:

$$\frac{\partial p_n}{\partial t} \;=\; -\frac{\partial}{\partial n} J_n \;=\; D\frac{\partial^2}{\partial n^2} p_n$$

Fick's law:

$$J_n \;=\; -D\frac{\partial}{\partial n} p_n$$

If we have a sample of length N then

$$J \;=\; -\frac{D}{N} \times [p_N - p_0]$$

$D/N \;=\;$ inverse resistance of the chain

If the w are not the same:

$$\frac{D}{N} \;=\; \left[\sum_{n=1}^{N} \frac{1}{w_{n,n-1}}\right]^{-1}$$

Hence

$D \;=\; \langle\langle w \rangle\rangle_h$ for n.n. hopping

Digression: Fermi Golden rule

The Hamiltonian in the standard representation:

$$\mathcal{H} = \{E_n\} - X(t)\Big\{V_{nm}\Big\}$$

The transformed Hamiltonian:

$$\tilde{\mathcal{H}} = \{E_n\} - \dot{X}(t)\left\{\frac{iV_{nm}}{E_n - E_m}\right\}$$

The FGR transition rate for $\omega \sim 0$ driving:

$$w_{nm} \quad = \quad 2\pi \left|\frac{V_{nm}}{E_n - E_m}\right|^2 \overline{|\dot{X}|^2}\; \delta_\Gamma(E_n - E_m)$$

Note that the spectral content of the driving is

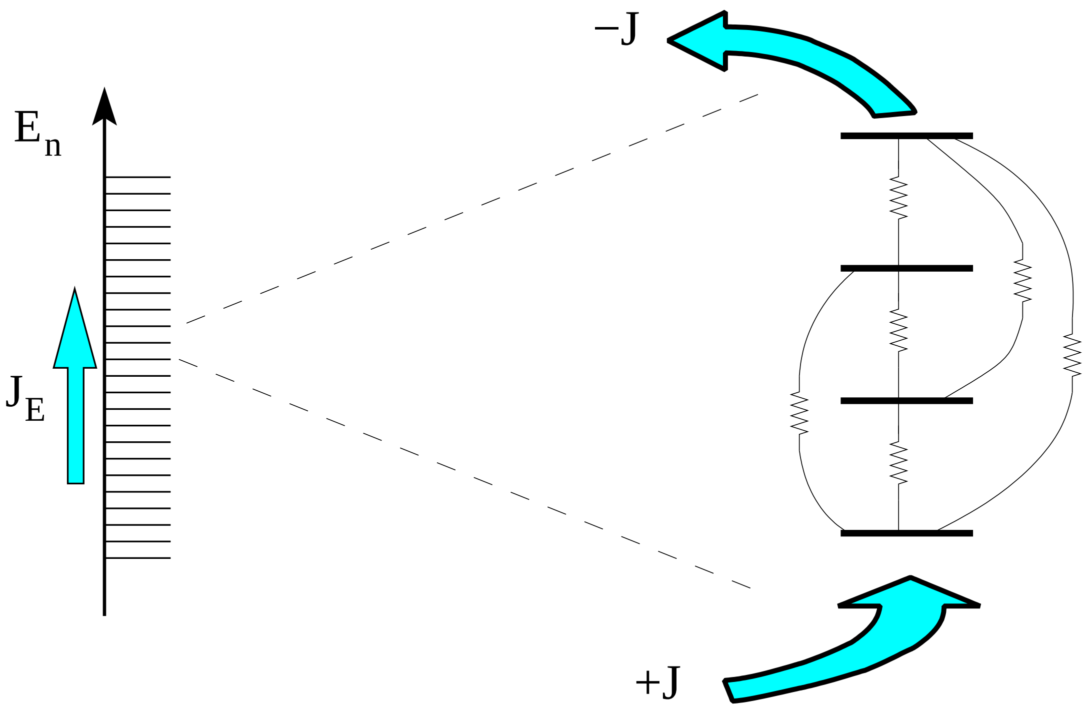
$$\tilde{S}(\omega) \quad = \quad \overline{|\dot{X}|^2}\; \delta_\Gamma(\omega - (E_n - E_m))$$

Semi Linear Response Theory (SLRT)

$$\mathcal{H} = \{E_n\} - X(t)\{V_{nm}\}$$

$$\frac{dp_n}{dt} = -\sum_m w_{nm}(p_n - p_m)$$

$$w_{nm} = \text{const} \times \mathrm{g}_{nm} \times \overline{\dot{X}^2}$$



$$\mathrm{g}_{nm} = 2\varrho^{-3} \frac{|V_{nm}|^2}{(E_n - E_m)^2} \delta_\Gamma(E_n - E_m)$$

$\langle\langle|V_{mn}|^2\rangle\rangle \equiv$ inverse resistivity of the network

$$\boldsymbol{D} = \pi\varrho \langle\langle|V_{mn}|^2\rangle\rangle \times \overline{\dot{X}^2} \equiv \boldsymbol{G}\, \overline{\dot{X}^2}$$

Example: cold atoms in vibrating trap

The Hamiltonian in the $\boldsymbol{n} = (n_x, n_y)$ basis:

$$\mathcal{H} = \mathrm{diag}\{E_{\boldsymbol{n}}\} + u\{U_{\boldsymbol{nm}}\} + f(t)\{V_{\boldsymbol{nm}}\}$$

The matrix elements for the wall displacement:

$$V_{\boldsymbol{nm}} \;=\; -\delta_{n_y,m_y} \times \frac{\pi^2}{\mathsf{M} L_x^3} n_x m_x$$

The Hamiltonian in the E_n basis:

$$\mathcal{H} = \mathrm{diag}\{E_n\} + f(t)\{V_{nm}\}$$

$$\langle\langle |V_{nm}|^2 \rangle\rangle_a \approx \frac{\mathsf{M} v_{\mathrm{E}}^3}{2\pi L_x^2 L_y}$$

$$\langle\langle |V_{nm}|^2 \rangle\rangle_g \approx \frac{4\mathsf{M}^2 v_{\mathrm{E}}^4}{L_x^3 L_y \omega_c^2} \exp\left[-\mathsf{M}^2 v_{\mathrm{E}}^2 (\sigma_x^2 + \sigma_y^2)\right] \times u^2$$

The SLRT result:

$$G_{\mathrm{SLRT}} \;=\; q \; \exp\left[2\sqrt{-\ln q}\right] \times G_{\mathrm{LRT}}$$

SLRT vs LRT

$$X \quad = \quad \text{some control parameter}$$
$$\dot{X} \quad = \quad \text{rate of the (noisy) driving}$$

The definition of the "conductance":

$$\boldsymbol{D} \; = \; \boldsymbol{G} \, \overline{\dot{X}^2}$$

LRT implies

$$\boldsymbol{D} \; = \; \int G(\omega) |\dot{X}_\omega|^2 \, d\omega \; = \; \int G(\omega) \tilde{S}(\omega) \, d\omega$$

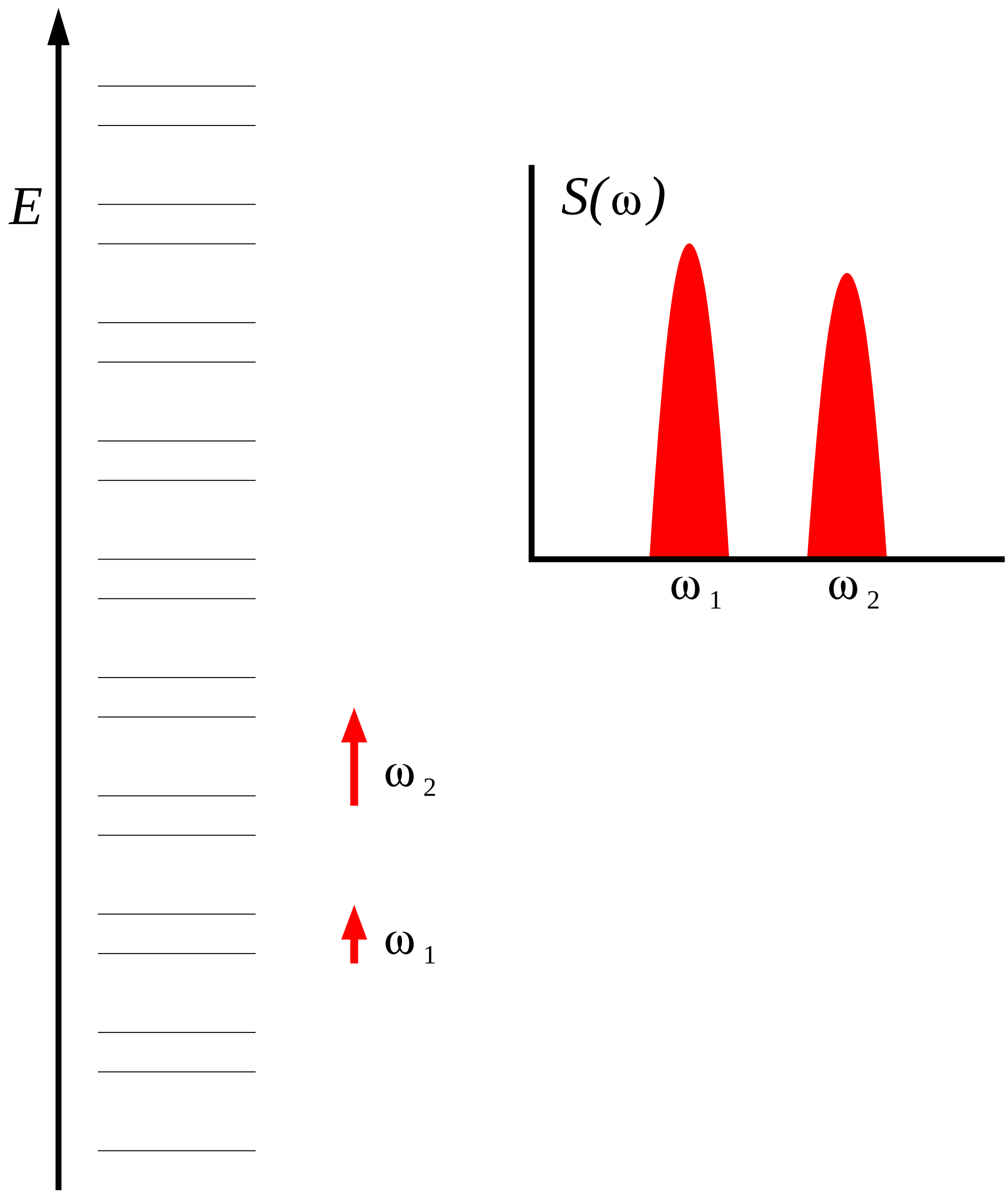
Within the framework of LRT

$$\tilde{S}(\omega) \mapsto \lambda \tilde{S}(\omega) \qquad \Longrightarrow \qquad \boldsymbol{D} \mapsto \lambda \boldsymbol{D}$$
$$\tilde{S}(\omega) \mapsto \sum_i \tilde{S}_i(\omega) \qquad \Longrightarrow \qquad \boldsymbol{D} \mapsto \sum_i \boldsymbol{D}_i$$

But there are circumstance such that e.g.

$$\boldsymbol{D} \; = \; \left[\int R(\omega) \left[\tilde{S}(\omega) \right]^{-1} d\omega \right]^{-1}$$

Simplest illustration



$$D \gg D_1 + D_2$$

Example: energy absorption by metallic grains

Linear response theory:

$$\boldsymbol{D} = \sigma^2 \hbar \varrho \int_0^{\infty} \mathrm{d}\omega \, \omega^2 \, R_2(\hbar\omega) \tilde{S}(\omega)$$

Semi-linear response theory:

$$\boldsymbol{D} \;=\; \frac{\sigma^2}{(\varrho\hbar)^3} \left[\int \frac{\mathrm{d}\mathbf{x} \, \mathrm{e}^{-\mathbf{x}^2/2}}{(2\pi)^{N/2} \mathbf{x}^2} \right]^{-1} \left[\int_0^{\infty} \mathrm{d}\omega \frac{P_2(\varrho\hbar\omega)}{\tilde{S}(\omega)} \right]^{-1}$$

Level spacing statistics:

$$P_2(s) \approx a_\beta s^\beta \exp(-c_\beta s^2) \qquad \text{with } \beta = 1, 2, 4$$

The LRT result of Gorkov and Eliashberg:

$$\boldsymbol{G} \;=\; C_\beta \sigma^2 (\hbar\varrho)^{\beta+1} \, T^{\beta+2}$$

Our SLRT result (large s statistics!):

$$\boldsymbol{G} \;=\; \frac{\sigma^2}{2\hbar\varrho} \frac{1}{(\hbar\varrho\omega_0)^{\beta-1}} \; \exp\left[-\frac{1}{\pi(\hbar\varrho T)^2} \right]$$

The conductance of metallic rings

Non interacting "spinless" electrons in a ring.

$$\mathcal{H}(r, p; \Phi(t))$$

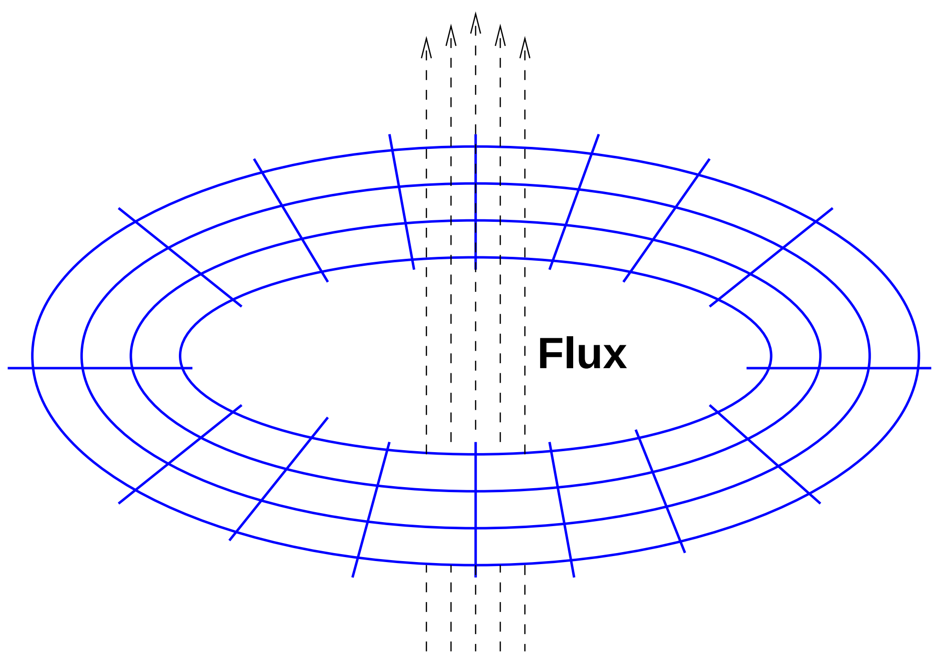
$$-\dot{\Phi} \quad = \quad \text{electro motive force (RMS)}$$

$$\boldsymbol{G}\,\dot{\Phi}^2 \quad = \quad \text{rate of energy absorption}$$

$$\boldsymbol{G} = \pi \left(\frac{e}{L}\right)^2 \mathrm{DOS}^2 \; \langle\langle |v_{mn}|^2 \rangle\rangle$$

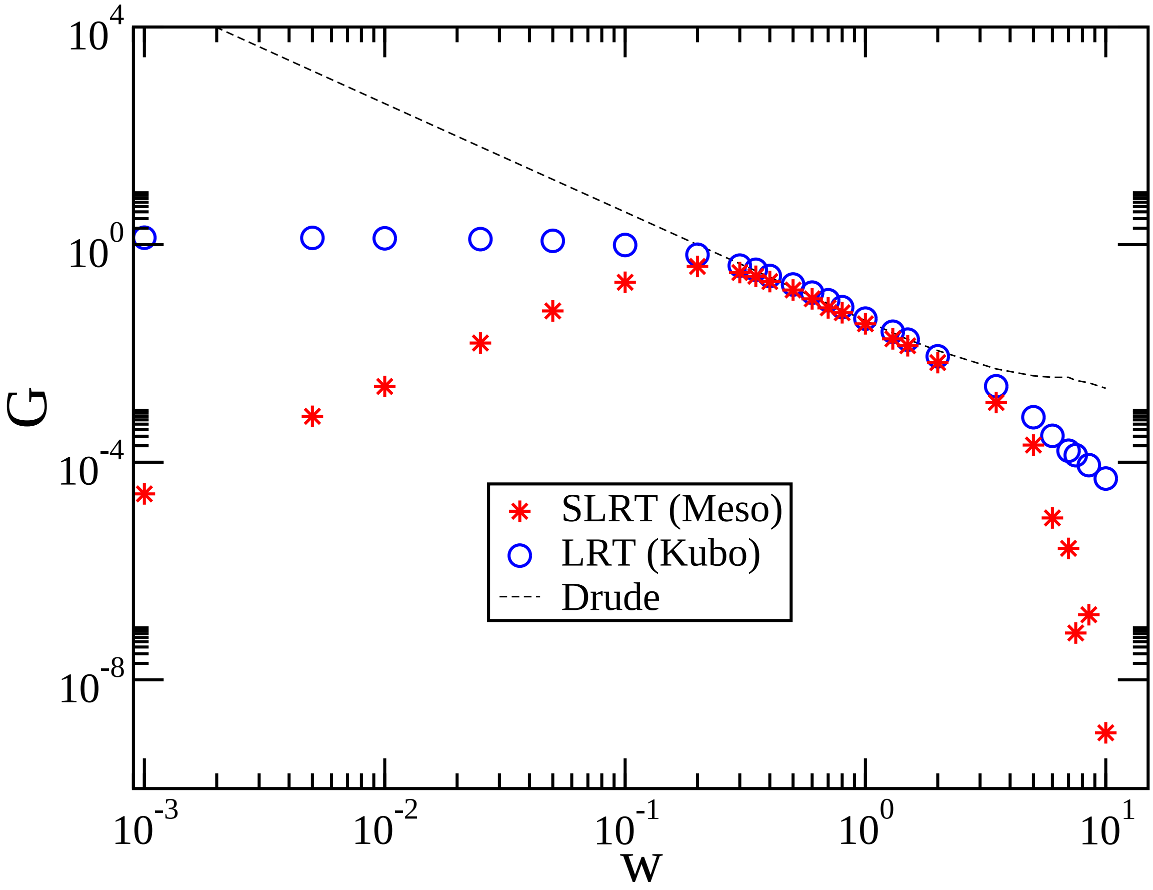
$$\langle\langle |v_{mn}|^2 \rangle\rangle_{\text{harmonic}} \ll \langle\langle |v_{mn}|^2 \rangle\rangle \ll \langle\langle |v_{mn}|^2 \rangle\rangle_{\text{algebraic}}$$

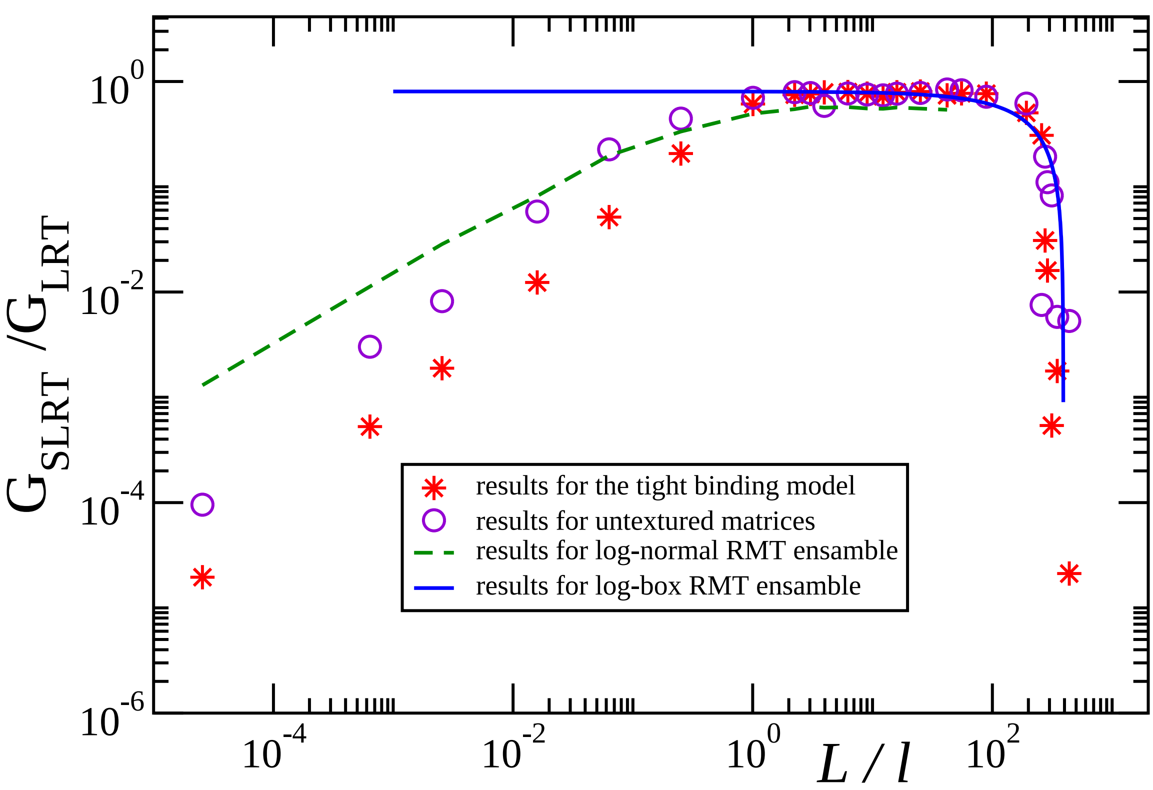
$\mathcal{M}$ mode ring of length L with disorder W



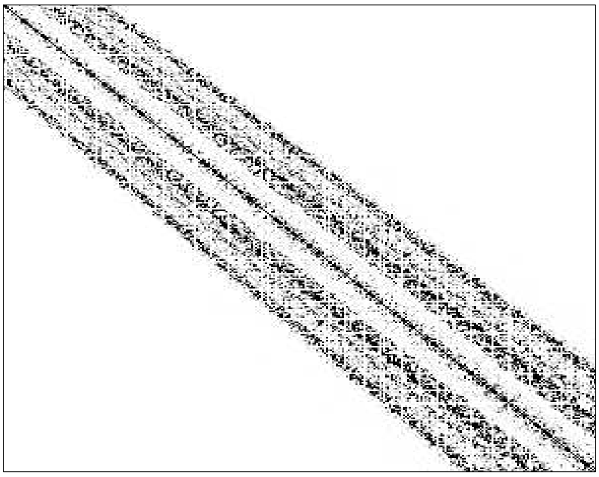
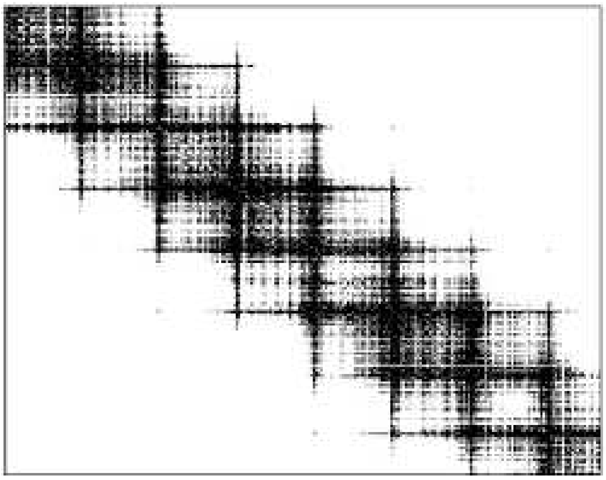
Numerical Results

Regimes: ballistic; diffusive; localizaion





Linear response theory (LRT)

$$\mathcal{H} = \{E_n\} - \frac{e}{L}\Phi(t)\{v_{nm}\}$$

$$\boldsymbol{G} \;=\; \pi\left(\frac{e}{L}\right)^2 \sum_{n,m} |v_{mn}|^2 \; \delta_T(E_n - E_F)\; \delta_\Gamma(E_m - E_n)$$

$$\boldsymbol{G} \;=\; \pi\left(\frac{e}{L}\right)^2 \mathrm{DOS}^2 \; \langle\langle |v_{mn}|^2 \rangle\rangle_{\mathrm{algebraic}}$$

applies if

EMF driven transitions $\ll$ relaxation

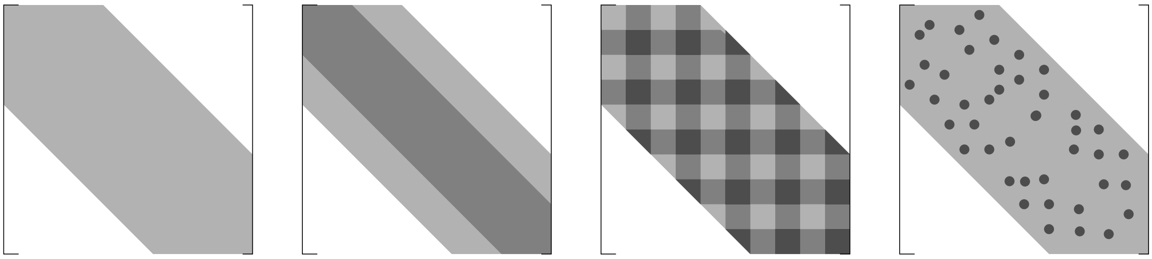
otherwise

connected sequences of transitions are essential.

leading to

Semi Linear Response Theory (SLRT)

Bandprofile, sparsity and texture



$$\boldsymbol{G} \;=\; \pi \left(\frac{e}{L}\right)^2 \mathrm{DOS}^2 \; \langle\langle |v_{mn}|^2 \rangle\rangle$$

$\langle\langle |v_{mn}|^2 \rangle\rangle \;\equiv\;$ inverse resistivity of the network

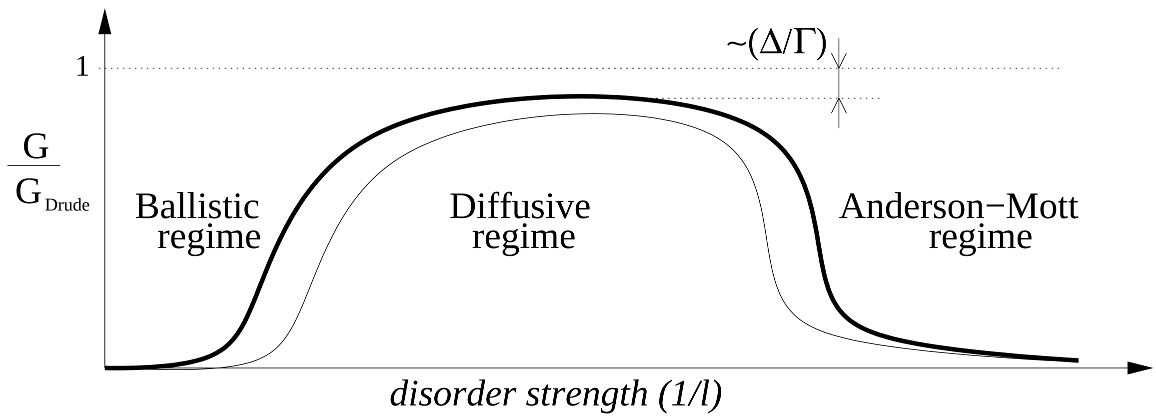
Bounds:

$$\langle\langle |v_{mn}|^2 \rangle\rangle_{\mathrm{harmonic}} \ll \langle\langle |v_{mn}|^2 \rangle\rangle \ll \langle\langle |v_{mn}|^2 \rangle\rangle_{\mathrm{algebraic}}$$

Analytical estimates:

- Mixed average scheme
- Variable range hopping scheme

Conductance versus disorder



Naive expectation (assuming $\Gamma > \Delta$):

$$G = \frac{e^2}{2\pi\hbar}\mathcal{M}\frac{\ell}{L} + \mathcal{O}\left(\frac{\Delta}{\Gamma}\right)$$

L = perimeter of the ring

ℓ = mean free path $\propto W^2$

ℓ_∞ = localization length $\approx \mathcal{M}\ell$

Ballistic regime: $L \ll \ell$

Diffusive regime: $\ell \ll L \ll \ell_\infty$

Anderson regime: $\ell_\infty \ll L$

Strategy of analysis

Given W...

Characterization of the eigenstates:

- participation ratio (PR)

Characterization of v_{nm} and RMT modeling

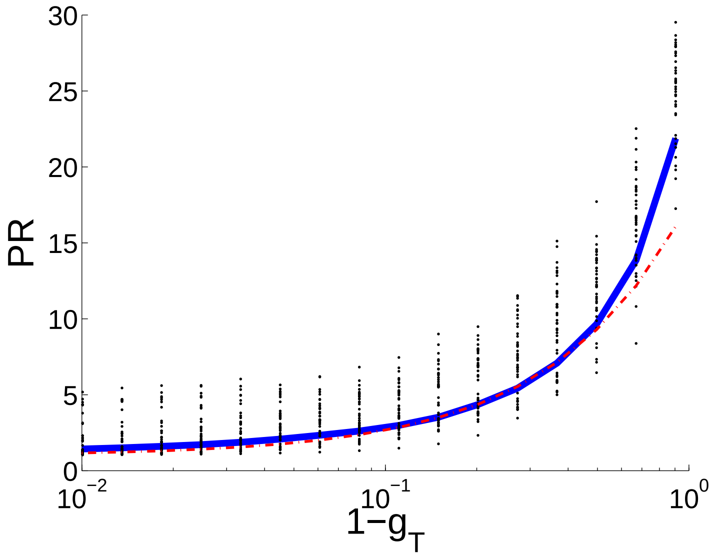
- bandwidth
- sparsity (p)
- texture

Approximation schemes for G

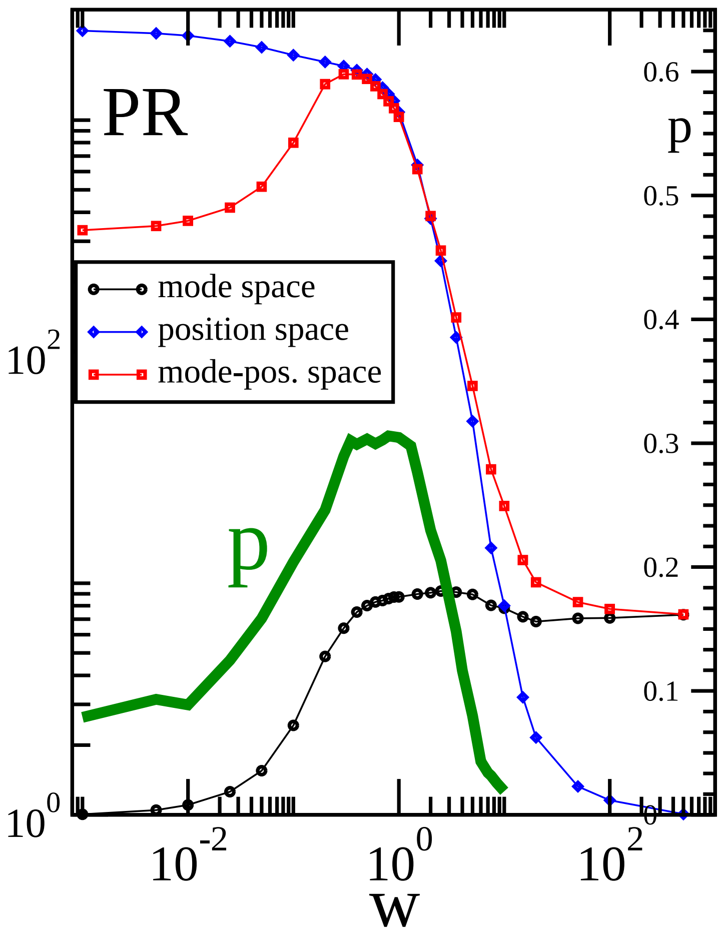
- Mixed average
- Variable range hopping estimate

Ergodicity of the eigenstates

- Weak disorder (ballistic rings):
  Wavefunctions are localized in mode space.
- Strong disorder (Anderson localization):
  Wavefunctions are localized in real space.



The PR of eigenstates of a ring with a single scatterer. The horizontal axis is the reflection of the scatterer.



The PR of eigenstates of a ring with disorder. The horizontal axis is W.

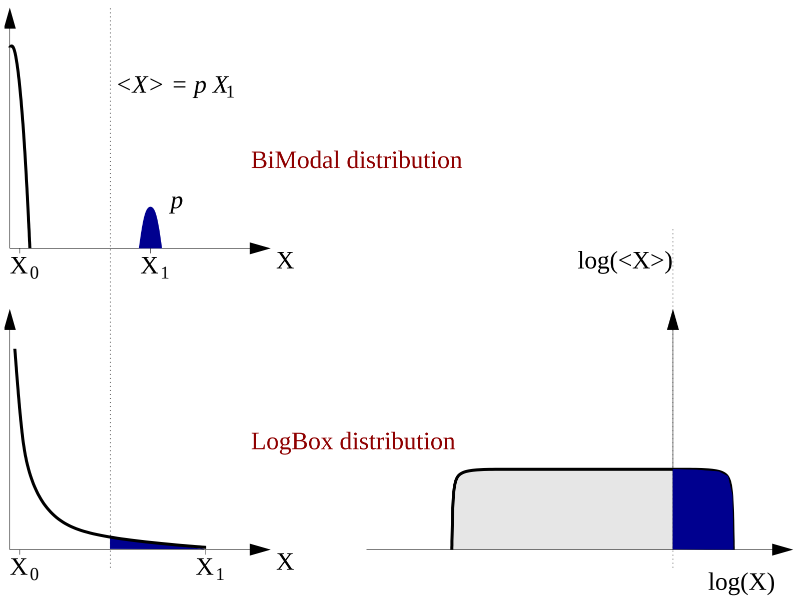
The sparsity (p) of the perturbation matrix is related to the ergodicity of the eigenstates.

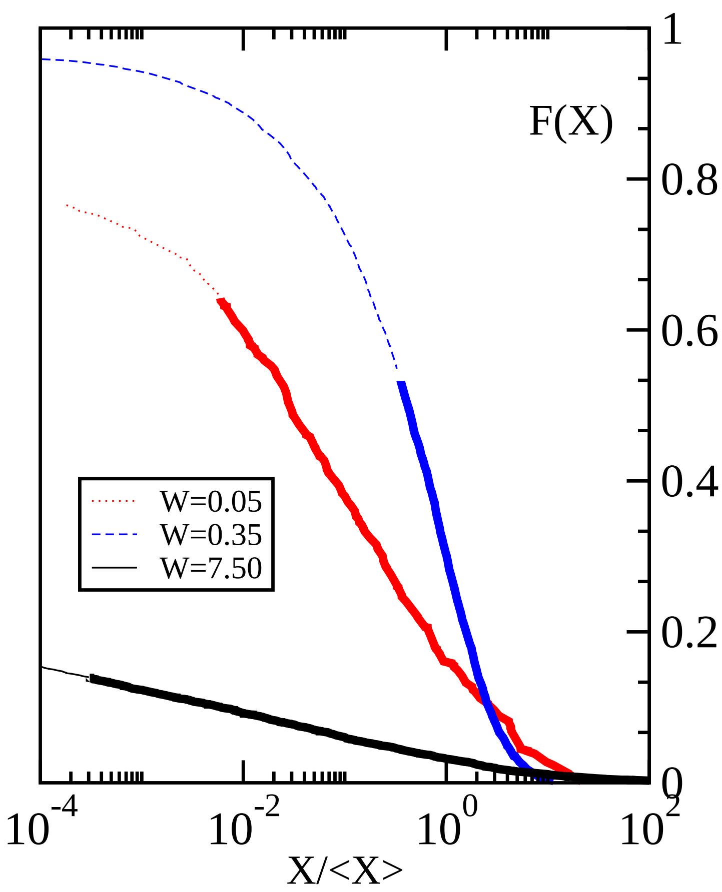
$\{|v_{nm}|^2\}$ **as a random matrix** $\{X\}$

The fraction of "large" elements:

$p \equiv \mathrm{F}(\langle X \rangle)$

Sparsity: $p \ll 1$.





Histograms of X:

Ballistic:

$X \sim \mathrm{LogNormal}$

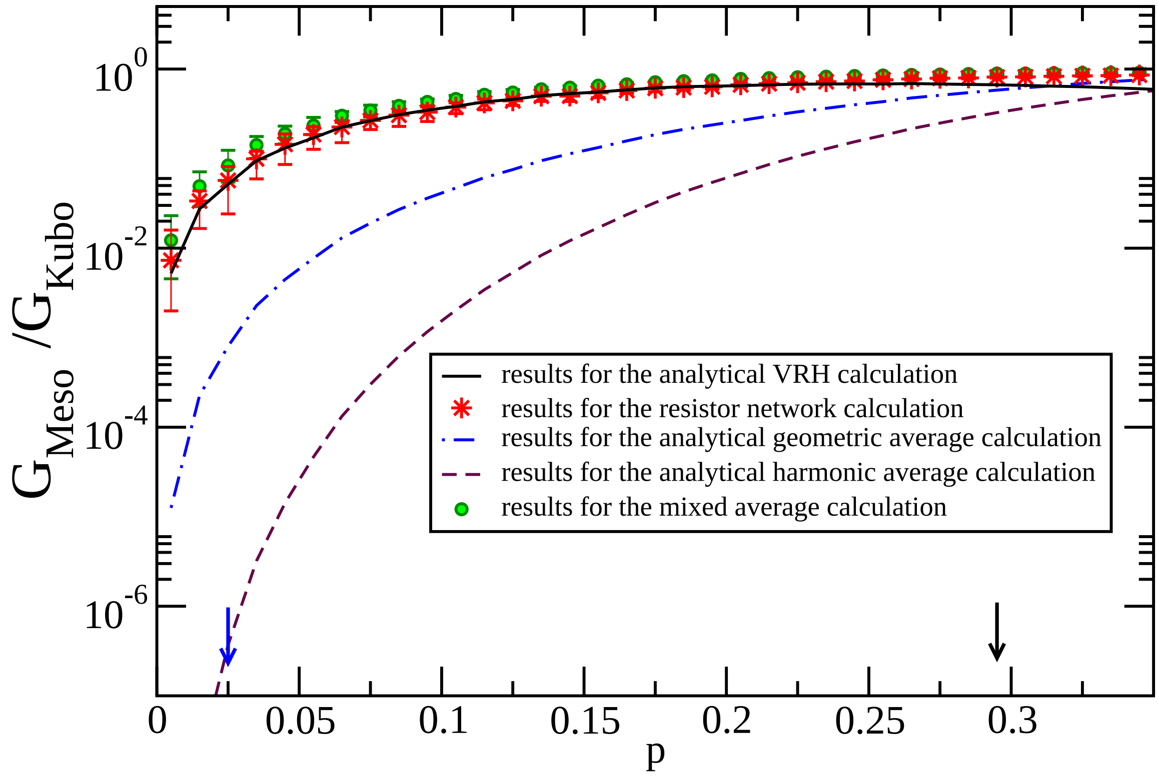
Localization:

$X \sim \mathrm{LogBox}$

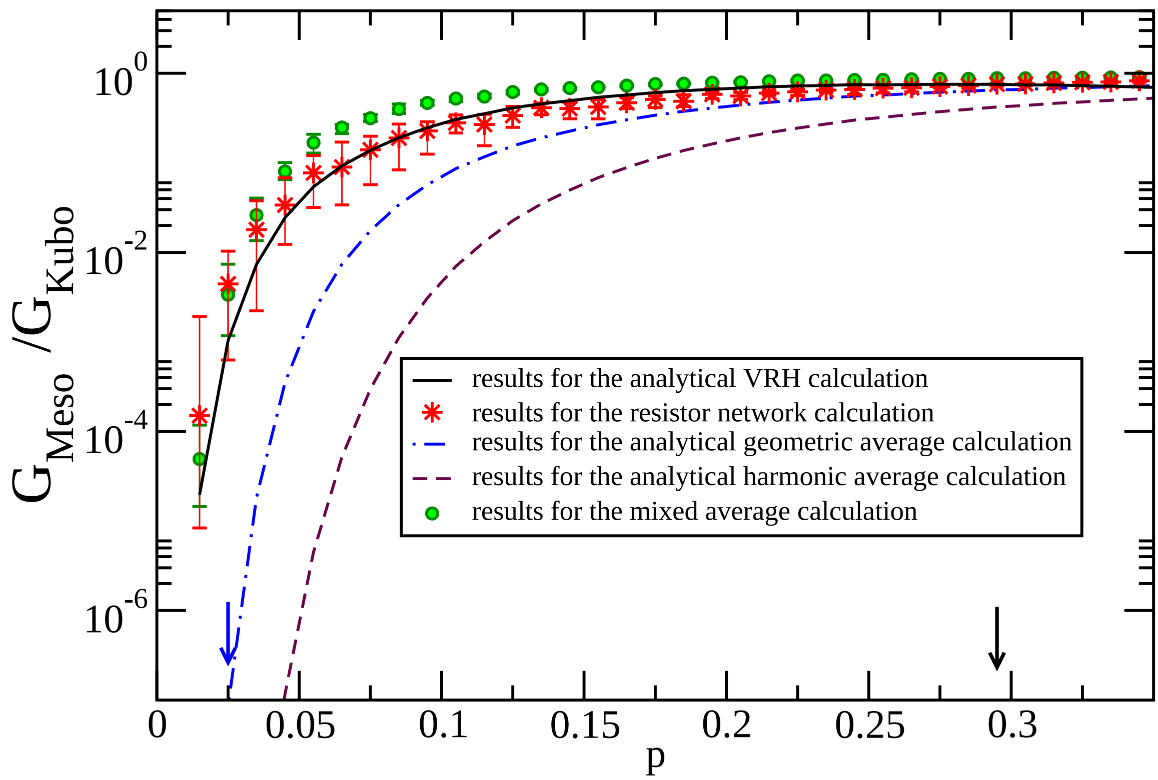
RMT based prediction for $G_{\mathrm{SLRT}}/G_{\mathrm{LRT}}$

RMT implied dependence on p

Log-normal distribution:



Log-box distribution:



The VRH estimate

$$\boldsymbol{G} \;=\; \pi\hbar\left(\frac{e}{L}\right)^2 \sum_{n,m} |v_{mn}|^2 \; \delta_T(E_n-E_F)\; \delta_\Gamma(E_m-E_n)$$

$$\boldsymbol{G} \;=\; \frac{1}{2}\left(\frac{e}{L}\right)^2 \varrho_{\text{F}} \int \tilde{C}_{\text{qm}}(\omega)\; \delta_\Gamma(\omega)\; d\omega$$

$$\tilde{C}_{\text{qm-LRT}}(\omega) \;\equiv\; 2\pi\varrho_{\text{F}}\,\langle X\rangle$$

$$\tilde{C}_{\text{qm-SLRT}}(\omega) \;\equiv\; 2\pi\varrho_{\text{F}}\,\overline{X}$$

where by definition: $\left(\frac{\omega}{\Delta}\right)\text{Prob}\Big(X > \overline{X}\Big) \;\sim\; 1$

For strong disorder we get:

$$\overline{X} \;\approx\; v_{\text{F}}^2\,\exp\left(-\frac{\Delta_\ell}{\omega}\right)$$

$$\boldsymbol{G} \;\propto\; \int \exp\left(-\frac{\Delta_\ell}{|\omega|}\right)\,\exp\left(-\frac{|\omega|}{T}\right)\,d\omega$$

LRT, SLRT and beyond

$$-\dot{\Phi} \quad = \quad \text{electro motive force (RMS)}$$

$$G\,\dot{\Phi}^2 \quad = \quad \text{rate of energy absorption}$$

Semi linear response theory

[1] D. Cohen, T. Kottos and H. Schanz,
"Rate of energy absorption by a closed ballistic ring",
(JPA 2006)

[2] S. Bandopadhyay, Y. Etzioni and D. Cohen,
The conductance of a multi-mode ballistic ring,
(EPL 2006)

[3] M. Wilkinson, B. Mehlig, D. Cohen,
The absorption of metallic grains,
(EPL 2006)

[4] D. Cohen,
"From the Kubo formula to variable range hopping",
(PRB 2007)

[5] A. Stotland, R. Budoyo, T. Peer, T. Kottos and D. Cohen,
The conductance of disordered rings,
(JPA / FTC 2008)

Beyond (semi) linear response theory

[6] D. Cohen and T. Kottos,
"Non-perturbative response of Driven Chaotic Mesoscopic Systems",
(PRL 2000)

[7] A. Stotland and D. Cohen,
"Diffractive energy spreading and its semiclassical limit",
(JPA 2006)

[8] A. Silva and V.E. Kravtsov,
Beyond FGR,
(PRB 2007)

[9] D.M. Basko, M.A. Skvortsov and V.E. Kravtsov,
Dynamical localization,
(PRL 2003)

Conclusions

(*) Wigner ($\sim$ 1955):
The perturbation is represented by a random matrix whose elements are taken from a Gaussian distribution.

Not always...

1. Ballistic ring $\Longrightarrow$ log-normal distribution.
2. Strong localization $\Longrightarrow$ log-box distribution.
3. Resistors network calculation to get G_{SLRT}.
4. Generalization of the VRH estimate
5. SLRT is essential whenever the distribution of matrix elements is wide ("sparsity") or if the matrix has "texture".
6. Other applications of SLRT...