BEC dynamics in a few site systems

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The Bose-Hubbard Hamiltonian (BHH) for a dimer

\[ H = \sum_{i=1,2} \left[ E_i \hat{n}_i + \frac{U}{2} \hat{n}_i(\hat{n}_i - 1) - \frac{K}{2} (\hat{a}_2^{\dagger} \hat{a}_1 + \hat{a}_1^{\dagger} \hat{a}_2) \right] \]

\( N \) particles in a double well is like spin \( j = N/2 \) system
\[ H = -\mathcal{E} \hat{J}_z + U \hat{J}_z^2 - K \hat{J}_x + \text{const} \]

Classical phase space
\[ \mathcal{H}(\theta, \varphi) = \frac{NK}{2} \left[ \frac{1}{2} u (\cos \theta)^2 - \varepsilon \cos \theta - \sin \theta \cos \varphi \right] \]
\[ \mathcal{H}(\hat{n}, \varphi) = \text{(similar to Josephson/pendulum Hamiltonian)} \]

\[ \hat{J}_z = (N/2) \cos(\theta) = \hat{n} = \text{occupation difference} \]
\[ \hat{J}_x \approx (N/2) \sin(\theta) \cos(\varphi), \quad \varphi = \text{relative phase} \]

Rabi regime: \( u < 1 \) (no islands)
Josephson regime: \( 1 < u < N^2 \) (sea, islands, separatrix)
Fock regime: \( u > N^2 \) (empty sea)

\( K = \text{hopping} \)
\( U = \text{interaction} \)
\( \mathcal{E} = \mathcal{E}_2 - \mathcal{E}_1 = \text{bias} \)

\[ u \equiv \frac{NU}{K}, \quad \varepsilon \equiv \frac{\varepsilon}{K} \]

Assuming \( u > 1 \) and \( |\varepsilon| < \varepsilon_c \)

Sea, Islands, Separatrix

\[ \varepsilon_c = (u^{2/3} - 1)^{3/2} \]
\[ A_c \approx 4 \pi \left( 1 - u^{-2/3} \right)^{3/2} \]
Wavepacket dynamics

Coherent state $|\theta \varphi\rangle$ is like a minimal Gaussian wavepacket
Fock state $|n\rangle$ is like equi-latitude annulus

Fock $n=0$ preparation - exactly half of the particles in each site
Fock coherent $\theta=0$ preparation - all particles occupy the left site
Coherent $\varphi=0$ preparation - all particles occupy the symmetric orbital
Coherent $\varphi=\pi$ preparation - all particles occupy the antisymmetric orbital

MeanField theory (GPE) = classical evolution of a point in phase space
SemiClassical theory = classical evolution of a distribution in phase space
Quantum theory = recurrences, fluctuations (WKB is very good!)
WKB quantization (Josephson regime)

\[ h = \text{Planck cell area in steradians} = \frac{4\pi}{N+1} \]

\[ A(E_n) = \left( \frac{1}{2} + n \right) h \]

\[ \omega(E) \equiv \frac{dE}{dn} = \left[ \frac{1}{h} A'(E) \right]^{-1} \]

\[ \omega_K \approx K = \text{Rabi Frequency} \]

\[ \omega_J \approx \sqrt{NUK} = \sqrt{u} \omega_K \]

\[ \omega_+ \approx NU = u \omega_K \]

\[ \omega_x \approx \left[ \frac{1}{2} \log \left( \frac{N^2}{u} \right) \right]^{-1} \omega_J \]
Recurrences and fluctuations

\[ \vec{S} = \vec{J} / (N/2) \]

\[
\begin{align*}
\langle S_x \rangle, \langle S_y \rangle, \langle S_z \rangle & = \text{Bloch vector} \\
\text{AverageOccupation} & = (N/2) \left[ 1 + \langle S_z \rangle \right] \\
\text{OneBodyPurity} & = \left( \frac{1}{2} \right) \left[ 1 + \langle S_x \rangle^2 + \langle S_y \rangle^2 + \langle S_z \rangle^2 \right] \\
\text{FringeVisibility} & = \left[ \langle S_x \rangle^2 + \langle S_y \rangle^2 \right]^{1/2}
\end{align*}
\]

Spectral analysis of the fluctuations
Dependence on \( u \) and on \( N \)
Various preparations, LDOS analysis
Some observations

Fock regime, Coherent $\varphi$ preparation:
Phase spreading diminishes coherence $\langle S_x \rangle_\infty \approx 0$ [Leggett’s “phase diffusion”]
Evolution leads to a Fock like $n = 0$ state.

But what about the Josephson regime?
$\langle S_x \rangle_\infty$ is determined by $u/N$.

$n = 0$ Fock preparation:
Self induced coherence leading to $\langle S_x \rangle_\infty \approx 1/3$;

Off-separatrix $\varphi = 0$ coherent preparation:
Coherence is maintained if $u/N < 1$ (phase locking).

On-separatrix $\varphi = \pi$ coherent preparation:
Fluctuations are suppressed by $u$.

On-separatrix $\varphi \neq \pi$ coherent preparation:
Fluctuations are suppressed by $N$ (classical limit).
The many body Landau-Zener transition

Dynamical scenarios:
adiabatic/diabatic/sudden
Adiabatic-diabatic (quantum) crossover
Diabatic-sudden (semiclassical) crossover
Sub-binomial scaling of $\text{Var}(n)$ versus $\langle n \rangle$
Quantum Stirring in a 3 site system

\[ \hat{H} = \sum_{i=0}^{2} \varepsilon_i n_i + \frac{U}{2} \sum_{i=0}^{2} n_i (n_i - 1) - k_c (b_1^\dagger b_2 + b_2^\dagger b_1) - k_1 (b_0^\dagger b_1 + b_1^\dagger b_0) - k_2 (b_0^\dagger b_2 + b_2^\dagger b_0) \]

The induced current: \[ I = -G \dot{\varepsilon} \]

The pumped particles: \[ Q = \int I dt = \int G \cdot dX \] (per cycle)
Stirring of BEC

strong attractive interaction: classical ball dynamics

negligible interaction (|U| \ll \kappa/N): mega-crossing

weak repulsive interaction: gradual crossing

strong repulsive interaction (U \gg N\kappa): sequential crossing
Results for the geometric conductance

\[ G(R) = -N \frac{(k_1^2-k_2^2)/2}{[(\varepsilon-\varepsilon_-)^2+2(k_1+k_2)^2]^{3/2}} \]

\[ G(J) \approx - \left[ \frac{k_1-k_2}{k_1+k_2} \right] \frac{1}{3U} \]

\[ G(F) = - \left( \frac{k_1-k_2}{k_1+k_2} \right) \sum_{n=1}^{N} \frac{(\delta\varepsilon_n)^2}{[(\varepsilon-\varepsilon_n)^2+(2\delta\varepsilon_n)^2]^{3/2}}, \]

where:

- \( R = \) Rabi regime \((U \ll \kappa/N)\)
- \( J = \) Josephson regime \((\kappa/N \ll U \ll N\kappa)\)
- \( F = \) Fock regime \((U \gg N\kappa)\)

Observation:

It is possible to pump \( Q \gg N \) per cycle.
Summary

- Semiclassical and WKB analysis of the dynamics
- Occupation statistics in a time dependent scenario
- The adiabatic / diabatic / sudden crossovers
- Quantum stirring: mega / gradual / sequential crossings