

Reduced Compressibility and Inverse Problem for Spinning Gas

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A spinning ideal gas in a cylinder with a smooth surface is shown to have unusual properties. First, under compression parallel to the axis of rotation, the spinning gas exhibits reduced compressibility because energy can be stored in the rotation. Second, the spinning breaks the symmetry under which partial pressures of a mixture of gases simply add proportional to the constituent number densities. Thus, remarkably, in a mixture of spinning gases, an inverse problem can be formulated such that the gas constituents can be determined through external measurements only.

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Introduction: The classic problem of rotating gases arises in a variety of physical contexts, both naturally and in device applications [1, 2]. Natural contexts include accretion discs or compact stars such as white dwarfs [3]. Device applications include vortices in turbines and centrifuges [4]. The equilibrium state can be characterized by conserved quantities, including both relativistic and quantum effects [5, 6]. In the absence of relativistic or quantum effects, an equation of state for radially adiabatically compressing spinning gases was recently obtained and numerically simulated [7].

What we consider here is a nonrelativistic, spinning Boltzmann gas, rotating with angular velocity ω , confined to a long cylinder of radius r_0 and length L , with $L \gg r_0$. The spinning gas can also be compressed adiabatically either longitudinally (parallel to the axis of rotation) or radially. The gas has very small slip on the smooth cylinder walls, so angular momentum is conserved on time scales of interest. We will assume that: (i) the cylinder is long, so that end effects are negligible; (ii) there is no friction on the walls (smooth wall condition); and (iii) the gas is at all times very close to equilibrium. From assumption (iii), it follows that the temperature T is constant in space, and that viscous friction assures that all rotating layers have the same angular velocity ω (solid body rotation). The rotation can be initialized even for vanishing viscosity by spinning the cylinder for a time long enough to reach equilibrium. The third assumption means that the slow adiabatic processes of gas compression and expansion can be approached through thermodynamic considerations, neglecting fast dynamics such as inhomogeneous flow and turbulence.

We show here that such a spinning gas has two hitherto unappreciated and unusual properties:

First, under longitudinal compression, the spinning gas features reduced compressibility. To see the reduced compressibility effect, consider that in the case of relatively low temperature, the spinning gas hugs the cylinder walls. But as the gas is compressed longitudinally, its temperature rises, so the spinning gas can no longer hug the cylinder walls. Thus, as the gas moves to lower radii, its rotational velocity must increase in order to conserve

angular momentum. The energy for the increased rotational energy must come from the temperature, so for a given decrease in length L , the pressure of the spinning gas increases less than the pressure of a stationary gas at the same temperature. In other words, because energy can be stored in rotation, the rotation introduces a reduced compressibility in the axial direction.

Second, because of the rotation, partial pressures of a mixture of gases no longer simply add proportional to the constituent number densities. Under rotation, heavier gases will sediment more towards larger radii, thereby capturing more of the angular momentum. That leaves the lighter gases less subject to, for example, the reduced compressibility effect. Since the partial pressures of different constituents now respond differently to rotation, the equation of state is more complicated. But this also means that, from the external parameters (such as pressure, radius, length, or temperature), it is possible to deduce the concentrations of individual constituents in the gas mixture. Thus, the spinning gas permits an inverse problem for gas composition, not available for the non-spinning gas.

This inversion can have practical implications: Given a cylinder with an unknown gas mixture that is unsafe to open or otherwise difficult to analyze spectroscopically, its components might be deduced nonetheless by spinning the cylinder and measuring external parameters, such as, for example, the pressure response to longitudinal or radial compression.

Basic equations: To see these unusual properties, consider first the equation of state for a single-component ideal gas, $P = nT$, with pressure P and density n . Using assumption (iii), namely the isothermal approximation, we have $dP = dnT$. From radial force balance we have $dP = drn(r)m\omega^2r$, which can be integrated to give

$$n = n_0 \exp\left(\frac{m\omega^2r^2}{2T}\right), \quad (1)$$

where the constant n_0 can be written in terms of radius

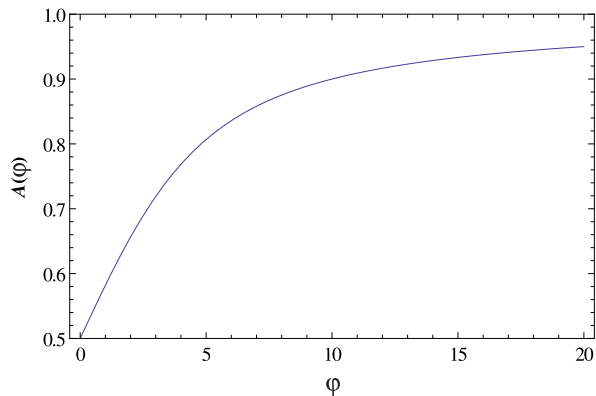


FIG. 1: Functional dependence of angular momentum on φ .

r_0 and total number of particles N as

$$n_0 = \frac{m\omega^2 N}{2\pi T L} \left(\frac{1}{e^\varphi - 1} \right) = \frac{N}{V} \frac{\varphi}{e^\varphi - 1}. \quad (2)$$

Here we introduce $\varphi = m\omega^2 r_0^2 / 2T$, the ratio of rotation kinetic energy at the boundary to the temperature, and the volume of the cylinder, $V = \pi r_0^2 L$. Note that small φ describes the limit of ideal gas without rotation, while large φ describes the limit of cold particles all rotating at frequency ω along the boundary at radius r_0 .

The total angular momentum M can be written as

$$M = mL \int_0^{r_0} 2\pi r dr n(r) \omega r^2 = Nm\omega r_0^2 A(\varphi), \quad (3)$$

where

$$A(\varphi) = \frac{e^\varphi(\varphi - 1) + 1}{\varphi(e^\varphi - 1)}, \quad (4)$$

and where the factor $Nm\omega r_0^2$ is the angular momentum if all particles were rotating with frequency ω at the boundary. As expected and as we see from Fig.(1), $A(\varphi)$ tends to 1/2 for φ small to and 1 for large φ . The most interesting behavior will result from intermediate φ .

Since the total angular momentum $M = M(\omega, r_0, \varphi)$ is conserved, while $\varphi = \varphi(\omega, r_0, T)$, we can write the variation of M in the form

$$\begin{aligned} d\omega \left(\frac{\partial M}{\partial \omega} + \frac{\partial M}{\partial \varphi} \frac{\partial \varphi}{\partial \omega} \right) + dT \frac{\partial M}{\partial \varphi} \frac{\partial \varphi}{\partial T} + \\ dr_0 \left(\frac{\partial M}{\partial \varphi} \frac{\partial \varphi}{\partial r_0} + \frac{\partial M}{\partial r_0} \right) = 0. \end{aligned} \quad (5)$$

For completeness, the partial derivatives required to cal-

culate $d\omega$ may be written as:

$$\frac{\partial M}{\partial \varphi} = M \frac{(1 + e^{2\varphi} - 2e^\varphi - \varphi^2 e^\varphi)}{\varphi(e^\varphi - 1)(e^\varphi(\varphi - 1) + 1)} = MH(\varphi), \quad (6)$$

$$\frac{\partial M}{\partial \omega} = \frac{M}{\omega}, \quad \frac{\partial M}{\partial r_0} = 2 \frac{M}{r_0}, \quad (7)$$

$$\frac{\partial \varphi}{\partial r_0} = 2 \frac{\varphi}{r_0}, \quad \frac{\partial \varphi}{\partial \omega} = 2 \frac{\varphi}{\omega}, \quad \frac{\partial \varphi}{\partial T} = -\frac{\varphi}{T}. \quad (8)$$

where for simplicity we introduced a function $H(\varphi)$.

In Eq. (5), the terms $\partial M/\partial \omega$ and $\partial M/\partial r_0$ are independent of the redistribution of the particle density, and would be the sole surviving terms in the cold gas limit. The term $\partial M/\partial \varphi$ reflects the changing particle distribution at finite temperature.

The energy E can be written as $E = E' + M\omega/2$, where E' is the internal energy in rotating coordinate system and rotation energy is given by the second term [2]. For an ideal, nonrelativistic gas, the particle internal energy depends on temperature only, so the energy can be written as $E = c_v NT + M\omega/2$, where c_v is the specific heat. The energy change, for an adiabatic process, can then be put as the work done on the system, so that

$$pdV = -dE = -c_v NdT - (M/2)d\omega, \quad (9)$$

The energy balance together with angular momentum conservation provides complete information in calculating how internal variables, such as T and ω , depend on external variables such as r_0 and L .

Longitudinal Compression: Slowly compressing the cylinder from the ends (keeping radius r_0 constant), increases the temperature. However, if the temperature increases, then φ decreases, so that, to conserve angular momentum, ω must increase. Thus, the rotation speed increases, even in the absence of radial compression, with Eq. (3) giving implicitly the dependence of ω on T . The energy distribution between heating and spinning may be found by first integrating the pressure on the surface of the cylinder end to find

$$pdV = \int_0^{r_0} 2\pi r dr p(r) dL = \frac{NT}{L} dL. \quad (10)$$

Note that the side compression acts only against the thermal motion of the particles, since the spinning velocity is perpendicular to the force exerted. Now using Eq. (10) together with Eq. (9), and substituting for $d\omega$, from Eq. (5), we get

$$\frac{dL}{L} + \frac{dT}{T} (c_v + B(\varphi)) = 0, \quad (11)$$

where in Fig. 2, we plot the function $B(\varphi)$,

$$B(\varphi) = \frac{\varphi^2 A(\varphi) H(\varphi)}{1 + 2\varphi H(\varphi)}, \quad (12)$$

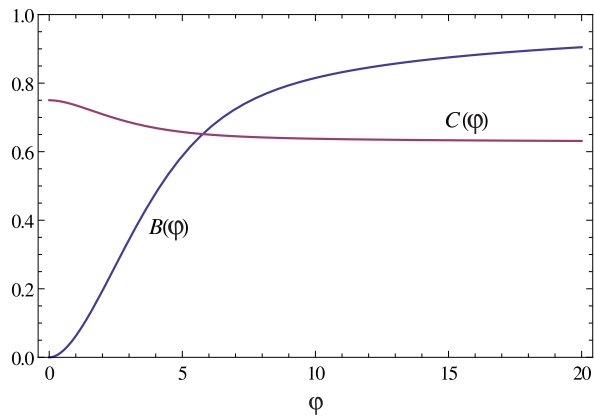


FIG. 2: Parallel and perpendicular compression functions, $B(\varphi)$ and $C(\varphi)$.

which governs the extent to which compression energy is converted into rotation energy. The maximum fraction, $1/(1 + c_v)$, is reached as $\varphi \rightarrow \infty$. In this limit, heating the gas changes the density distribution, resulting in faster spinning. In the opposite limit, $\varphi \rightarrow 0$, the radial density distribution cannot change significantly, so neither can the rotation energy. Clearly, the rotation facilitates compression; because some energy is absorbed by the spinning degree of freedom, the temperature and hence the pressure, grow more slowly with compression, which might be called a *reduced compressibility* effect.

Perpendicular Compression: Perpendicular compression is somewhat more complicated than longitudinal compression, because r_0 is no longer invariant. The work done by radial compression can be written as

$$pdV = dr_0 2\pi L p(r_0) = dr_0 \frac{Nm\omega^2 r_0 e^\varphi}{e^\varphi - 1}. \quad (13)$$

Note that, while the pressure is isotropic, the force acting on the radial walls is different from that is acting in axial direction since it includes also the centrifugal force. Now using Eq. (13) together with Eq. (9), and substituting for $d\omega$, from Eq. (5), we get

$$\frac{dr_0}{r_0} + \frac{dT}{T} C(\varphi) = 0, \quad (14)$$

where

$$C(\varphi) = \frac{\frac{1}{2}(c_v + B(\varphi))}{\frac{\varphi e^\varphi}{e^\varphi - 1} - \varphi A(\varphi) \frac{1 + \varphi H(\varphi)}{1 + 2\varphi H(\varphi)}}. \quad (15)$$

As we see from Fig. 2, in the non-rotating limit, $\varphi \rightarrow 0$ a $C(\varphi)$ approaches $c_v/2$. On the other hand, for the strongly rotating case, $C(\varphi)$ has a finite limit $C(\varphi) \rightarrow (c_v + 1)/4$, even though most of pdV work increases the rotation energy, namely the ratio of the energy going to heat over the work against centrifugal force is

$$D = \frac{c_v N dT}{pdV - c_v N dT} = \frac{c_v (e^\varphi - 1)}{2\varphi e^\varphi C(\varphi) - c_v (e^\varphi - 1)}, \quad (16)$$

and $D \rightarrow 0$ as $\varphi \rightarrow \infty$.

Commutative property: The adiabatic parallel and perpendicular compression will possess a commutative property. To see this, consider differential changes of radius and parallel length (dr_0 , dL), with the goal to demonstrate that the order of dr_0 and dL is immaterial. Call compression first in dr_0 the *RL process* and compression first in dL the *LR process*. For adiabatic processes, by the second law of thermodynamics, the processes must be reversible. Hence, since energy is a state function, monotonic with respect to both T and ω , the work done through *LR* and *RL* must be identical. For, if not, then we could set up a cycle that eventually gives us positive work, by pumping energy through one process and retrieving it through the other. However, this would violate the second law of thermodynamics because, as the total energy is conserved, work can be obtained only through cooling of the gas. Since any macroscopic change can be constructed through adding differential changes, clearly parallel and perpendicular compression are commutative. This result may also be shown directly by calculating the work done through Eq. (13) and Eq. (10).

Inverse problem: While the spinning preserves the commutative property of parallel and perpendicular compression, it does break the property that the total pressure in a system with different mass gases is simply proportional to the total number of gas molecules. Note that the degree to which the spinning affects the compressibility is governed by the ratio of spinning energy to thermal energy φ for each species; since temperature T and rotation ω are common to all gases, the φ for each gas depends linearly on the mass. This, in turn, means that at the same temperature and rotation, different mass gases will behave differently to differential changes in rotation, temperature, or volume.

To be specific, consider that, by Dalton's law, the total pressure is a sum of partial pressures of each gases,

$$p = \sum_{i=0}^K P_i = \frac{T}{V} \sum_{i=0}^K \frac{N_i \varphi_i e^{\varphi_i}}{e^{\varphi_i} - 1}, \quad (17)$$

where $\varphi_i = m_i \omega^2 r_0^2 / 2T$. For the vanishing rotation case, where all the φ_i vanish, we have

$$p \rightarrow \frac{T}{V} \sum_{i=0}^K N_i = \frac{TN_{\text{tot}}}{V}, \quad (18)$$

so, clearly, measurements of external parameters of the system inform only on the total number of particles N_{tot} , but not the individual N_i . Similarly, for supersonic rotation, in the limit that all the φ_i are large, we have

$$p \rightarrow \frac{r_0^2 \omega_0^2}{2V} \sum_{i=0}^K N_i m_i = \frac{r_0^2 \omega_0^2}{2V} m_{\text{tot}}, \quad (19)$$

where m_{tot} is the total confined mass. Thus, external measurements inform only on the total mass, but, as in the opposite limit, not on the individual N_i .

However, for finite rotation, the coefficients weighting the N_i in Eq.(17) differ for different mass molecules, and depend on both temperature T and rotation ω . Thus, by varying either temperature or rotation velocity, in principle, K independent measurements of total pressure could be made to resolve K species through a linear system of equations to find N_i . Of course, the degree to which these measurements are informative will depend upon the sensitivity of the partial pressures to the variations in temperature or rotation velocity. Hence, we can imagine that the most informative measurements occur for intermediate rotation speeds when, for any two species i and j , there is at least one measurement in which not both species are in the same φ limit.

In principle, any number of inverse problems could be constructed in which, given masses m_i , the pressure as a function of external parameters such as L or r_0 are used to calculate the N_i . As but one example, we show here a class of inverse problems that admit analytic solutions, specifically, the distribution of masses $N(m)$ can be found from measuring the radial pressure as a function of T or ω .

First, write the total pressure as

$$p(x) = x \frac{T}{V} \int_0^{\infty} \frac{N(m) m e^{xm} dm}{e^{xm} - 1}, \quad (20)$$

where we introduced the parameter $x = \omega^2 r_0^2 / 2T$. Note that the limit of small x give the total number of particles

$$p(0) = \frac{T}{V} \int_0^{\infty} N(m) dm = \frac{T}{V} N_{\text{tot}}, \quad (21)$$

and the limit of large x gives total mass of particles m_{tot}

$$p(\infty) = x \frac{T}{V} \int_0^{\infty} N(m) m dm = x \frac{T}{V} M_{\text{tot}}. \quad (22)$$

Define now the function $G(x)$

$$G(x) = p(x) - p(\infty) = x \frac{T}{V} \int_0^{\infty} \frac{N(m) m dm}{e^{xm} - 1}. \quad (23)$$

Eq.(23) is Fredholm integral equation of first kind for unknown $N(m)$, with kernel $K(m, x) = m / (e^{xm} - 1)$, and known $G(x)$. Multiply $G(x)$ by x^{s-1} and integrate with respect to x

$$\begin{aligned} \int_0^{\infty} x^{s-1} G(x) dx &= \frac{T}{V} \int_0^{\infty} \left(\int_0^{\infty} \frac{(xm)^s d(xm)}{e^{xm} - 1} \right) dm N(m) m^{-s} \\ &= \frac{T}{V} \int_0^{\infty} \frac{dy y^s}{e^y - 1} \int_0^{\infty} dm N(m) m^{-s}. \end{aligned} \quad (24)$$

Since $G(0) = T N_{\text{tot}} / V$, in order for the first integral to converge we demand $\text{Re}(s) > 1$. The second integral converges when $\text{Re}(s) > 0$. For the last integral to converge we need to have $N(0) = 0$, *i.e.* the realistic condition that all molecules have finite mass. To solve for $N(m)$ we employ an integral Mellin transform, given by

$$\phi(s) = \int_0^{\infty} x^{s-1} f(x) dx, \quad (25)$$

$$f(x) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} x^{-s} \phi(s) ds. \quad (26)$$

Now we have

$$\hat{K}(s) = \int_0^{\infty} y^{s-1} \frac{y}{e^y - 1} dy = \Gamma(s+1) \zeta(s+1), \quad (27)$$

$$\hat{G}(s) = \int_0^{\infty} x^{s-1} G(x) dx, \quad (28)$$

$$\hat{N}(1-s) = \int_0^{\infty} m^{-s} N(m) dm, \quad (29)$$

where, $\Gamma(s)$ and $\zeta(s)$ are the Euler Gamma function and Riemann Zeta function respectively. Thus, the solution for the image is

$$\hat{N}(s) = \frac{V}{T} \frac{\hat{G}(1-s)}{\Gamma(2-s) \zeta(2-s)}, \quad (30)$$

giving $N(m)$

$$N(m) = \frac{V}{2\pi i T} \int_{C-i\infty}^{C+i\infty} \frac{\hat{G}(1-s)}{\Gamma(2-s) \zeta(2-s)} m^{-s} ds. \quad (31)$$

Note that the class of inverse problems solved here began with knowledge of $p(x)$ only, namely the pressure measured at the radial periphery as a function of x , which in turn depends explicitly on rotation speed, temperature and radius. However, these parameters in turn might be interdependent, depending on the particular way in which the parameters are varied. Perhaps the easiest realization of the inverse problem is to vary the the frequency, putting the cylinder in a heat bath, thus leaving the radius and temperature constant. The frequency can be varied until a steady state is reached at constant temperature.

However, insofar as this class of inverse problems is concerned, any way in which pressures are measured as a function of x admits the inverse solution, no matter how the range of x is obtained. For example, we can vary not angular velocity, but temperature, and measure again the pressure on the wall. Alternatively, the spinning gas

can be compressed longitudinally or radially. In each case, a range of x is obtained. The precise range of x expected may be found by using Eq. (14) in the case of perpendicular compression and Eq. (11) in the case of parallel compression. In either case, once the pressure function $p(x)$ is measured, the inversion of the Fredholm equation gives us the distribution function $N(m)$.

However, for the inversion to be robust against noise in the measurements, we expect that, however the range in x is obtained, it will be important to reach intermediate, or approximately sonic (when mean gas flow velocity is equal to thermal velocity), rotation speeds for each gas constituent. As discussed, it is at the sonic speeds that the distribution function of any constituent gas is most sensitive to changes in temperature or rotation speeds. Additional robustness might be obtained through other sources of information, for example, that the molecular masses must take on discrete values only, or that the molecular masses are chosen from a certain finite subset of all possible masses. The quantification of these other sources of information, however, is outside the scope of our effort here. Our main goal here is simply to point out that there is a whole class of inverse problems that now can now be formulated and solved.

Conclusion: The spinning gas possesses a number of significant and apparently unnoticed features. A finite rotation velocity changes the compressibility of the gas in both the perpendicular and axial directions, with the axial direction exhibiting an unusual reduced compressibility effect. The rotation plays the role of energy storage similar to a change in specific heat. However, because this storage effect is most sensitive to external parameters at sonic rotation, as opposed to subsonic or supersonic rotation, constituent gases in a mixture may be distin-

guished by molecular weight. Thus, as opposed the case of no rotation, for finite rotation, a variety of inverse problems can be constructed in which external parameters can resolve the individual constituent gas densities. Remarkably, these inverse problems admit direct solutions by posing the inversion as a Fredholm integral of the first kind. The practical merit of these inverse problems is that gas constituents in a closed cylinder, lacking spectroscopic access, might nonetheless be deduced by external measurements.

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