BH,PLD, cond-mat/0002037

Disorder Induced Transitions in Layered Coulomb Gases and Superconductors Phys. Rev. Lett. 84 5395 (2000)

BH,PLD, cond-mat/0108143

Freezing transitions and the density of states of 2D random Dirac Hamiltonians Phys. Rev. B 65, 125323 (2002)

A. Morozov,BH,PLD, cond-mat/0211080

Decoupling and decommensuration in layered superconductors with columnar defects Phys. Rev. B67, 140505(R) (2003)

BH,PLD, cond-mat/0410019

Disorder induced transitions in layered Coulomb gases and application to flux lattices in superconductors Phys. Rev. B71, 134202 (2005)

BH,PLD, cond-mat/0602391

Interference in presence of Dissipation

Next ??

Are 2D Ising spin glass domain walls described by a Stochastic Loewner Evolution?

Denis Bernard, PLD, Alan A. Middleton

cond-mat/0611433

Christian Hagendorf

Pure 2d critical systems Conformal Invariance (CFT)

interfaces *SLE*

Q: Can some property (interfaces)
of a system w. quenched disorder (ising spin glass)
be described using conformal invariance (SLE) ??

idea:ISG domains may exhibit Conf.Inv

C. Amoruso, A.K. Hartmann,M.B. Hastings, M.A. Moore

cond-mat/0601711

Date (v1): Tue, 31 Jan 2006 16:59:37 GMT (22kb)
Date (revised v2): Wed, 12 Jul 2006 16:49:04 GMT (26kb)
Date (revised v3): Sat, 28 Oct 2006 20:45:22 GMT (51kb)

Conformal Invariance and SLE

in Two-Dimensional Ising Spin Glasses

We did several direct tests of SLE

Outline

- what is SLE ?
- 2d Ising spin glass DW
- numerical evidence

Schramm 1999

Probability measure on set of curves in upper half plane $\mathbb{H}$ such that:

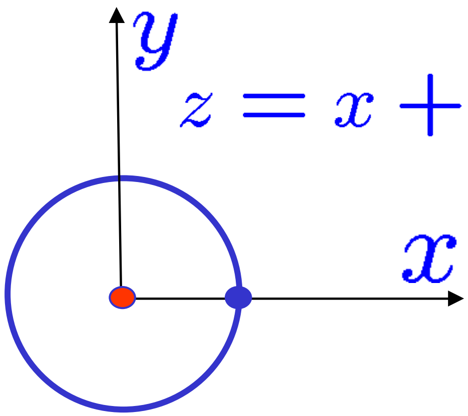
- conformal invariant
- Markov property

Is SLE_κ

Greg Lawler,Ode Schramm,Wendelin Werner

conformal maps, conformal invariance

$z = x + iy$

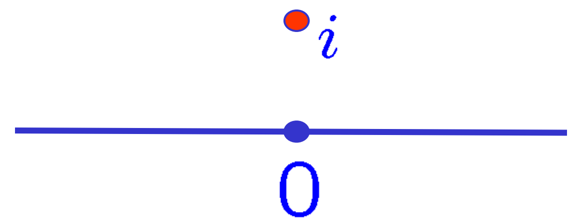


$|z| < 1$

unit disk

$$w = g(z) = i\frac{1-z}{1+z}$$

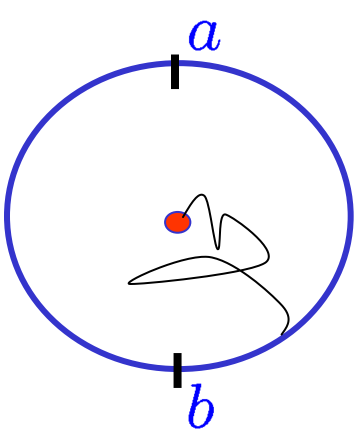
$w = w_1 + iw_2$



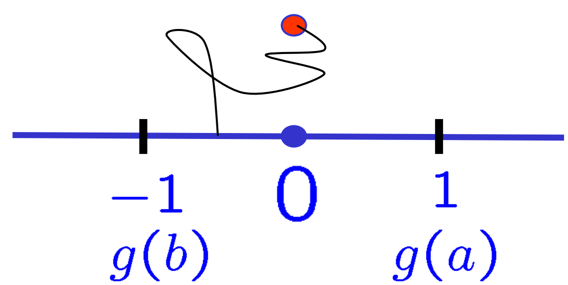
$w_2 > 0$

upper half-plane

Brownian Motion has a CI:



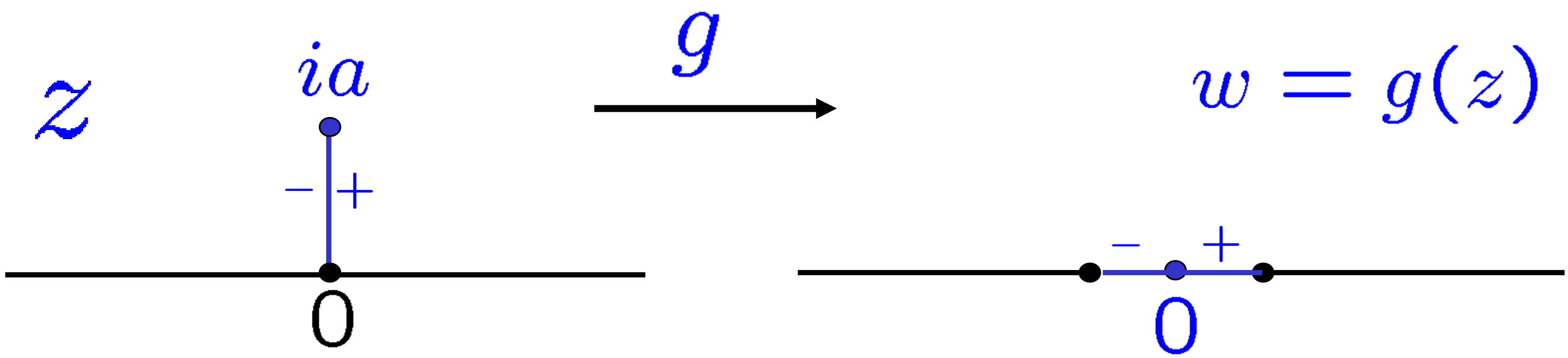
same exit proba



Outline SLE

- uniformizing maps
- Loewner equation
- Markov property:SLE
- Critical interfaces

uniformizing map (segment)



$a > 0$

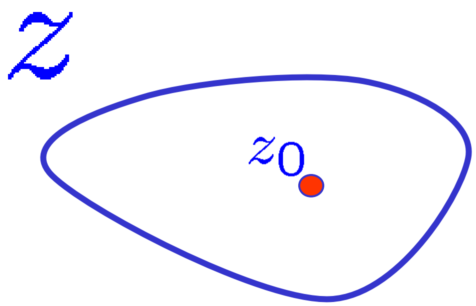
$$g(z) = \sqrt{z^2 + a^2}$$

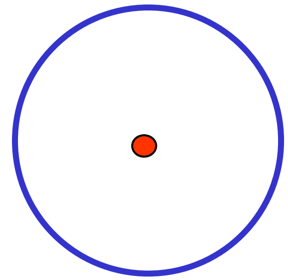
$$g^{-1}(w) = \sqrt{w^2 - a^2}$$

$$g(z) \sim_{z \to \infty} z + \frac{a^2}{2z} + ..$$

map from $\mathbb{H} \backslash [0, ia]$ to $\mathbb{H}$

Riemann mapping theorem





$w = g(z)$

unit disk

$|w| < 1$

simply connected

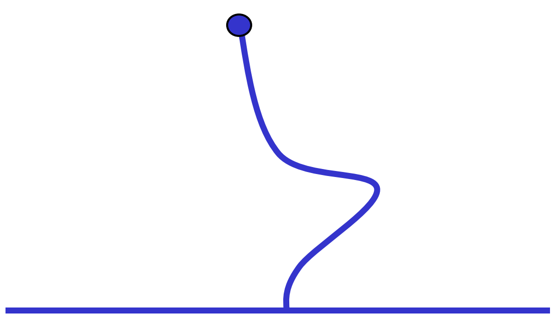
open subset of $\mathbb{C}$

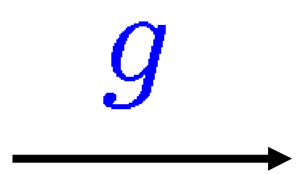
unique if $g(z_0) = 0$

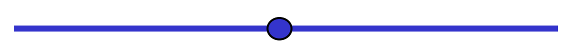
$Arg g'(z_0)$ fixed

3 real conditions

$w' = i\frac{1-w}{1+w}$

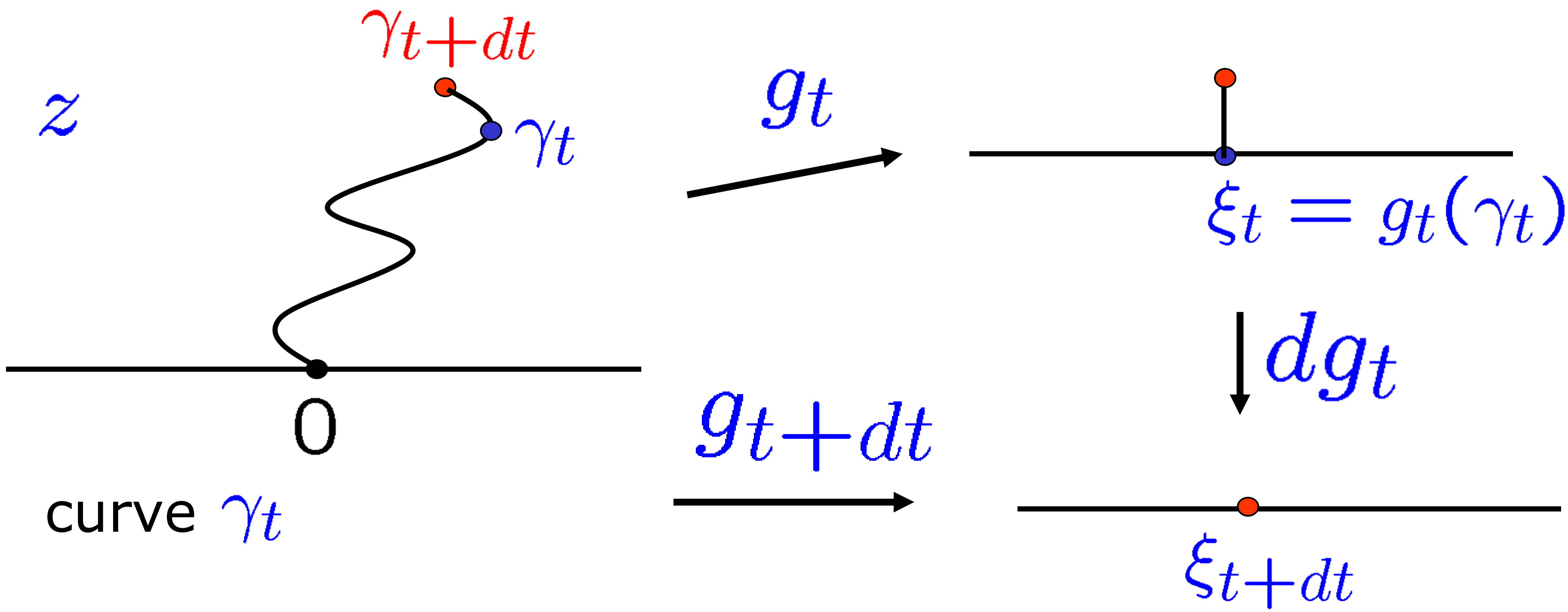






$$g(z) = z + O(\tfrac{1}{z}) \qquad z \to \infty$$

uniformizing map and growth



$$dg_t(z) = \xi_t + \sqrt{(z-\xi_t)^2 + 4dt}$$

$$g_{t+dt}(z) = dg_t(g_t(z))$$
$$= g_t(z) + \frac{2dt}{g_t(z)-\xi_t}$$

Loewner equation

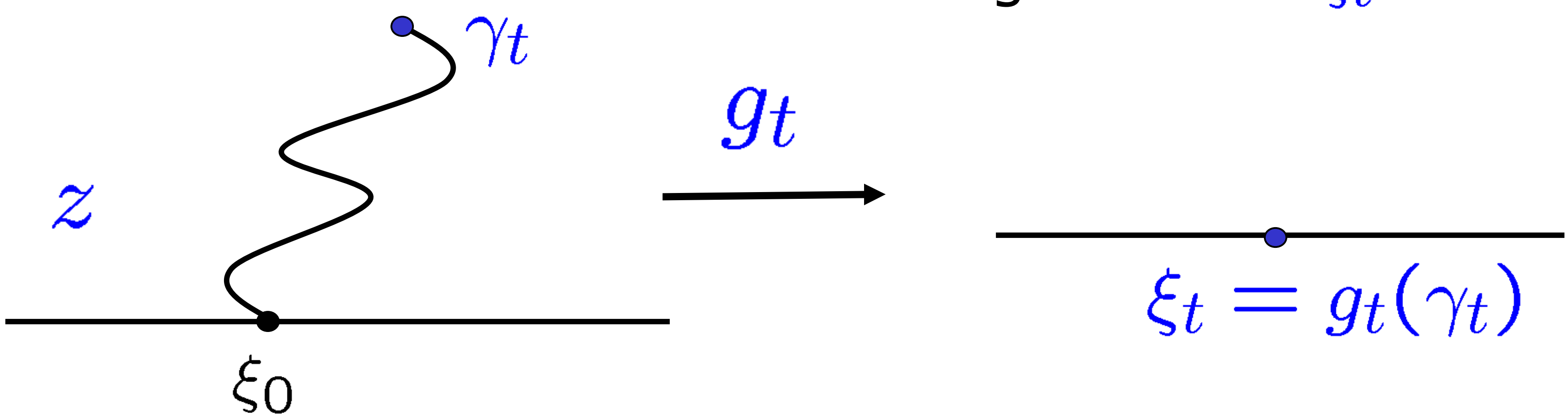
$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t}$$

$$g_{t=0}(z) = z$$

curve $\gamma_t \in \mathbb{H}$

↕

driving function $\xi_t \in \mathbb{R}$



$$g_t(z) \sim_{z \to \infty} z + \frac{2t}{z} + ..$$

t is capacity

Schramm Loewner Evolution

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t}$$

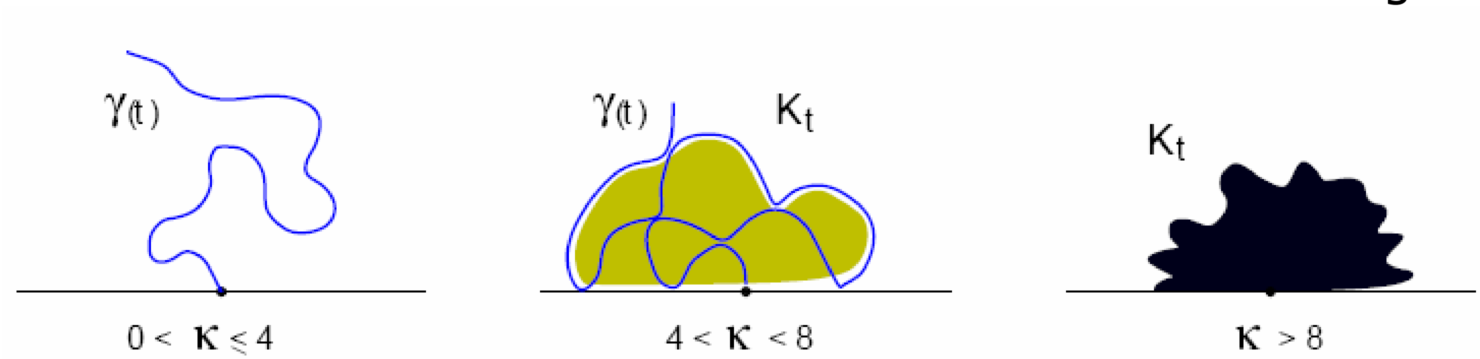
driving function
is Brownian Motion

$\langle \dot{B}(t)\dot{B}(t')\rangle = \delta(t - t')$

$$g_{t=0}(z) = z \qquad \xi_t = \sqrt{\kappa} B_t$$

resulting probability measure on curve γ_t is SLE_κ

non crossing



trace is simple curve no touching

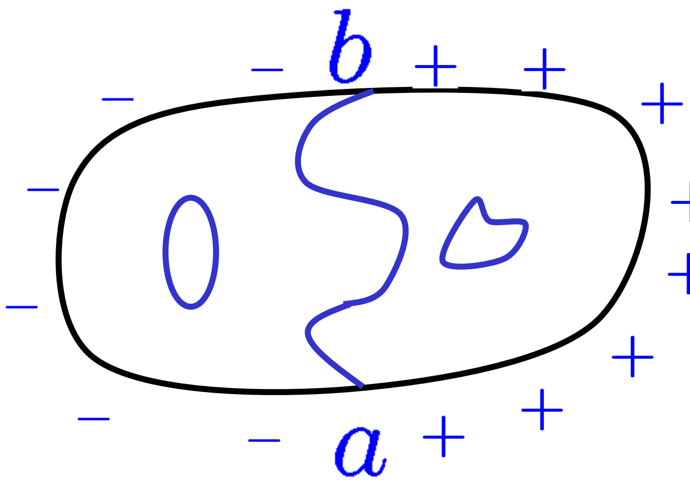
trace touches itself infinitely many double points

space-filling

$$d_f = 1 + \frac{\kappa}{8}$$

$\kappa \leq 8$

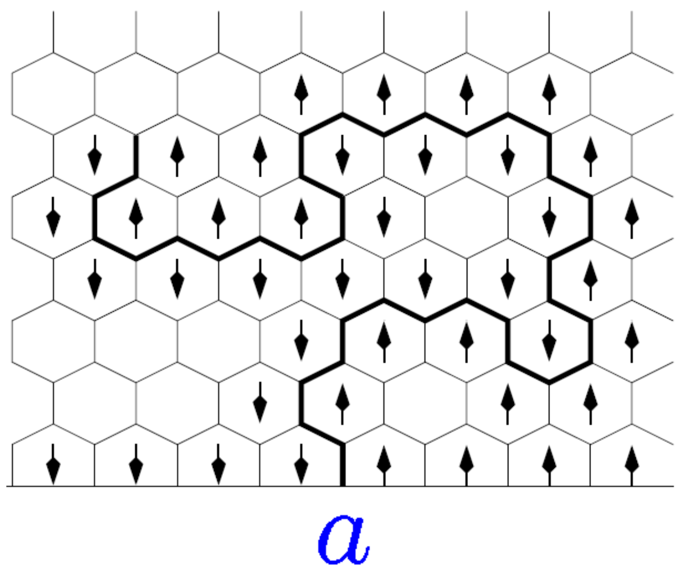
Critical interfaces



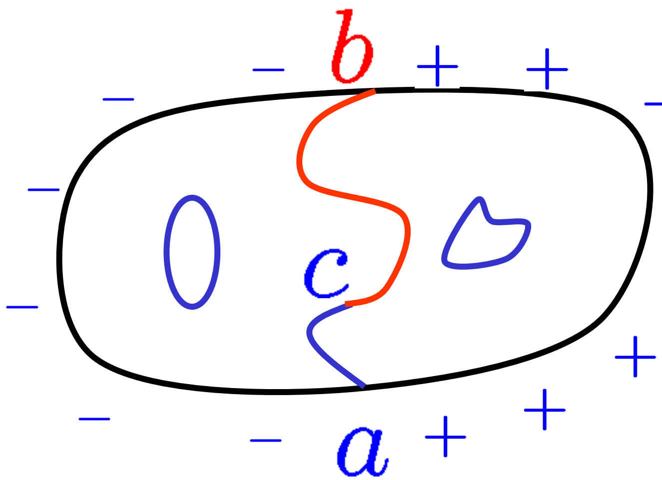
Critical Ising model $K = K_c$

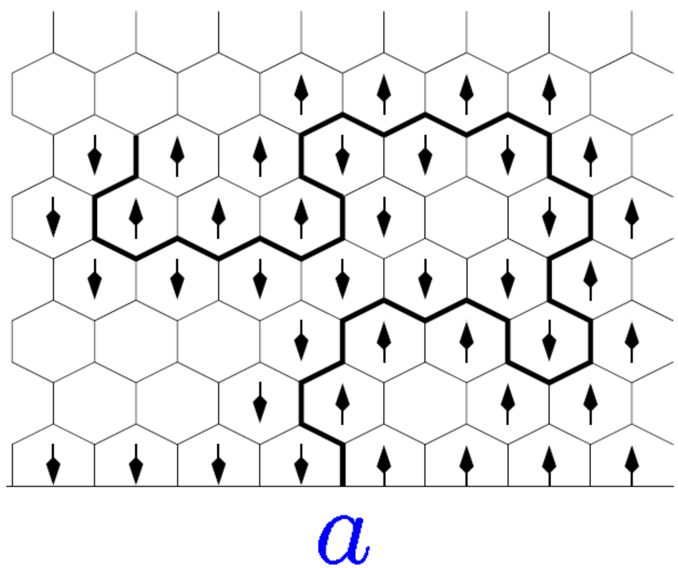
$Z = \sum_{S_i=\pm 1} \exp(K \sum_{<ij>} S_i S_j)$

$$P_D(\gamma_{ab}) = \frac{Z_D(\gamma_{ab})}{Z_D}$$



Critical interfaces





Critical Ising model $K = K_c$

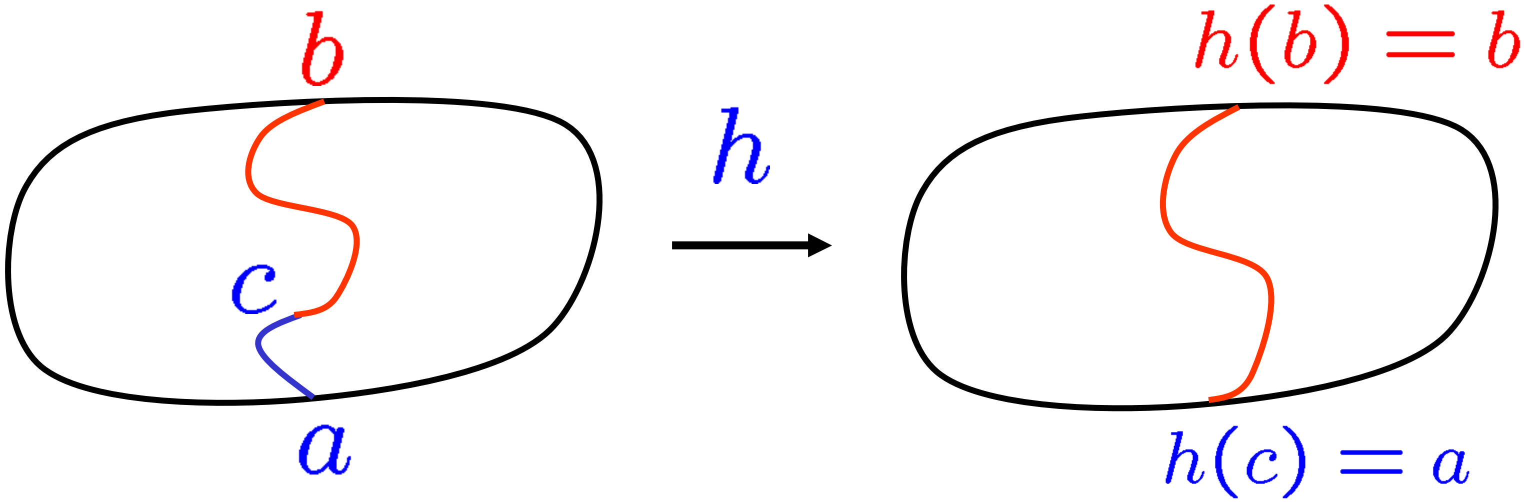
$Z = \sum_{S_i=\pm 1} \exp(K \sum_{<ij>} S_i S_j)$

$$P_D(\gamma_{ab}) = \frac{Z_D(\gamma_{ab})}{Z_D}$$

$$P_D(\gamma_{ab}|\gamma_{ac}) = P_{D-\gamma_{ac}}(\gamma_{cb})$$

(Domain) Markov property

Conformal invariance



$$P_D(\gamma_{cb}|\gamma_{ac}) = P_{D-\gamma_{ac}}(\gamma_{cb})$$

$$= P_{h(D-\gamma_{ac})}(h(\gamma_{cb}))$$

$$= P_D(h(\gamma_{cb})) \longrightarrow \text{SLE}$$

SLE and 2D critical models

Lattice model	κ	c_κ	$d_f(\kappa)$
Loop-erased random walk	2	-2	5/4
Self-avoiding random walk	8/3	0	4/3
Ising model spin cluster boundaries	3	1/2	11/8
Dimer tilings	4	1	3/2
Harmonic explorer	4	1	3/2
Level lines of Gaussian field	4	1	3/2
Ising model FK cluster boundaries	16/3	1/2	5/3
Percolation cluster boundaries	6	0	7/4
Uniform spanning trees	8	-2	2

$$d_f = 1 + \frac{\kappa}{8}$$

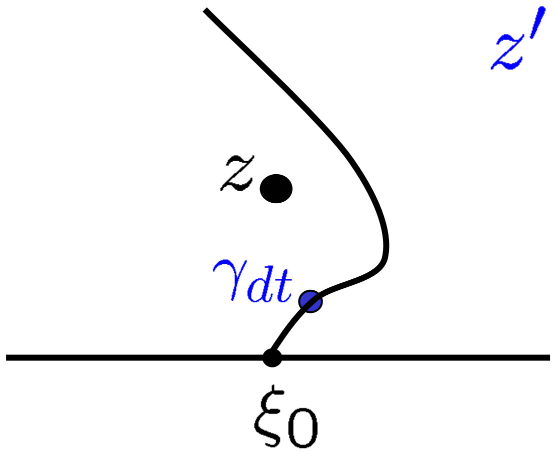
$$c = 1 - 3\frac{(\kappa-4)^2}{2\kappa}$$

$$\kappa' = 16/\kappa \qquad c_{\kappa'} = c_\kappa$$

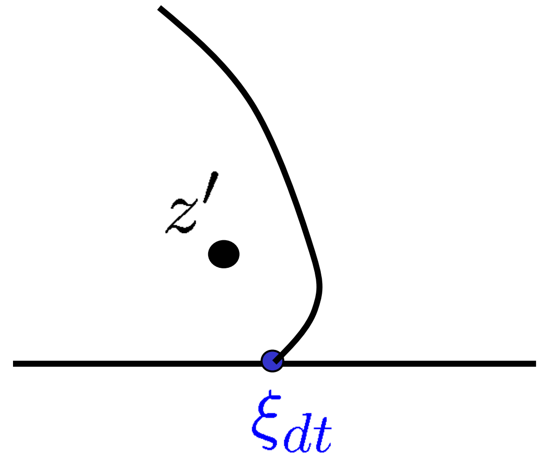
duality: hull boundary ∂K_t of SLE_κ is $SLE_{\kappa'}$

Also O(n) and Potts model, ..

Probability trace passes left of point

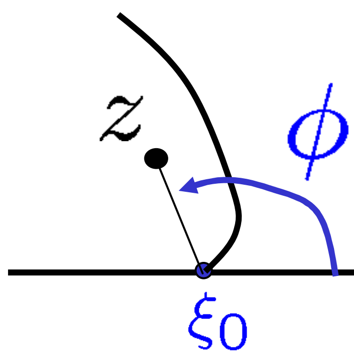


$$z' = z + \frac{2dt}{z - \xi_0}$$



$$p(z, \bar{z}, \xi_0) = < p(z + \frac{2dt}{z-\xi_0}, \bar{z} + \frac{2dt}{\bar{z}-\xi_0}, \xi_0 + d\xi) >_{d\xi}$$

$$\left(\frac{2}{z-\xi_0}\frac{\partial}{\partial z} + \frac{2}{\bar{z}-\xi_0}\frac{\partial}{\partial \bar{z}} + \frac{\kappa}{2}\frac{\partial^2}{\partial \xi_0^2}\right) p(z, \bar{z}, \xi_0) = 0$$



$$P_\kappa(\phi) = \frac{1}{2} + \frac{\Gamma(\frac{4}{\kappa})}{\sqrt{\pi}\,\Gamma(\frac{8-\kappa}{2\kappa})} \cot(\phi)\, {}_2F_1(\tfrac{1}{2}, \tfrac{4}{\kappa}, \tfrac{3}{2}, -\cot^2(\phi))$$

can SLE describe other phenomena?

- 2D turbulence
  inverse cascade

Bernard,Boffetta,
Celani,Falkovich,2006

Surface quasi-geostrophic
turbulence

nlin.cd/0609069 $\kappa = 4$

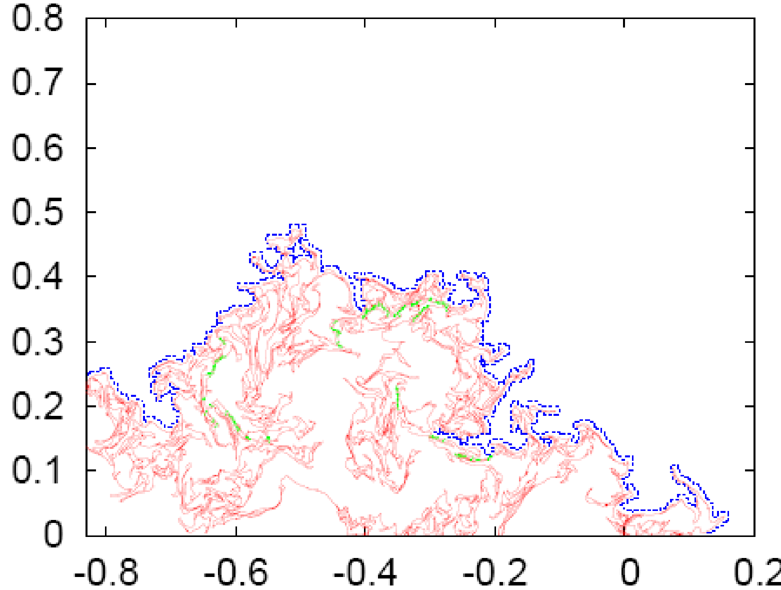


FIG. 3: A portion of a candidate SLE trace obtained from the vorticity field. The red curve is a zero-vorticity line in the upper half-plane. The dashed blue line is the "outer boundary" of the red curve, i.e. the boundary of the region that can be reached from infinity without getting closer than L_f to the red curve. The green dots mark the necks of large fjords and peninsulae.

$d_f = \frac{4}{3}$

$\kappa' = 8/3$

$d_f = \frac{7}{4}$

$\kappa = 6$

- systems with quenched disorder ?

2d Ising spin glass

$H = -\sum_{<ij>} J_{ij} S_i S_j$

$S_i = \pm 1$

J_{ij} i.i.d.

$P(-J) = P(J)$ unit gaussian

T > 0 disordered

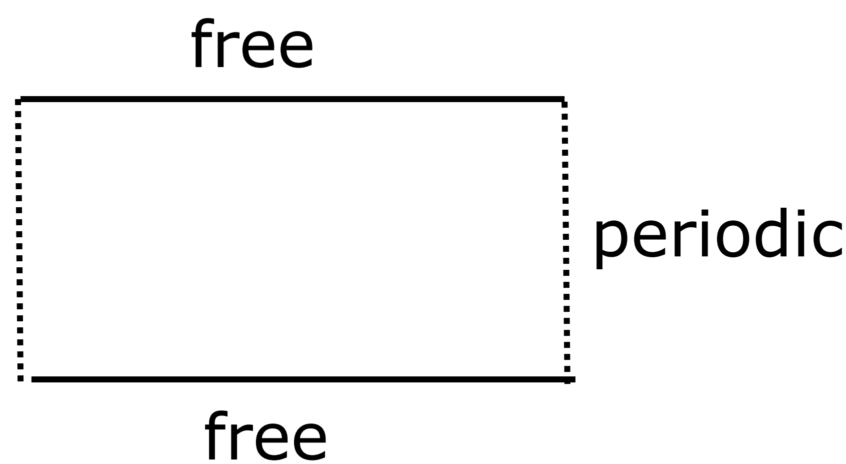
T=0 for given J find ground state pair α $S_i = S_i^0$

β $S_i = -S_i^0$

excitations size l

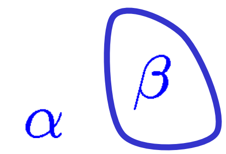
$\Delta E \sim l^\theta$

$\theta < 0$



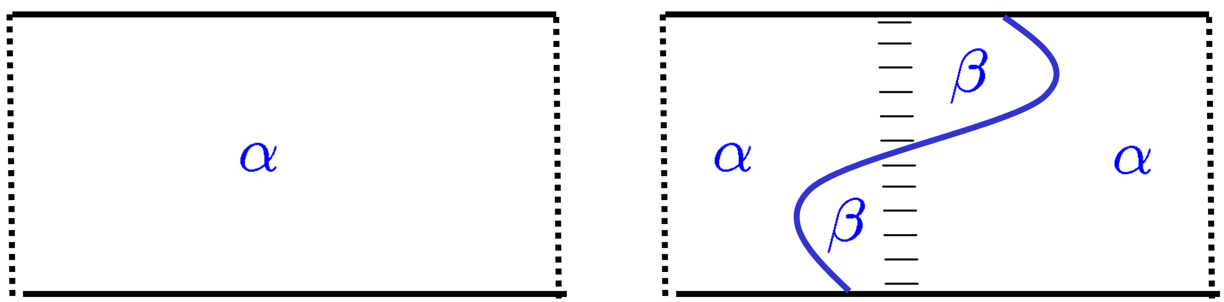
Ground state and domain walls

$F^0_{ij} = J_{ij} S^0_i S^0_j$ bond satisfactions in G.S.



flip spins in a block of frontier γ

$\Delta E = 2 \sum_{ij \in \gamma} F^0_{ij} > 0$ all closed loop γ



$J_{ij} \rightarrow -J_{ij}$

along a column

$E_{antiper} - E_{per} = \text{min}_\gamma\, 2 \sum_{ij \in \gamma} F^0_{ij}$ over all γ

top to bottom paths

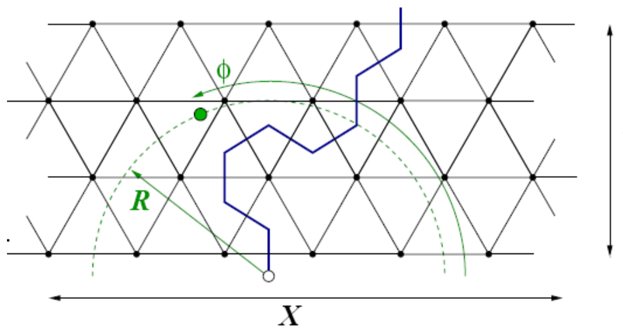
study distribution (over J) of γ_{opt}

DW is where bond satisf changed

numerics

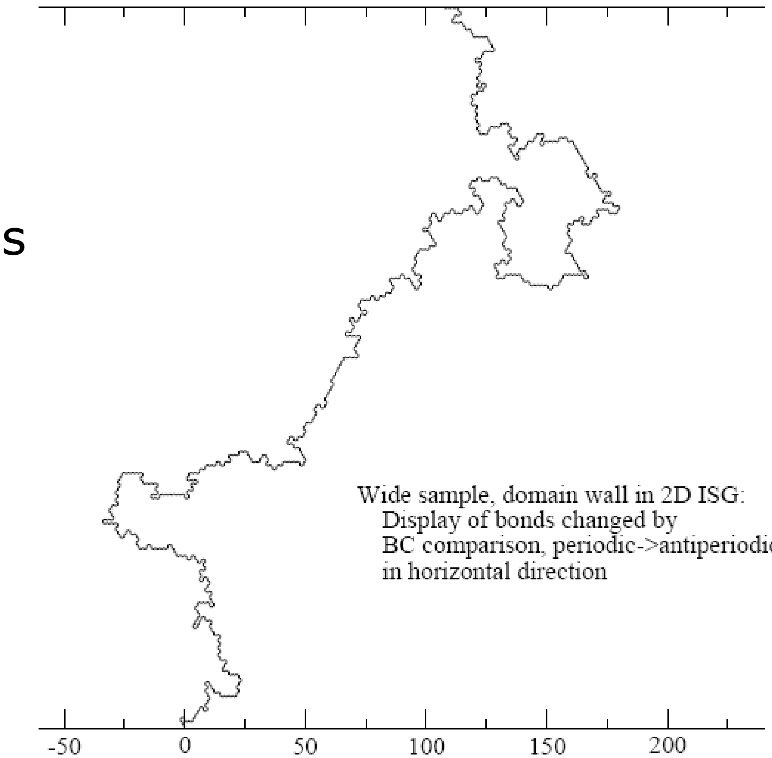
Find exact GS (map to matching problem) and DW

$t \sim N^2$ A. Middleton



Up to 720²
or 1024*512

$> 10^4$ samples



Wide sample, domain wall in 2D ISG:
Display of bonds changed by
BC comparison, periodic->antiperiodic
in horizontal direction

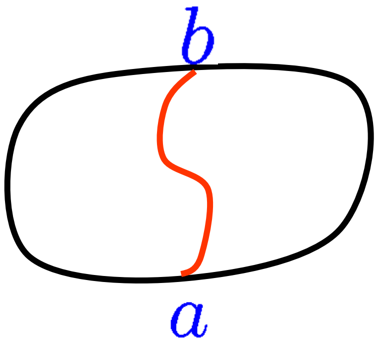
two types of BC: free-free

localized-free

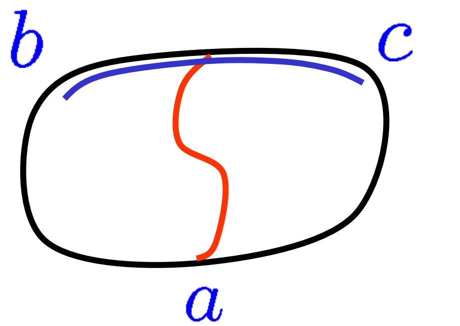
$$E_{DW} \sim L^{\theta} \quad \theta = -0.28(1)$$

$$|\gamma| \sim L^{d_f} \quad d_f = 1.28(1)$$

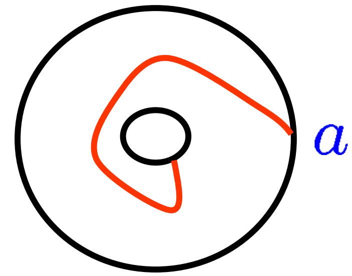
compare with what?



chordal SLE



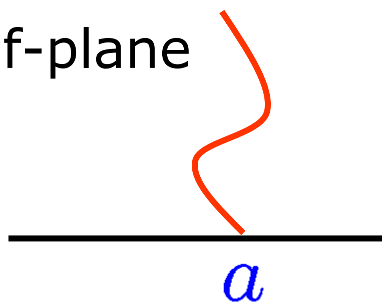
dipolar SLE



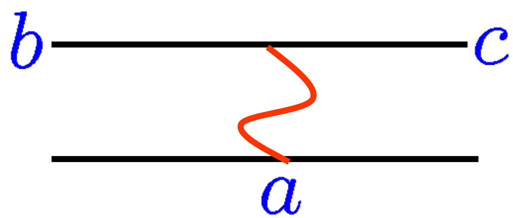
annulus SLE

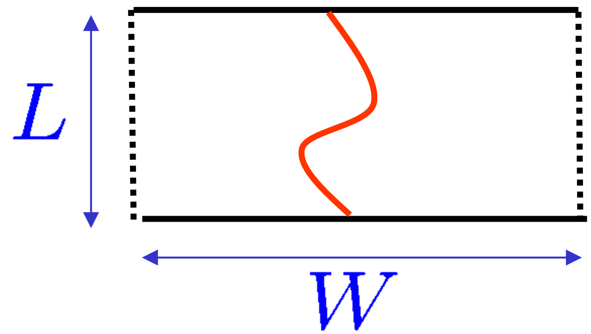
→ radial SLE

half-plane



infinite strip





- chordal near origin
- dipolar when $W \gg L$
- radial when $L \gg W$

For SGlass:

find SLE only free-free

fixing endpoint is bad

SLE tests

- probability DW passes left of point (chordal SLE)
- winding around long cylinder (radial SLE)
- hitting probability of top (dipolar SLE)
- iterated slit maps – is driving funct Brownian Motion?

$\kappa = 8(d_f - 1) = 2.24(8)$ $d_f = 1.28(1)$

free-free BC: find consistent value all tests $\kappa = 2.32(8)$

fixed endpoint BC: tests 1,3,4 give $\kappa \approx 2.8$

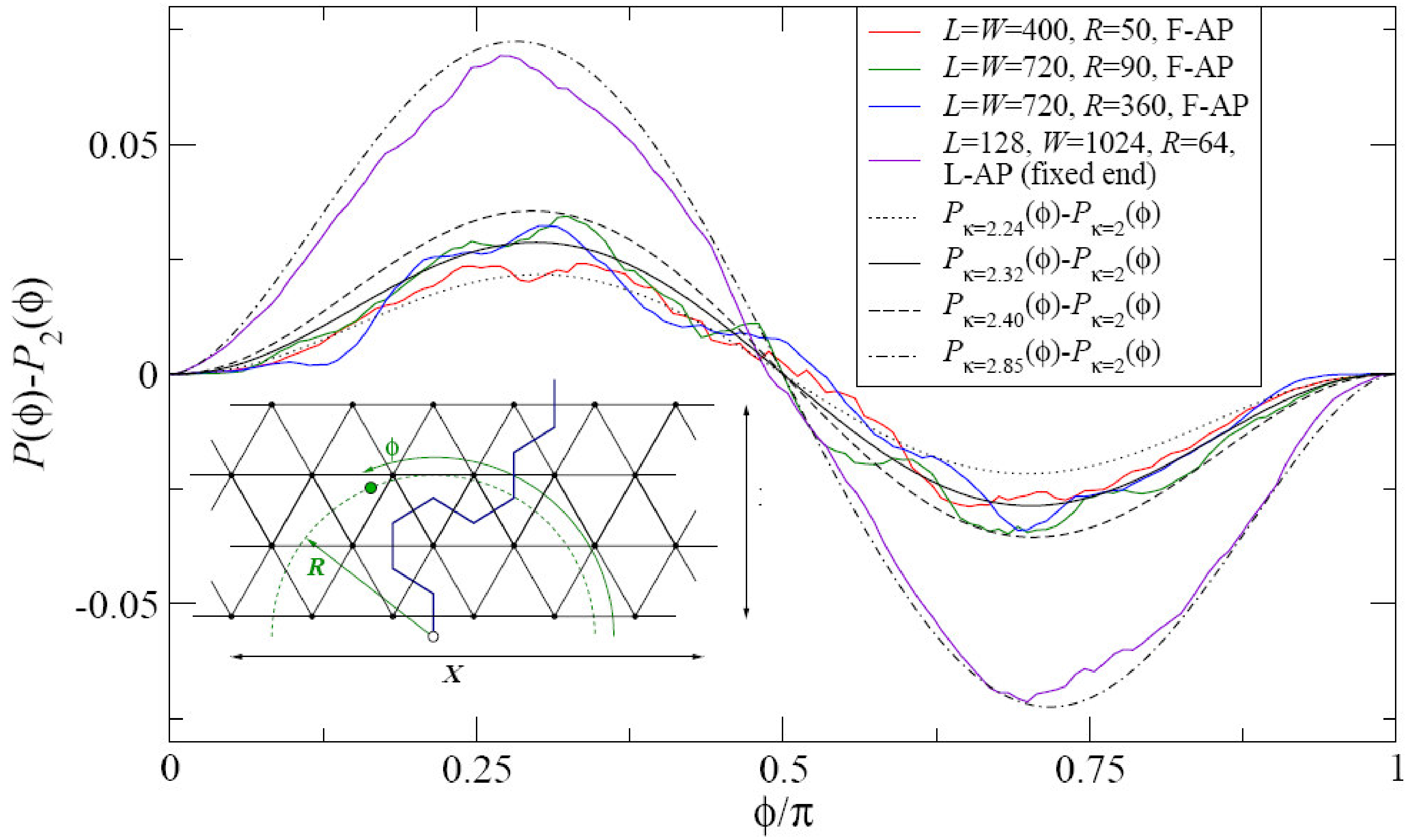
→ Loop Erased RW $d_f = 1.25$ absorbing BC: $SLE_{\kappa=2}$

reflecting BC: not Markov
not SLE

→ Minimal spanning tree

$d_f = 1.217(3)$ not conformal inv. (Wilson) fails test 2

probability of passage right



Chordal SLE

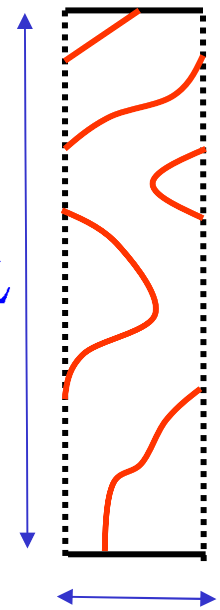
Schramm $P_\kappa(\phi) = \frac{1}{2} + \frac{\Gamma(\frac{4}{\kappa})}{\sqrt{\pi}\Gamma(\frac{8-\kappa}{2\kappa})} \cot(\phi)\, {}_2F_1(\frac{1}{2}, \frac{4}{\kappa}, \frac{3}{2}, -\cot^2(\phi))$

winding around long cylinder

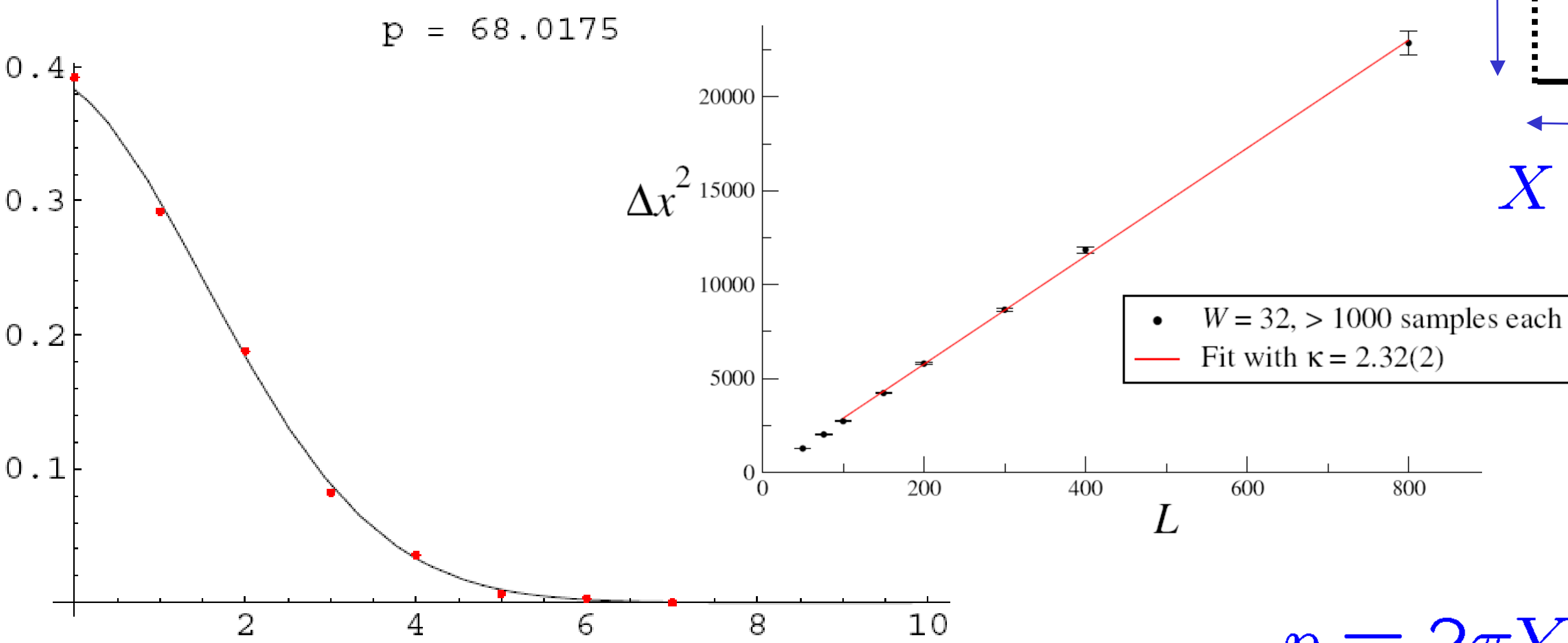
$$< x^2 >= \frac{\kappa}{2\pi} XY$$

x gaussian

$$Y = \frac{\sqrt{3}}{2} L$$



$$X = W$$

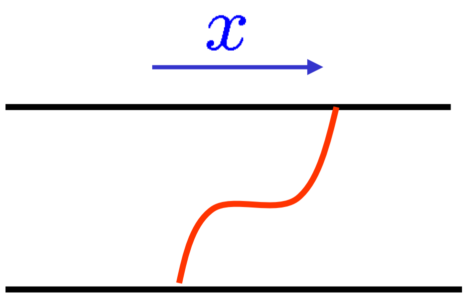


(c)

$$p = 2\pi Y/X$$

Figure 2: Winding probabilities w_n for different large p as a function of the winding number n. The continuous curves are the predictions from the Gaussian model (a guideline for the eyes) with $\kappa = 2.32$.

endpoint distribution on a strip

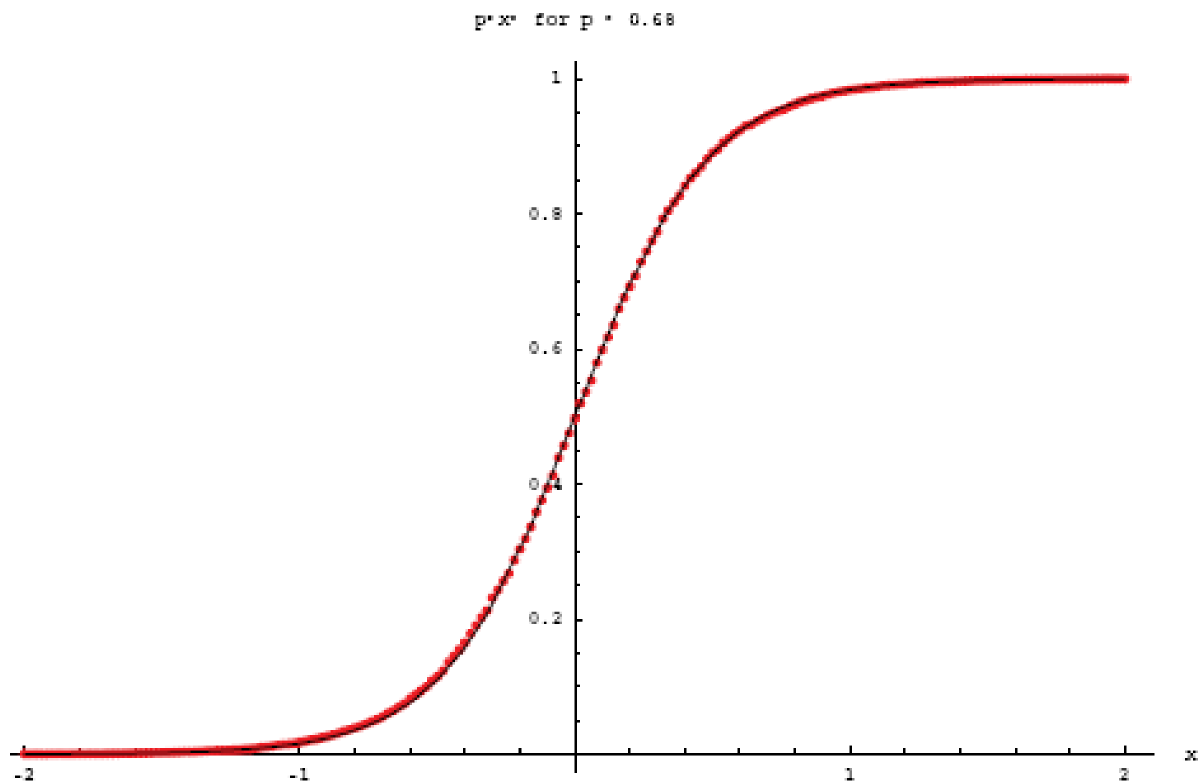


location of endpoint relative to start

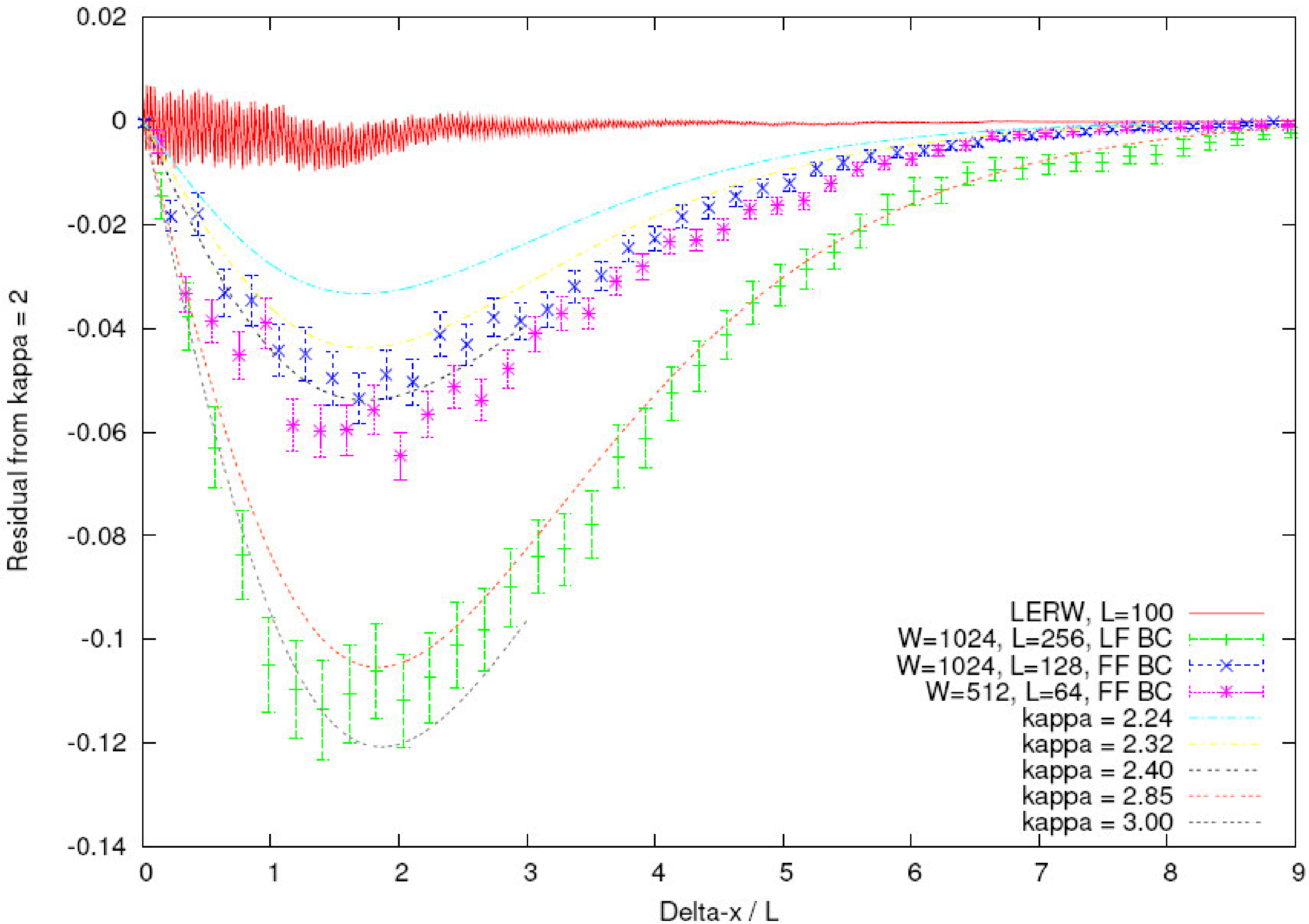
prediction from dipolar SLE

Bauer,Bernard

$$P(x) = A(\cosh(\frac{\pi x}{2Y}))^{-\frac{4}{\kappa}}$$



endpoint distribution P(x) residuals



endpoint distribution for various aspect ratio

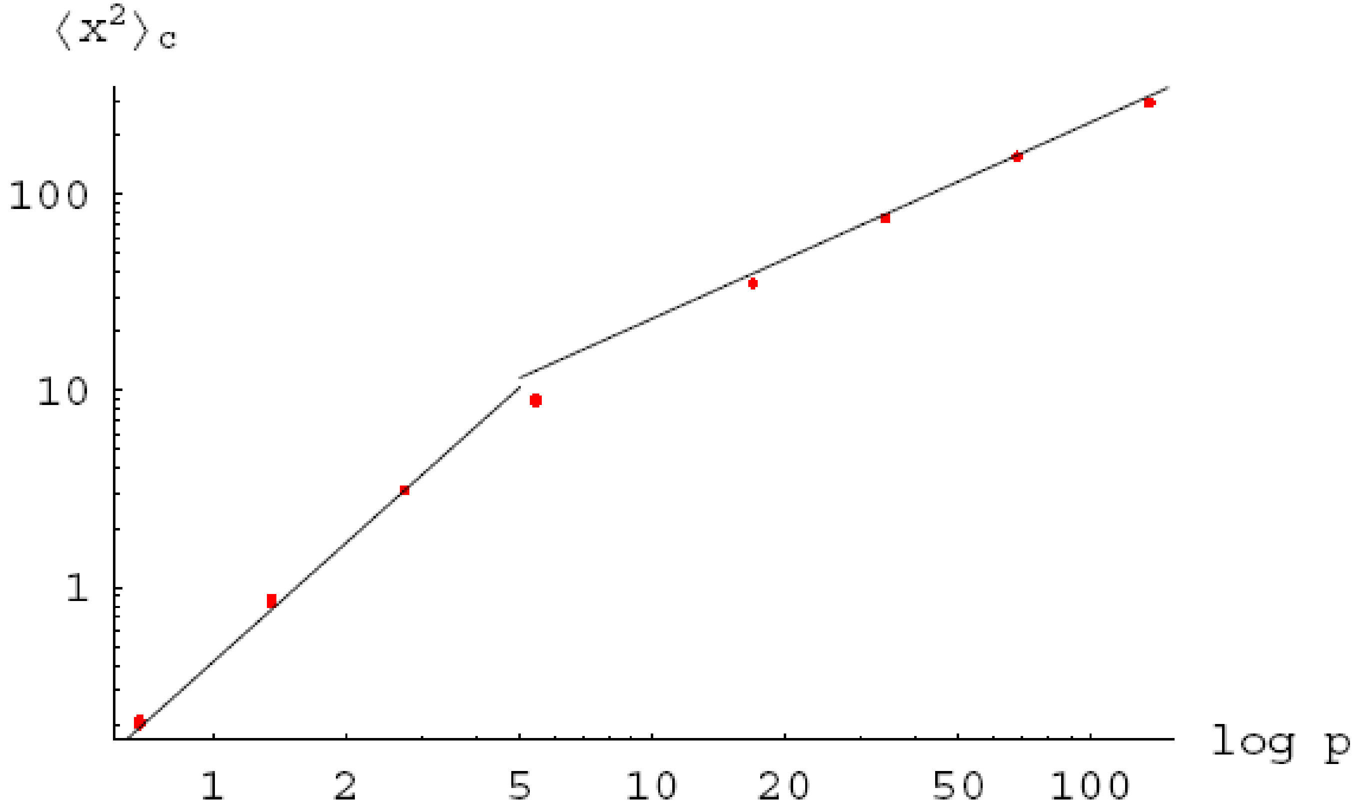
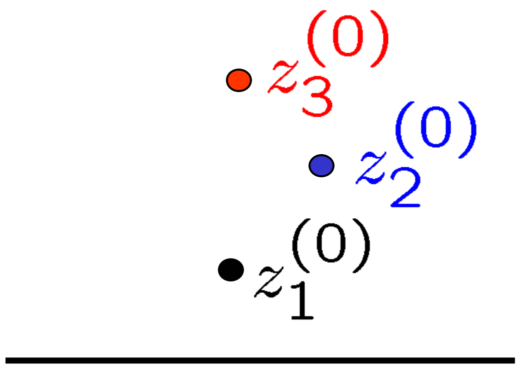


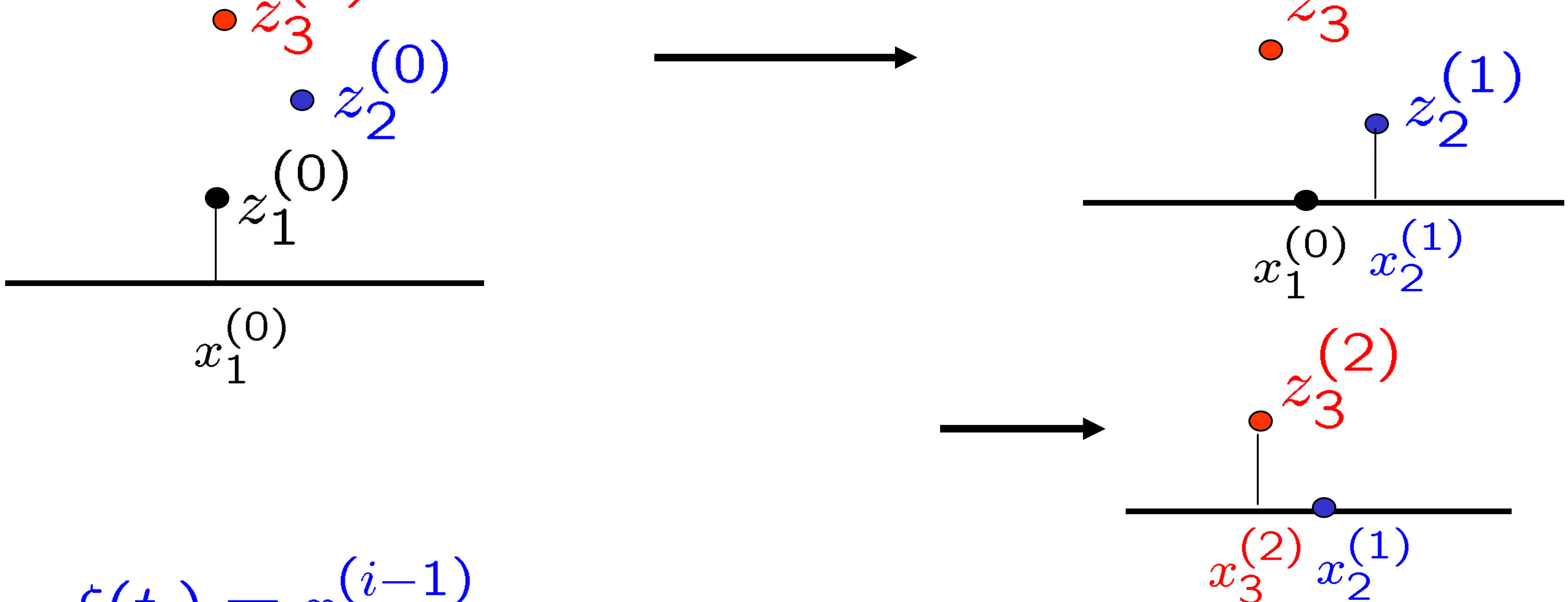
Figure 1: Log-log-plot for the second cumulant $\langle x^2 \rangle_c = \langle x^2 \rangle - \langle x \rangle^2$ as a function of the modulus p. The straight black lines correspond to the predictions from dipolar and radial SLE with $\kappa = 2.32$ for small and large p respectively.

iterated slit map



iterated slit map

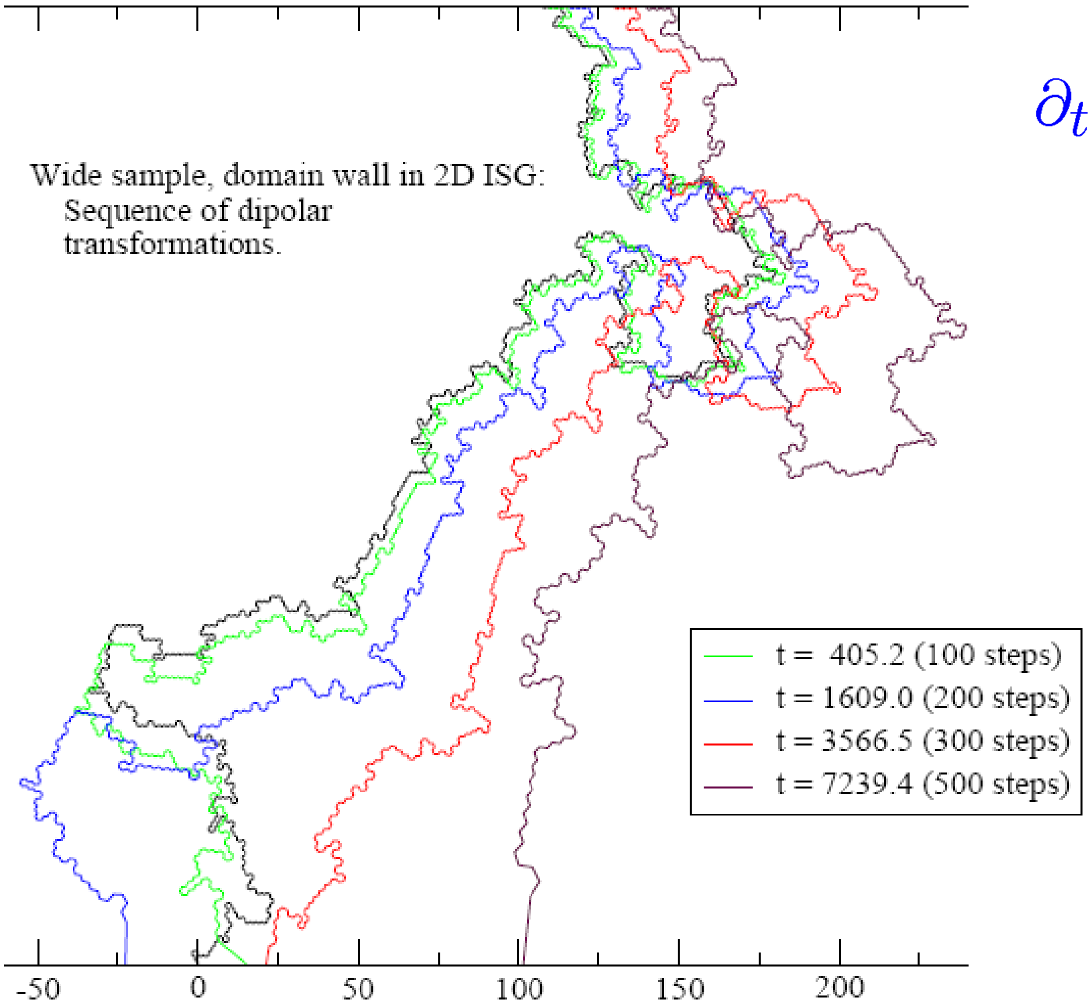
$$z' = x_1^{(0)} + \sqrt{(z - x_1^{(0)})^2 + (y_1^{(0)})^2}$$



$$\xi(t_i) = x_i^{(i-1)}$$

$$t_i - t_{i-1} = (y_i^{(i-1)})^2/4$$

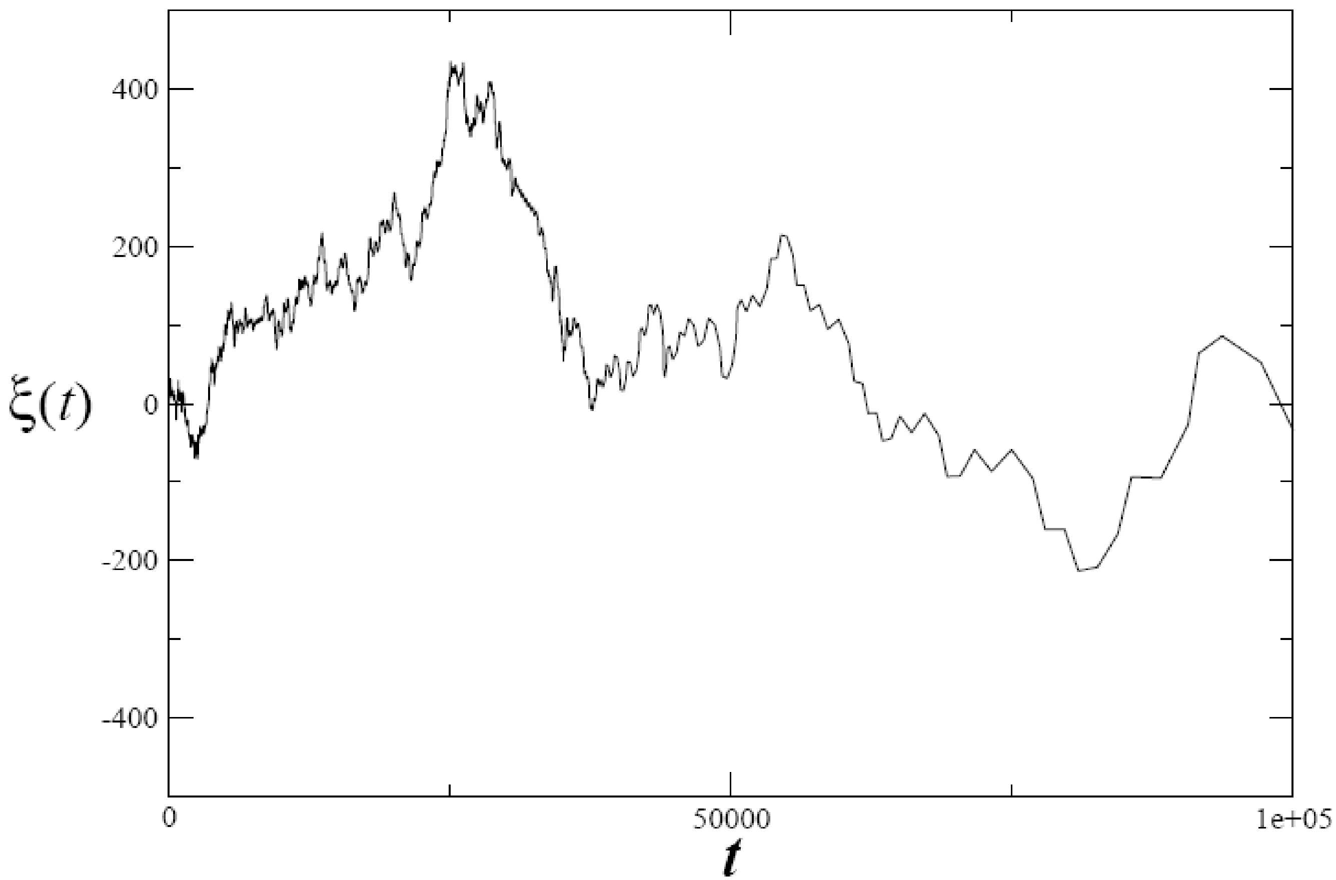
Dipolar maps



$$\partial_t g_t(z) = \frac{2}{\tanh(g_t(z) - \xi_t)}$$

strip height $\pi/2$

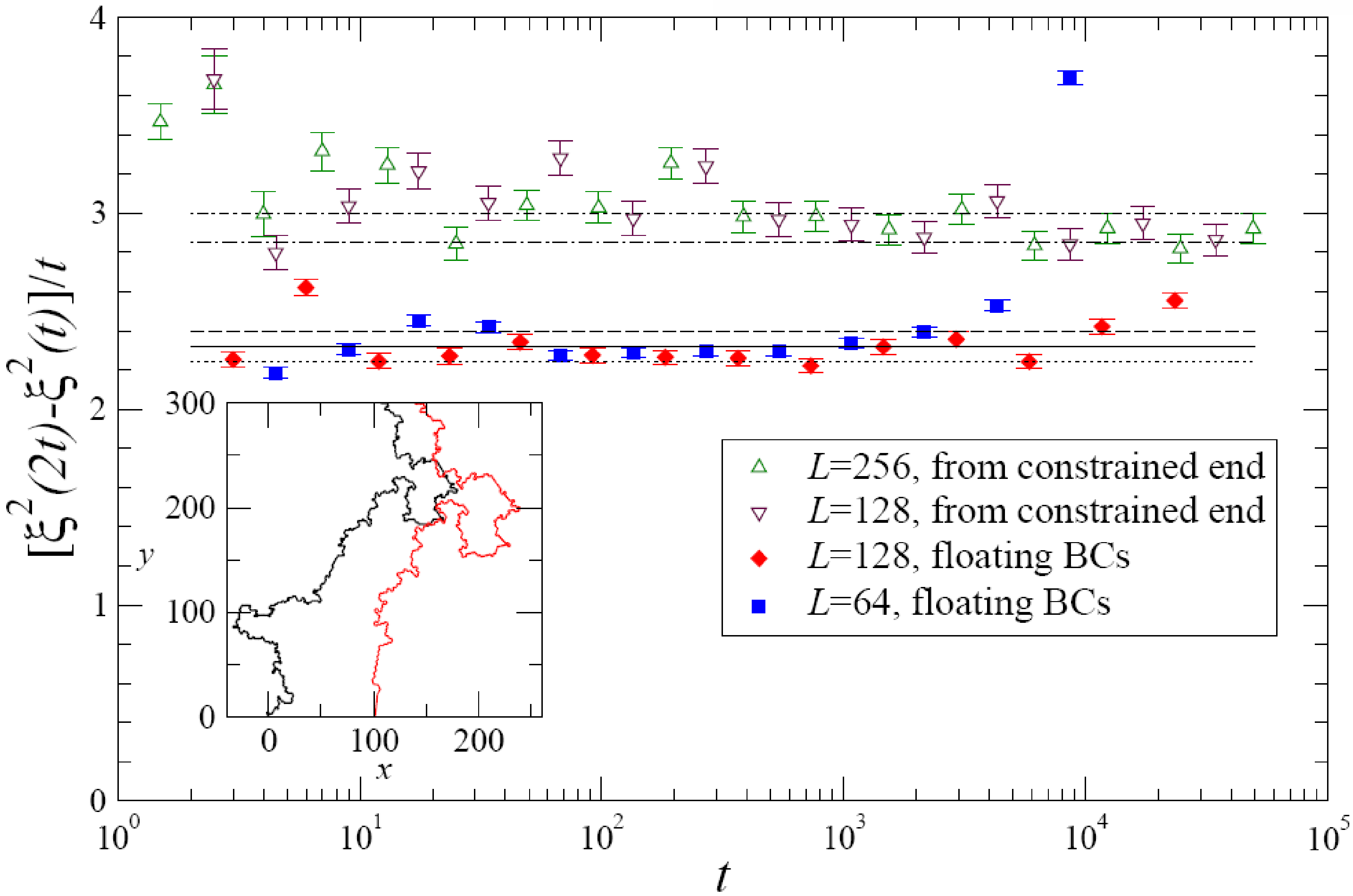
driving function



tests of Brownian motion

driving function

Figure 2: Plot of an effective diffusion constant $\kappa_{\text{eff}} = \xi^2(2t) - \xi^2(t)/t$, for $W \geq 4L$. Lines indicate $\kappa = 2.24$, 2.32, 2.40, 2.85, and 3.00. The range $2.24 < \kappa < 2.40$ fits the data for curves with F-AP BCs, while $2.85 < \kappa < 3.00$ describes the diffusion measured from a constrained domain wall end. Inset: Part of a sample conversion of a domain wall in the 2D Ising spin glass to a sequence $\xi(t_i)$, $i = 1 \ldots S$. The left curve is the initial domain wall with $\xi(0) = 0$, while the red [lighter] curve is the remainder after 500 applications of the dipolar map, giving $\xi(t_{500} \approx 7239.4) \approx 101.5$.



conclusion

- multiple tests of SLE suggest

AP/P domain walls in 2D Ising Spin Glass

are SLE_κ $\kappa = 2.32(8)$ free-free BC

NOT for fixed endpoints

surprising because:

- exact domain Markov property on lattice

$$P_D(\gamma_{ab}|\gamma_{ac}) = P_{D-\gamma_{ac}}(\gamma_{cb})$$

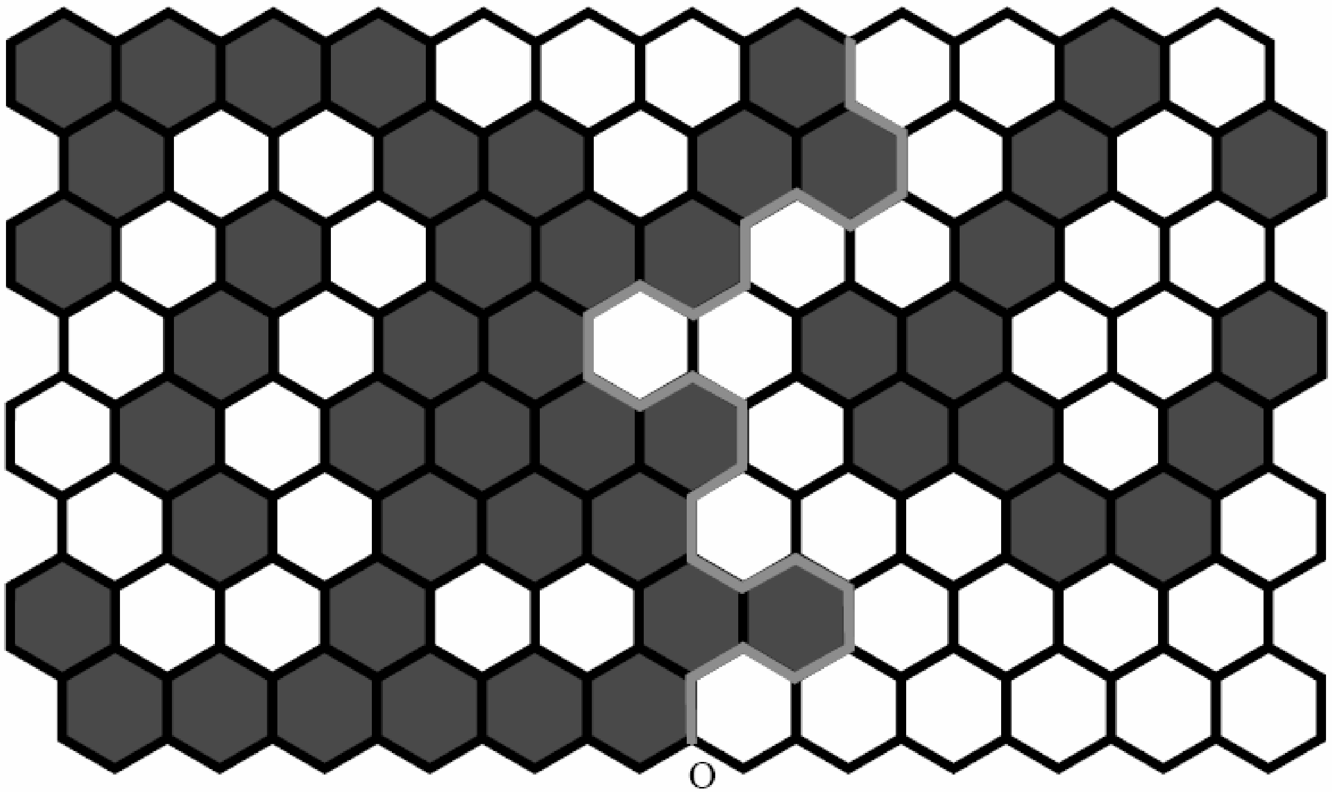
arise in continuum limit ?

- no known conformal field theory

correlation of boundary changing operators ?

look at other geometrical observables,
numerics in various domains

look for SLE in other 2d systems


O

Markov property

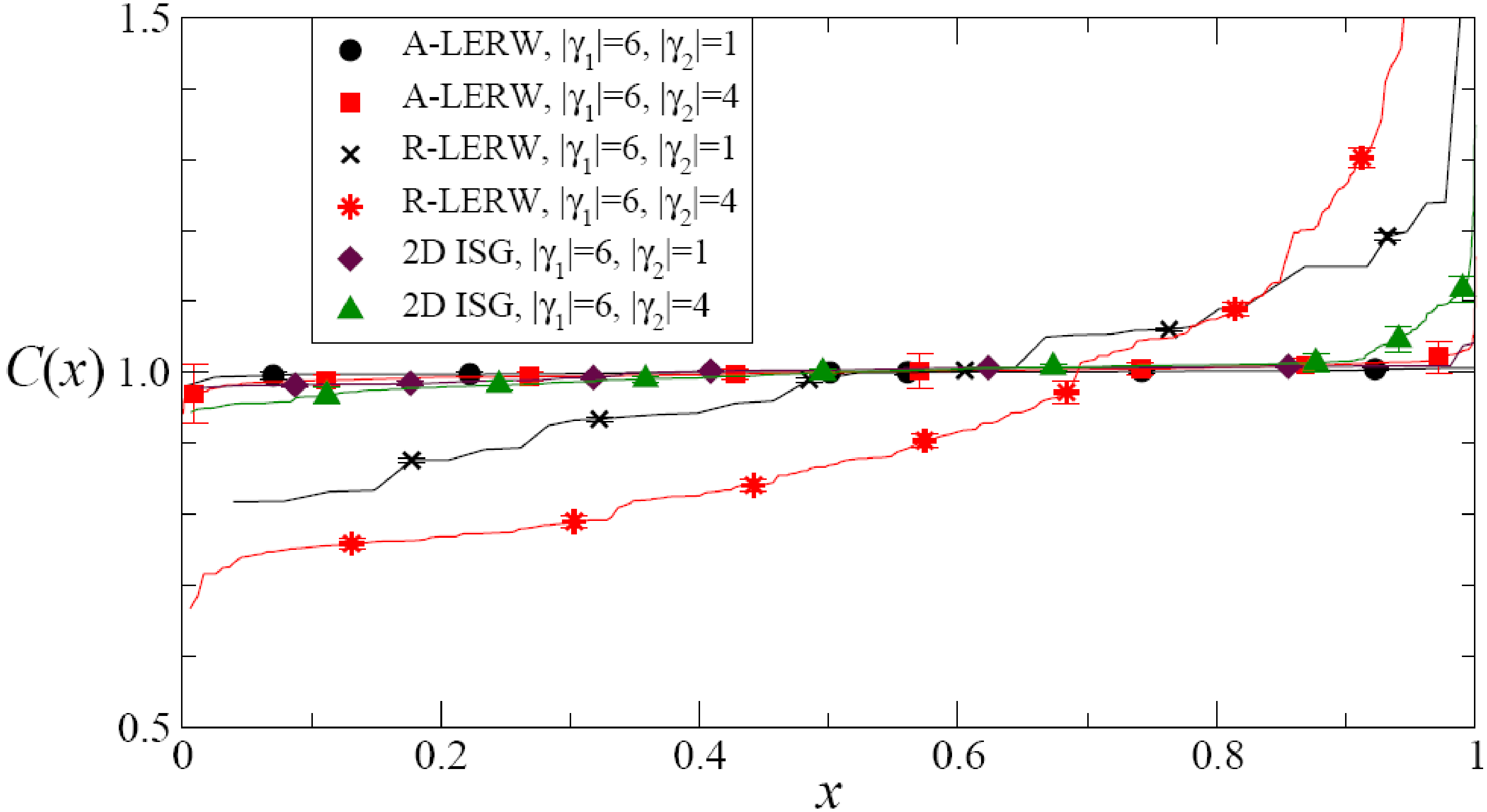
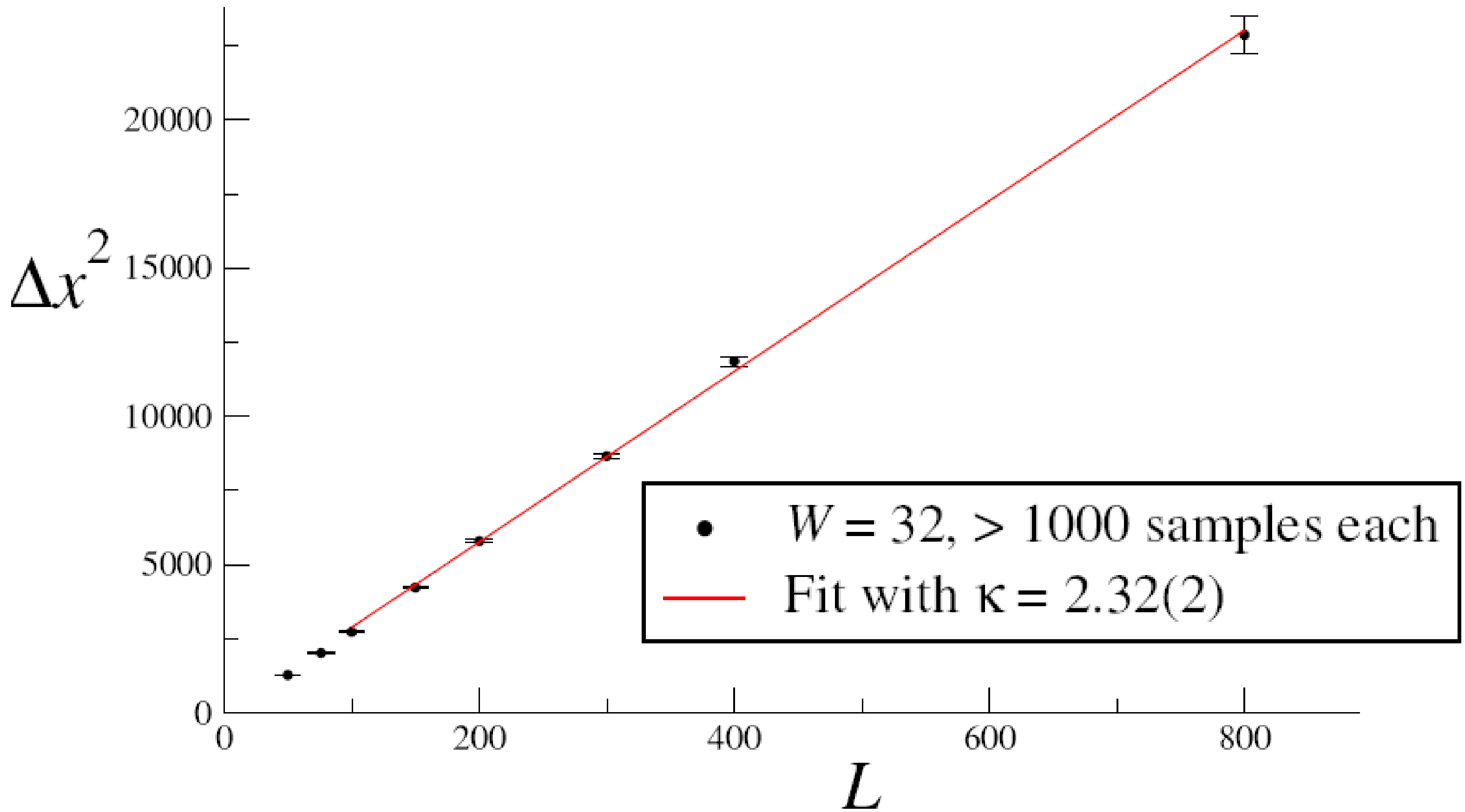
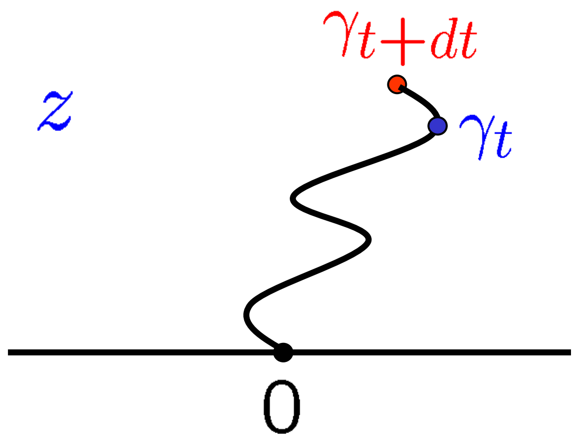


Figure 3: Plot of $C(x)$, cumulative probability of ranked values for $r(\gamma_1, \gamma_2)$, as defined in the text. Large deviations from $r = 1$, as clearly seen for R-LERW, indicate a failure of the domain Markov property.



Δx^2
20000
15000
10000
5000
0
0
200
400
600
800
L
W = 32, > 1000 samples each
Fit with κ = 2.32(2)



$\xi_s - \xi_t$ independent of $\xi_{t'}$

$s > t$ $\quad$ $t' < t$

$\xi_s - \xi_t := \xi_{t-s}$

implies conformal Markov property