Membrane

Write the mode as

$$\Psi(x,y,t) = f(x,y)\cos(\omega t + \phi) \tag{1}$$

and write $f(x,y)$ as a Fourier expansion

$$f(x,y) = \sum_{nm} A_{nm} \sin(k_n x + \phi_n)\sin(k_m y + \phi_m) \tag{2}$$

Apply the boundary conditions (without the fixed point)

$$f(0,y) = f(L,y) = f(x,0) = 0 \tag{3}$$

$$\frac{\partial f}{\partial y}|_{y=L} = 0 \tag{4}$$

This gives

$$k_n = \frac{\pi n}{L} \tag{5}$$

$$k_m = \frac{\pi m - \pi/2}{L} \tag{6}$$

$$\phi_n = \phi_m = 0 \tag{7}$$

so that

$$f(x,y) = \sum_{nm} A_{nm} \sin\left(\frac{\pi n}{L}x\right)\sin\left(\frac{\pi m - \pi/2}{L}y\right) \tag{8}$$

Substitution into the wave equation gives

$$\omega^2 = v^2\left[\left(\frac{\pi n}{L}\right)^2 + \left(\frac{\pi m - \pi/2}{L}\right)^2\right] \tag{9}$$

A mode of frequency ω thus involves a sum over pairs n and m that satisfy Eq. (9)

Add the fixed point $(x_0, y_0) = (L/2, L)$

$$f(x_0,y_0) = -\sum_{nm} A_{nm}\sin\left(\frac{\pi n}{2}\right)\cos(\pi m) = 0 \tag{10}$$

There are two families of solutions.

The first one is for *even* n's. For even n Eq.(10) is satisfied for arbitrary m, and the mode is

$$\Psi_{nm}(x,y,t) = A_{nm}\sin\left(\frac{\pi n}{L}x\right)\sin\left(\frac{\pi m - \pi/2}{L}y\right)\cos(\omega t + \phi) \tag{11}$$

The second family is for *odd* n's. In this case the product of the sines is either $+1$ or -1, so this family will consist of pairs, such that one is $+1$ and the other one is -1. Let us take odd n_1, n_2 and arbitrary integers m_1, m_2, and derive the condition for sataisfying Eq.(10):

$$A_{n_1m_1}\sin\left(\frac{\pi n_1}{2}\right)\cos(\pi m_1) + A_{n_2m_2}\sin\left(\frac{\pi n_2}{2}\right)\cos(\pi m_2) = 0 \tag{12}$$

which gives for odd n_1, n_2

$$\frac{A_{n_1m_1}}{A_{n_2m_2}} = -\frac{\sin\left(\frac{\pi n_2}{2}\right)\cos(\pi m_2)}{\sin\left(\frac{\pi n_1}{2}\right)\cos(\pi m_1)} = -(-1)^{n_2/2 - n_1/2 + m_2 - m_1} \tag{13}$$

Then the mode is

$$\Psi_{n_1 m_1 n_2 m_2}(x,y,t) = \tag{14}$$
$$A_{n_2 m_2}\left[-(-1)^{n_2/2-n_1/2+m_2-m_1}\sin\left(\frac{\pi n_1}{L}x\right)\sin\left(\frac{\pi m_1-\pi/2}{L}y\right)+\sin\left(\frac{\pi n_2}{L}x\right)\sin\left(\frac{\pi m_2-\pi/2}{L}y\right)\right]\cos(\omega t+\phi)$$

The possible values of n_1, n_2, m_1, m_2 follow from the condition (9). In order to know what the possible values of n_1, n_2, m_1, m_2 are, substitute the mode into the wave equation, compare coefficients and obtain:

$$n_1^2 - n_2^2 + (m_1 - 1/2)^2 - (m_2 - 1/2)^2 = 0 \tag{15}$$

So, for example, if $n_1 = n_2$, then $(m_1 - 1/2)^2 = (m_2 - 1/2)^2$; another example $(n1, n2, m1, m2) = (1, 5, 6, 3)$.

In order to find the general solution, one can write an odd n as $n = 2n' - 1$ and get

$$4(n_1' + n_2' - 1)(n_1' - n_2') - (m_1 + m_2 - 1)(m_1 - m_2) = 0 \tag{16}$$
$$4N_1N_2 - M_1M_2 = 0 \tag{17}$$

where the last equation is for new redefined integers $N_1 = n_1' + n_2' - 1, N_2 = n_1' - n_2', M_1 = m_1 + m_2 - 1, M_2 = m_1 - m_2$. So for a given N_1, N_2, M_1, there will be a solution for M_2 if $4N_1N_2/M_1$ is an integer.