SCARS IN BILLIARDS: THE PHASE SPACE APPROACH

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Although scars in the eigenfunctions of classically chaotic systems were originally observed in the configuration space representation, we show that these can be better visualized in phase space. On the quantitative side, a recent theory of scars is extended to billiards. For the stadium detailed agreement between theory and numerical experiment is found.

Condensed matter physics, atomic physics, quantum optics and other subfields of contemporary physics are largely based upon quantum mechanics. Unfortunately, except for a small number of very simple systems, we can only solve the Schrödinger equation numerically. Approximate solutions can be obtained either by perturbation methods (these are limited to systems close to the ones which are exactly solvable) or semiclassically. The latter approach is quite a bit more powerful since in principle it should apply to all systems. However, at present a full semiclassical theory exists only for integrable Hamiltonians.

One of the few semiclassical results which apply in the case of classically chaotic motion is the trace formula of Gutzwiller [1], which gives the density of states in terms of classical periodic orbits. Recently, this formalism has received considerable attention and has been shown to be in good agreement with numerical experiment for a variety of chaotic systems [2]. On the other hand, the eigenfunctions of strongly chaotic systems were believed for a long time to correspond to Wigner functions which are homogeneous over the energy shell [3]. In billiards, this conjecture leads to eigenfunctions which in turn are homogeneous over configuration space. This simple picture was undermined by the finding of high intensity patterns resembling periodic orbits ("scars") in the eigenfunctions of the stadium billiard [4]. Consequently, a theory of scars in configuration space was constructed by Bogomolny [5]. Latter on, Berry [6] derived a semiclassical phase space formula for the spectral Wigner function

$$W(z, E, \epsilon) \equiv h^N \sum_n \delta_\epsilon(E-E_n) W_n(z) , \qquad (1)$$

where $z \equiv (q, p)$, $W_n(z)$ is the Wigner function corresponding to the nth eigenstate, N is the number of degrees of freedom and $\delta_\epsilon(E)$ is a normalized Lorentzian of width ϵ. As will be shown, Berry's formula is restricted to the case of smooth Hamiltonians. Accordingly, the purpose of this paper is twofold. First we shall argue that phase space is a more natural environment for the study of scars. In order to support this statement, the Wigner and Husimi functions for the stadium are compared with the corresponding scarred eigenfunction, $\Psi_n(x, y)$. Second, a billiard formula for $W(z, E, \epsilon)$ is derived and compared with the numerical results.

For two-degrees-of-freedom systems the energy shell is three-dimensional while configuration space has only two dimensions. As a consequence, in phase space one has better resolution when attempting to distinguish between the contributions of different periodic orbits to a given state. The importance of phase space has been especially appreciated by Davis [7], who studied Wigner and Husimi distributions numerically for systems with a mixture of regularity and chaos. Here, we shall illustrate this statement using the stadium billiard, a system which is globally chaotic. We set the radius of the two semicircular caps, r, to be equal to half the length of the straight segments, a. In fig. 1a, an eigenfunction is compared with periodic orbits numbers 2 (full line) and 26 (dashed line). (We order the periodic orbits in ascending sequence by their linear stability eigenvalue.) To each there corresponds a second symmetry-related periodic orbit obtained by a reflection with respect to the x-axis. Although the second orbit seems to resemble better the shape of the scar in the eigenfunction, the assignment is not compelling. In phase space however, the correspondence between periodic orbits and scars is rather transparent. In fig. 1b the associated Husimi distribution in the surface of section at $x=r=1$ is compared with the periodic orbits which neighbor its local maxima [8]. In this representation, two additional orbits, numbers 25 and 52, can be seen to contribute to this state. Notice that the inherent limitation of this procedure is due to the finite width of the individual contributions to the Husimi distribution. The major advantage of working in phase space is that the relationship between scars and orbits suggested in fig. 1b can be verified by requiring consistency with a different surface of section. Accordingly, in fig. 1c the same comparison is performed in the $x=r/2$ section. We find that the assignment of periodic orbits to maxima in the corresponding Husimi distribution matches the one in fig. 1b.

In both fig. 1a and fig. 1b the stable and unstable manifolds of orbit number 2 are shown. Although the high intensity contours do follow the manifolds, the effect is not pronounced [9]. This phenomenon is strongly enhanced, however, if the Wigner function rather than the Husimi distribution is studied (see fig. 1d). On the other hand, due to the very rich structure of the Wigner function it cannot be easily employed to visually identify the periodic orbits which scar the state.

We now proceed to the second part of the paper and to a quantitative discussion of scarring. We start by shortly summarizing the derivation which led Berry to a trace formula for $W(z, E, \epsilon)$. For details we refer the reader to ref. [6]. From eq. (1) one can show that

$$W(z, E, \epsilon)=\frac{2}{h}\,\mathrm{Re}\left(\int_0^\infty \mathrm{d}t\,\exp[\mathrm{i}(E+\mathrm{i}\epsilon)t/\hbar]\right.$$
$$\left.\times\int \mathrm{d}s\,\exp(-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{s}/\hbar)G(\boldsymbol{q}_\mathrm{A},\boldsymbol{q}_\mathrm{B},t)\right), \qquad (2)$$

where G is the propagator, evaluated at $\boldsymbol{q}_\mathrm{A}=\boldsymbol{q}-\boldsymbol{s}/2$, $\boldsymbol{q}_\mathrm{B}=\boldsymbol{q}+\boldsymbol{s}/2$. Berry first replaces the propagator by its WKB approximation, the Van Vleck formula [10], and then performs the $\boldsymbol{s}$ and t integrations by the stationary phase approximation. As a consequence of the $\boldsymbol{s}$ integration, it is found that trajectories which satisfy the midpoint rule, that is, $z=(z_\mathrm{A}+z_\mathrm{B})/2$, dominate the spectral Wigner function. The remaining integral over time has the form

$$W(z, E, \epsilon)=\frac{2^N}{\pi\hbar}$$
$$\times\sum_{\substack{\text{midpoint}\\ \text{orbits}}}\mathrm{Re}\int_0^\infty \mathrm{d}t\,\exp\{(\mathrm{i}t/\hbar)[E+\mathrm{i}\epsilon-H(z_\mathrm{A})]$$
$$+(\mathrm{i}/\hbar)A(z,t)+\mathrm{i}\gamma\}[\det(m_\mathrm{AB}+\mathbf{1})]^{-1/2}, \qquad (3)$$

where $A(z, t)$ is the symplectic area enclosed between the midpoint orbit and a straight segment connecting its endpoints (the chord area), and where $m_\mathrm{AB}=\mathrm{d}z_\mathrm{B}/\mathrm{d}z_\mathrm{A}$. If we denote the phase in eq. (3) by $\phi/\hbar$ then the stationary phase condition takes the form $\partial\phi/\partial t=E-H(z_\mathrm{A})=0$. Accordingly, the midpoint orbits which lie in the E-energy shell will dominate the time-integral of eq. (3). As was shown by Berry [6], a crucial point for scarring is that for periodic orbits $\partial^2\phi/\partial t^2=0$. Consequently, the periodic orbit contributions to $W(z, E, \epsilon)$ result from a degenerate stationary phase integral and therefore dominate the contributions from non-periodic orbits. However, trajectories which are almost periodic are also important because they determine the way in which the dominant contributions to $W(z, E, \epsilon)$ decay as a function of $\boldsymbol{\xi}$, the displacement away from the periodic orbit in the surface of section. For these trajectories the correction to the chord area is

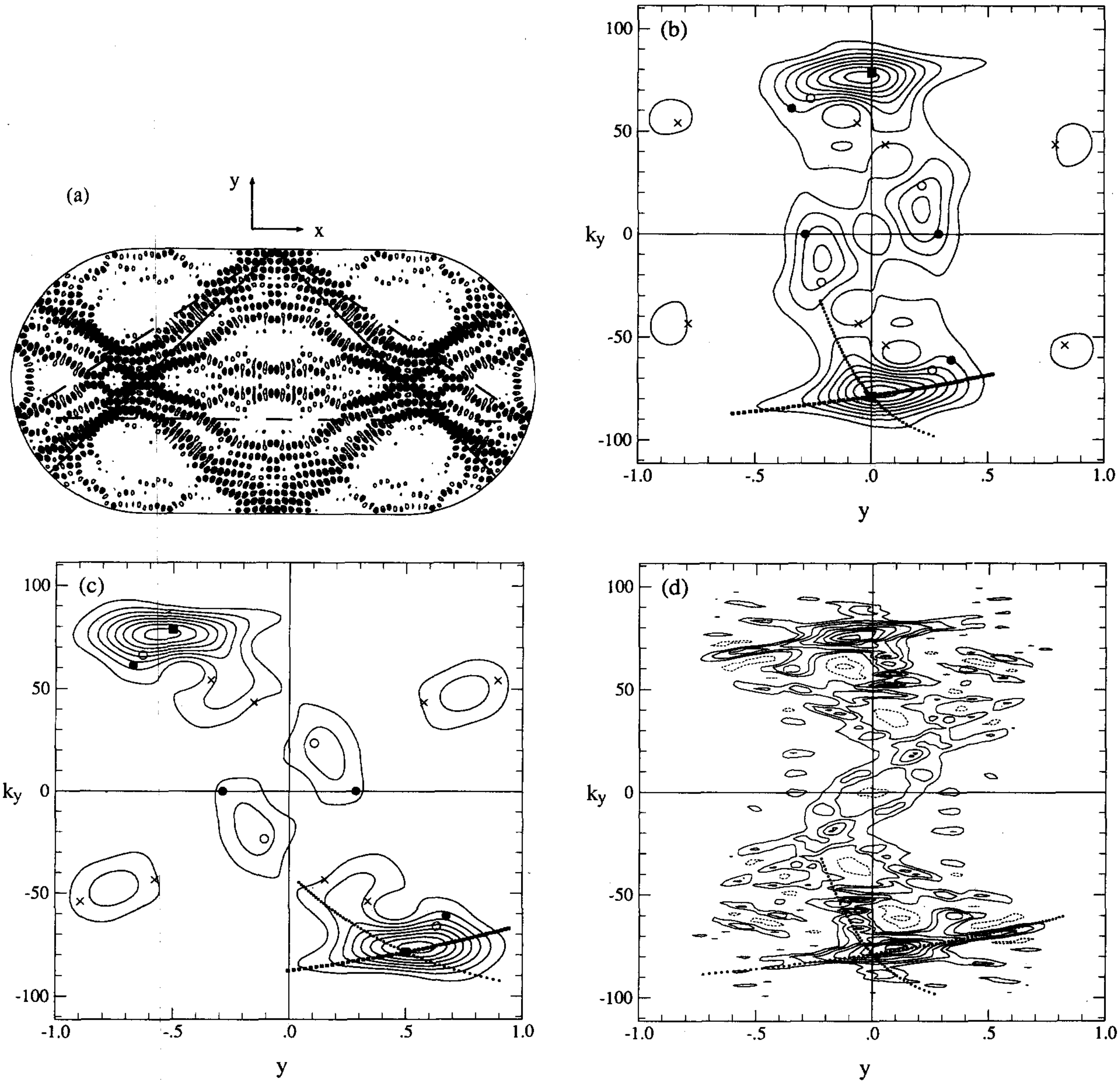

Fig. 1. Configuration versus phase space for the 744th state. (a) The eigenfunction. (b) The Husimi distribution on the $x=r$ surface of section. The relevant periodic orbits are also shown: the second (■), the 25th (○), the 26th (●) and the 52nd (×). The dashed curves which emerge from the second periodic orbit are its stable and unstable manifolds. (c) As in (b) only that here $x=r/2$. (d) The Wigner function on the same surface of section as in (b) (see text).

$$\Delta A = \tilde{\xi} J \frac{M-I}{M+I} \xi , \tag{4}$$

where J is the unit symplectic matrix and $M = \mathrm{d}\xi_B / \mathrm{d}\xi_A$ is the linearized mapping about the periodic orbit in the surface of section. Finally, in order to perform the degenerate stationary phase integration the third derivative of the phase in eq. (3) is needed, $\partial^3\phi/\partial t^3 = \tilde{\dot{\xi}} J \dot{\xi}$. The resulting contribution from an individual periodic orbit of action S is

$$W_{\mathrm{po}}(z,E,\epsilon)=\frac{2^N}{\sqrt{\det(M+I)}}\,\mathrm{e}^{-\epsilon T/\hbar}$$
$$\times\cos\left(\frac{1}{\hbar}S(E)+\frac{1}{\hbar}\xi J\frac{M-I}{M+I}\xi+\gamma\right)$$
$$\times\frac{2}{(\hbar^2|\xi J\dot{z}|)^{1/3}}\,\mathrm{Ai}\left(\frac{2[H(z)-E]}{(\hbar^2|\xi J\dot{z}|)^{1/3}}\right). \tag{5}$$

For billiard systems, $\ddot{z}=(\dot{p},\ddot{p})=0$ and accordingly W_{po} of eq. (5) needs to be modified. This is most easily done when $\xi=0$, $\epsilon=0$ (point z on the periodic orbit, and no smoothing in energy). In this case, the only way to satisfy the midpoint requirement is to have the endpoints of the midpoint orbit, z_{A} and z_{B}, also lie on the periodic orbit at symmetrically displaced positions from z. This displacement uniquely determines a midpoint orbit and is proportional to $\tau\equiv t-T$, where t is the time it takes to get from z_{A} to z_{B} and T is the period. It is convenient to change the integration variable in eq. (3) from t to τ. Then, in order to perform this integration, we need to express the properties of the midpoint orbits as a function of τ. Since here $H(z_{\mathrm{A}})=H(z_{\mathrm{B}})=H(z)$ and $A(z,t)$ is just the action of the periodic orbit, S, the argument of the exponent in eq. (3) is independent of τ. As a consequence, there is no stationary point and the integration has to be performed over all possible values of τ. As τ is increased, eventually either z_{A} or z_{B} will reach the boundary. We shall refer to this value of τ as τ_0. For $\tau>\tau_0$ the midpoint relation cannot be satisfied any longer. Accordingly, the integral in eq. (3) runs only over the $(-\tau_0,\tau_0)$ interval. We now use the facts that $M_{\mathrm{AB}}=T_{\tau/2}MT_{\tau/2}$ where T_τ is a translation for time τ and that in billiards $S=pL$ where L is the length of the periodic orbit, to obtain

$$W_{\mathrm{po}}(z,E,\xi=\epsilon=0)$$
$$=\frac{2^N}{\pi\hbar}\frac{1}{M_{21}}\,[(\mathrm{Tr}\,M+2+2T_0M_{21})^{1/2}$$
$$-(\mathrm{Tr}\,M+2-2T_0M_{21})^{1/2}]\cos(kL+\gamma)\,. \tag{6}$$

Notice that in eq. (6), $W_{\mathrm{po}}=\mathrm{O}(\hbar^{-1})$ while Berry's result of eq. (5) is only $\mathrm{O}(\hbar^{-2/3})$. *Accordingly, phase space scars for billiards will more strongly diverge in the semiclassical limit.*

When either ϵ or ξ is nonzero the formula for $W(z,E,\epsilon)$ becomes rather complex and we shall postpone its derivation for a future publication. The major difference with respect to eq. (5) is that the elegant factorization of the behaviors in the surface of section and that perpendicular to the energy shell is lost. Instead, in the billiard result the two behaviors are mixed in the form of a new special function.

We now proceed to quantitatively check the predictions of eq. (6) for the case of the stadium billiard. For this purpose we have numerically calculated the values of the Wigner function at the point $\mathrm{P}_3=(x=0.51,\ y=0.745,\ k_x/k_y=-2)$ for the first 1000 odd–odd states and have taken the Fourier transform with respect to k,

$$F(z,L)=\left|\int \mathrm{d}k\,\exp(\mathrm{i}kL)kW(z,k)\right|. \tag{7}$$

In practice, the spectral Wigner function is a complex superposition of many contributions generated by the multitude of periodic orbits. In the light of eq. (6), the purpose of the Fourier transform is to filter out all the unwanted orbits and therefore allow us to study the component due to a particular periodic orbit. The factor of k which appears in eq. (7) was introduced in order to remove any k-dependence in the amplitude of eq. (6). Since P_3 lies on the third periodic orbit we should obtain peaks in the Fourier transform at integer multiples of its length, $L_3=4.4721$. Indeed, in fig. 2a, among several other peaks the $n=1$ and $n=2$ multiples of L_3 can be located with a precision of 0.1%. The corresponding heights however, agree to the theoretical prediction of eq. (6) only to within 17% and 12% error respectively. The additional peaks are due to other periodic orbits which are neighboring in phase space. For these contributions ξ is finite and therefore the full fledged billiard theory is needed in order to analyze them. The prediction is that if the Fourier transform is performed over an infinitely large range which is restricted to the semiclassical regime (large k), then each periodic orbit corresponds to two δ-function peaks rather than a single one. Moreover, the locations of the two peaks are immediate results of the complete theory. On the other hand, for the k-interval which is used in the present calculation only one broad peak should be observed and its height and position can only be obtained by a different numerical integration for each ξ. We will not attempt to

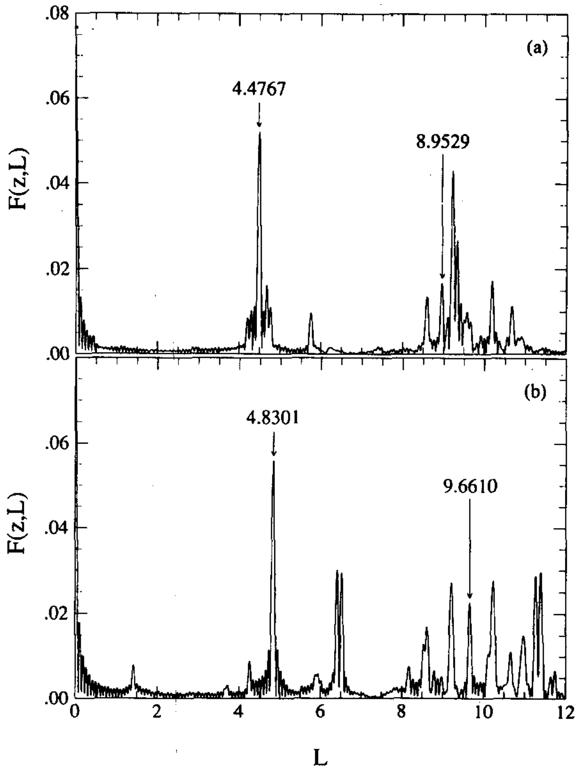

Fig. 2. The $F(z, L)$ function: (a) $z=P_3$, (b) $z=P_2$ (see text).

study the off-orbit peaks in a quantitative way. An additional source of structure in fig. 2a is that due to the finite k-integration interval, all the peaks are broadened and accordingly strongly interfere with each other. The effects of the interference can be observed by comparing the fringes associated with $L=0$ with those at $L=L_3$. It is obvious that for the latter they are strongly perturbed by a nearby finite-ζ peak. This nearby peak is also the source of the relatively large errors which we found in the heights and locations of the nL_3 peaks. In fig. 2b, $F(z, L)$ for $z=P_2=(x=0.51, y=0.49, k_x/k_y=-1)$ is shown. P_2 lies on the second periodic orbit for which $L_2=4.8482$. Clearly, the $n=1$ peak is quite well isolated from its neighbors. As a consequence, an error of only 0.03% in the position and 6% in the height is obtained.

In summary, we have shown that Berry's semiclassical theory for the spectral Wigner function is not only more convenient because of being a phase space theory but also that it can be extended to hold for billiards. Moreover, we have quantitatively checked the modified theory against numerical results from the stadium. We feel that such checking is important, because many studies of the scarring phenomenon have been qualitative or semiquantitative at best.

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References

[1] M.C. Gutzwiller, J. Math. Phys. 12 (1971) 343.
[2] D. Wintgen, Phys. Rev. Lett. 58 (1987) 1589;
M.L. Du and J.B. Delos, Phys. Rev. Lett. 58 (1987) 1731;
R. Aurich, M. Sieber and F. Steiner, Phys. Rev. Lett. 61 (1988) 483.
[3] B. Eckhardt, Phys. Rep. 163 (1988) 205.
[4] E.J. Heller, Phys. Rep. Lett. 53 (1984) 1515.
[5] E.B. Bogomolny, Physica D 31 (1988) 169.
[6] M.V. Berry, Proc. R. Soc. A 423 (1989) 219.
[7] M.J. Davis, J. Phys. Chem. 92 (1988) 3124.
[8] P. Leboeuf, M. Saraceno, M. Baranger, J. Mahoney and D. Provost, unpublished.
[9] R.L. Waterland et al., Phys. Rev. Lett. 61 (1988) 2733.
[10] M.V. Berry and K.E. Mount, Rep. Prog. Phys. 85 (1972) 315.