Two-Parameter Scaling in the Wigner Ensemble

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The Wigner ensemble of band random matrices describes the statistical properties of strongly chaotic Hamiltonians; it may also be viewed as a disordered tight-binding model with an electric field. We investigate the scaling properties of the localization of eigenstates and that of the distribution of level spacings, $P(s)$, for *finite matrices.* We show that both quantities are uniquely determined by *two scaling parameters.*

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Band random matrices (BRM) are now under close consideration mainly due to their relevance to different physical problems [1-6]. First, it was shown that the BRM ensemble can be used to describe statistical properties of the so-called quantum chaos for systems which are fully chaotic in the classical limit [2-4]. In this approach, the size of the band was found to relate to the degree of quantum localization. Therefore, the BRM give the possibility of studying the influence of quantum effects on statistical properties of both eigenfunctions and spectrum. Another important application of the BRM is in solid state physics where a similar band structure of matrices naturally occurs in the study of localization in 1D and quasi-1D models with random potential [5]. Recent studies [3,4] of the BRM ensemble have discovered scaling properties for both the localization length of eigenstates and the spectral statistics.

The BRM are defined as the set of real symmetric matrices of order N with nonzero matrix elements restricted to a band of width $2b-1$, and chosen as independent Gaussian random variables. For $N\rightarrow\infty$, the eigenstates of such matrices are exponentially localized and as a consequence, the level spacing distribution $P(s)$ is Poissonian. Using supersymmetry and relating averages of Green functions of BRM to known results about Anderson and nonlinear sigma models, Fyodorov and Mirlin [6] were able to prove that the localization length l_∞ is proportional to b^2. This result was also expected from the theory of a simple model of quantum chaos, the kicked rotator, whose time evolution is described by a band matrix [2] and also from a derivation given in Ref. [7].

For finite N, it was shown in Refs. [3,4] that the statistical properties of BRM are characterized by a single parameter $x=b^2/N$. More precisely, both the average localization length of eigenvectors, $l(N,b)$, and the level spacing distribution are functions of the variable x. For the eigenvectors, the scaling relation was found to be very simple:

$$\frac{l(N,b)}{N}=\frac{cx}{1+cx}, \quad c\approx 1.35\,. \tag{1}$$

To deal with both localized and extended states, a definition of l based on information entropy was used [see Eq. (6)]. Recently, Eq. (1) was analytically derived [8].

Let us first picture BRM as tight-binding Hamiltonians for 1D disordered systems; this approach was fruitful during early investigations of "dynamical localization" in quantum deterministic chaotic systems [9]. For the Anderson model, where eigenstates are exponentially localized, Pichard [10] has given the general scaling hypothesis

$$\xi_N/N=g(\xi_\infty/N)\,, \tag{2}$$

where ξ_N and ξ_∞ are, respectively, the localization length for the sample of length N and the infinite sample, defined as the rate of exponential decay and computed by means of transfer matrices. This scaling law takes the simple form (1), found for BRM, when the entropy length l_N is used, for the Anderson and Lloyd model [11]. The scaling variable x is just ξ_∞/N, and one can show that $l_\infty=\kappa\xi_\infty$.

One further expects that the detailed structure of quasi-1D tight-binding Hamiltonians, which, e.g., in the case of only nearest-neighbor hopping include many vanishing off-diagonal matrix elements within the band, does not significantly alter either the scaling variables or the global behavior of the scaling functions which are therefore similar to those for the BRM. In other words, there is evidence that the one-parameter scaling of Eq. (2) holds for quasi-1D systems of Anderson type [12].

On the other hand, the occurrence of band structure in a strongly chaotic Hamiltonian has been investigated by Feingold, Leitner, and Piro [13] with semiclassical methods. It was shown that, in the basis of the eigenvectors $\mathbf{v}_i$ of a canonical operator $A(\hat{\mathbf{q}},\hat{\mathbf{p}})$, arranged in increasing order of the corresponding eigenvalues a_i, the off-diagonal matrix elements of $H(\hat{\mathbf{q}},\hat{\mathbf{p}})$ decrease from the diagonal, giving rise to a band structure. Moreover, the diagonal elements have a smooth, classical variation, with additional small and rapid quantum fluctuations [14]. Correspondingly, a BRM model was studied which

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in addition has diagonal matrix elements with nonzero mean [7,15,16]: $\langle A_{ii}\rangle = i\alpha$. In the picture of disordered solids, one may think of this requirement as the application of an electric field to a 1D lattice of length N with random couplings of each site to the nearest b-1 left sites and b-1 right sites. The case of nearest-neighbor interaction and diagonal disorder has been investigated in Ref. [17], and the eigenvectors were shown to have a factorial decay.

The BRM model with increasing diagonal elements was originally introduced by Wigner [18] almost forty years ago, with a motivation similar to that of Refs. [7, 15,16]. We therefore shall refer to this model as the Wigner ensemble. It was then quickly discarded in favor of the well-known Gaussian ensembles when it seemed that it was mathematically difficult. Recently, however [7], it was shown that the localization length l_∞ should obey the relation

$$l_\infty = b^2 f(y), \quad y = \alpha b^{3/2}. \tag{3}$$

The function $f(y)$ was numerically found in Ref. [7], and in Ref. [6] an argument is given to show that it behaves like ky^{-1} at large y [19].

The level spacing distribution also scales with y, $P(s,y)$ [16]. It becomes Poissonian for $y \to 0$ and Wigner type in the limit $y \to \infty$. The gradual crossover occurs as semicircle densities of states corresponding to diagonal blocks of size l_∞ become displaced in energy due to the increase in α. While the $P(s)$ of each block is Wigner type, eigenvalues in different blocks are uncorrelated and unrepelling. Therefore, Poissonian behavior is lost as the degree of overlap between consecutive semicircles decreases. This degree of overlap is determined by $\gamma \equiv \alpha l_\infty / k\sqrt{b} = k^{-1} y f(y)$, which tends to unity as $y \to \infty$.

Until now, studies of this model have been restricted to infinitely long samples, $N \to \infty$. The purpose of this paper is to study the behavior of the Wigner ensemble at finite N. In particular, we show that

$$l_N / N = F(x,y), \tag{4}$$

and obtain the main properties of F, which include a transition between two different regimes. Moreover, we find that the spacing distribution also scales in both x and y, $P(s,x,y)$.

The Wigner ensemble is given by real symmetric $N \times N$ matrices with nonzero matrix elements restricted to a band centered on the diagonal: $A_{ij} = 0$ if $|i-j| > b$. The matrix elements are chosen as independent Gaussian random variables with the following moments:

$$\langle A_{ij}\rangle = i\alpha\delta_{ij}, \quad \langle A_{ij}^2\rangle = \delta_{ij} + 1. \tag{5}$$

The ensemble is therefore described by three parameters: N, b, and α. Our results, however, are restricted to the case where $1 \ll b \ll N$.

The entropy localization length of a normalized vector $\mathbf{u} = (u_1, \ldots, u_N)$ is defined as

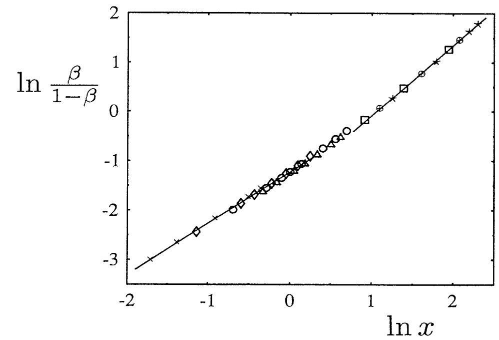

FIG. 1. Scaling of localization in x, at the fixed value of $y = 20$. The various symbols correspond to different values of b: $b = 12$ (×), $b = 16$ (◇), $b = 20$ (○), $b = 24$ (△), $b = 40$ (□), $b = 45$ (⊕), and $b = 50$ (∗). The straight lines are the best fits to the data in the asymptotic regimes.

$$l(\mathbf{u}) = N \exp\left[-\sum_{i=1}^{N} (u_i^2 \ln u_i^2) - H_{\rm ref}\right]. \tag{6}$$

One recognizes the definition of information entropy; the reference term $H_{\rm ref}$ is chosen in order that $l = N$ in the case of maximal delocalization, and in our case this occurs for $b = N$ and $\alpha = 0$, which corresponds to Gaussian orthogonal ensemble matrices of size N. For large N, one computes $H_{\rm ref} \approx \ln N - 0.73$. The definition (6) is very convenient because it applies to both localized and extended states, and gives results which correspond to the common intuition of length of a state. We average the entropy over all eigenvectors of a number of matrices in the ensemble. This average value is then used to evaluate l_N by means of (6).

The validity of the two-parameter scaling hypothesis has been tested numerically by plotting $\beta_N \equiv l_N/N$ as a function of one variable, keeping the other fixed (see Figs. 1-3). We find two different types of behavior, separated by a small crossover regime centered around the line

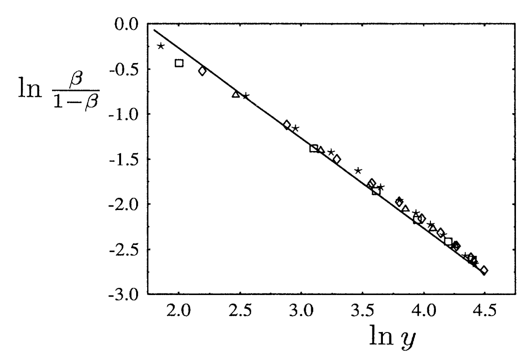

FIG. 2. Scaling of localization in y, at a fixed value of $x = 1$. The various symbols denote different values of N: $N = 256$ (∗), $N = 400$ (◇), $N = 576$ (△), and $N = 800$ (□). The straight line represents the prediction of Eq. (10) with $k = 4\sqrt{2}$.

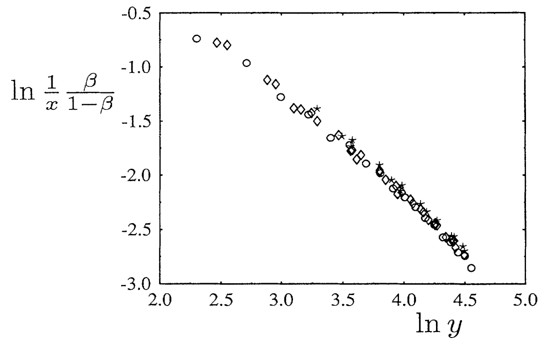

FIG. 3. The numerically obtained $C_2(y)$ function [see Eq. (8)]. The three symbols correspond to different values of x: $x=0.5$ (○), $x=1$ (◇), and $x=2$ (∗).

$$x=x_{\rm cr}(y)=y/2k\,, \tag{7}$$

where $k=4\sqrt{2}$ [19]. The behavior in each of the two regimes results from the competition between the characteristic length of the corresponding infinite system, l_∞, and the actual size of the sample, N. The first regime, where $x<y/2k$, is the "electrical regime." Here, it is the electric field characteristic length [19], $l_{\infty,{\rm el}}$, which competes with N. On the other hand, in the second regime, where $x>y/2k$, the situation is analogous to that of Eq. (1). We refer to this regime as the "Anderson regime." Here, $l_{\infty,A}$ is given by Eq. (3).

In order to understand the behavior of β_N in both regimes 1 and 2, one needs to interpolate between the limiting behaviors at large [19] and vanishing y [see Eq. (1)] using the insight obtained from the numerical data. Accordingly, we find that

$$F(x,y)=\frac{x^{a_i(y)}C_i(y)}{1+x^{a_i(y)}C_i(y)},\quad i=1,2\,, \tag{8}$$

where $i=1$ corresponds to the electrical regime and $i=2$ to the Anderson one. From Eq. (3), taking $x\to 0$, we have $C_1(y)=f(y)$. The numerical study indicates that $a_1(y)=1$ for all y, rather than only for large y or large N (see Fig. 1 and Table I). In order that Eq. (8) agree with Eq. (1), $a_i(0)=1$ and $C_2(0)=c$. Moreover, assuming continuity of β_N between the two regimes leads to

$$C_2(y)=(y/2k)^{-a_2(y)}C_1(y)\,. \tag{9}$$

Finally, we observe that $a_2(y)$ is monotonically increasing.

It is worthwhile to describe the way in which the limiting behavior of Eq. (8) is approached when either x or y vanishes. In both cases we are left with a single regime and, accordingly, no transition. In very large samples, that is, when $x\to 0$, we have $\beta_N\to\beta_\infty$ [see Eq. (3)]. Since always, $x<y/2k$ for all b and α, we are in the electric regime. This is in agreement with the finding of Ref. [17] that in infinite systems the electric field is dominant in determining the localization length. On the other hand, when $y\to 0$, we have $a_2(y)\to 1$ and $x_{\rm cr}\to 0$, such that the Anderson regime of Eq. (1) is recovered for all b and N.

In Table I, a quantitative comparison between the predictions of Eq. (8) and the numerical results is presented for four different values of y. Two different methods were used to obtain $x_{\rm cr}$. The value of $x_{{\rm cr},a}$ represents the value of x at the intersection of the best fitting straight lines corresponding to the two regimes (see Fig. 1). On the other hand, $x_{{\rm cr},1}$ is obtained in a similar way only this time with the constraint $a_1=1$. The agreement between the numerical and the theoretical $x_{\rm cr}$'s is reasonable considering the arbitrariness in the definition of the former.

The argument given in Refs. [6,19] for the asymptotic behavior of $f(y)$ can be extended to the case of a finite system of length N. The corresponding energetically allowed range in the space-energy plane changes shape when $r=N\alpha/2$, that is, $x=y/k$. It turns out, however, that the resulting localization length,

$$\beta_N=kx/(y+kx)\,, \tag{10}$$

is the same on both sides of the transition. This and the fact that the resulting $x_{\rm cr}$ is twice larger than in Eq. (7) are a consequence of the strong assumptions entering this derivation. Nevertheless, this simplified model is a good starting point towards a theory for Eq. (8).

One expects that in addition to β_N, which is basis dependent, other quantities of the Wigner ensemble also display scaling analogous to that of Eq. (4). In particular, it is important to verify whether such scaling holds for spectral properties. Accordingly, we studied the behavior of the spacing distribution $P(s)$. Figure 4 represents the analog of Fig. 2, with a parametrization of the spectrum, the q variable of the best fitting Brody distribution [20] to the numerically obtained spacing distributions, replacing the parametrization of eigenstates, namely, the entropy localization length. As in Fig. 2, we keep x fixed and vary y. Notice that while for $q=1$ the Brody distribution is identical to the one of Wigner, at $q=0$ it

TABLE I. The exponents a_i and crossover values of x, $x_{\rm cr}$, at various y's.

y	a_1	a_2	$x_{{\rm cr},a}$	$x_{{\rm cr},1}$	$x_{{\rm cr,th}}$
5	1.08 ± 0.02	1.145 ± 0.004	0.4 ± 0.2	0.27 ± 0.02	0.4
10	1.06 ± 0.02	1.305 ± 0.006	1.4 ± 0.2	1 ± 2	0.9
20	1.040 ± 0.006	1.433 ± 0.006	2.11 ± 0.07	1.9 ± 0.2	1.8
40	1.042 ± 0.007	1.40 ± 0.02	3.4 ± 0.6	2.9 ± 0.5	3.5

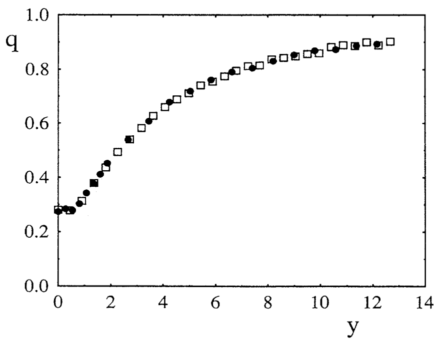

FIG. 4. The scaling of the Brody parameter q with y. Here $x = 0.18$. Moreover, $b = 10$ (□) and $b = 12$ (●).

takes the Poisson form. Since for $l_N = O(N)$ there is only a small number of independent blocks, $q > q_0 > 0$, even when $y \rightarrow 0$.

The two-parameter scaling we find in the Wigner ensemble is expected to show up in disordered short wires in strong electric fields. The latter should be not much longer than the square of the number of sites in their cross section, such as to keep x finite. For a particular sample, N and b are fixed. Then, as the electric field α is increased, a transition between the Anderson and electric regimes is to be expected. Since the behavior of the localization length is explicitly manifest in that of the corresponding conductance, these predictions can be directly verified in experiment.

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