PHYSICAL REVIEW B VOLUME 31, NUMBER 10 15 MAY 1985

Statistics of quasi-energy separations in chaotic systems

Mario Feingold and Shmuel Fishman
Department of Physics, Technion–Israel Institute of Technology, 32000 Haifa, Israel

D. R. Grempel
Institute Laue-Langevin, 38042 Grenoble Cedex, France

R. E. Prange
Department of Physics and Astronomy, Institute for Physical Sciences and Technology and Center for Theoretical Physics, University of Maryland, College Park, Maryland 20742

(Received 16 January 1985)

The distribution of the quasi-energy separations for the periodically kicked quantum rotor is calculated. This distribution is different from that associated with chaotic systems whose wave functions are extended throughout the system. Rather, it agrees with the statistics expected for random band-diagonal Hamiltonians, which describe quantum motion in a disordered medium, where the wave functions are localized. Novel statistical measures based on this localization are devised and applied.

The profound work of Wigner[1] and his successors on the statistics of energy levels at random matrices makes it natural to study similar statistics in problems of quantum chaos.[2] Wigner's argument for the use of random matrices is that for many-body systems like nuclei[3] the true Hamiltonian is very complicated, and therefore can be considered random. Moreover, there is no preferred representation in which to define the random distribution. Thus, an ensemble (of real symmetric matrices) invariant under, say, orthogonal transformations is studied. The main result is the Wigner-level repulsion, namely, the probability $P(S)dS$ of finding two levels separated by S vanishes as SdS for small S.

The Hamiltonians of simple systems of low symmetry that are chaotic in the classical limit,[4-6] like the stadium or Sinai billiard problem, or nonlinearly coupled harmonic oscillators, can be represented by a matrix which is pseudorandom, that is, the matrix elements can be calculated by a deterministic algorithm, but the final result is similar to a "typical" matrix chosen from an ensemble. In particular, the levels at high energy are studied where the quasiclassical approximation is valid. Since these systems are known to be chaotic (and nonintegrable) in the classical limit, it was expected[4-6] and found[7,8] that Wigner statistics would apply, i.e., there is level repulsion.

However, it is generally not obvious what ensemble of matrices corresponds to a given problem. An example of a quite different class of random matrices is given by the problem of quantum motion in a (one-dimensional) random medium.[9] This corresponds to an ensemble of band-diagonal matrices, matrices all of whose nonvanishing elements lie in a band close to the diagonal. There is here a special and natural representation, namely, the usual real-space representation. The energy statistics in this case are Poissonian, with no level repulsion, a rigorously proven result.[10] The reason is clear; the eigenstates are Anderson localized (falling off exponentially away from their centers), and there is thus negligible overlap between states with far-separated centers. Since nearly all pairs of states have far-separated centers in a large system, there is no level repulsion. However, the existence of a preferred representation allows meaningful new statistical questions to be explored, for example, the distribution of level separation of pairs of states whose centers are separated by a given distance. Here Wigner repulsion is expected, with the scale of such repulsion decreasing (exponentially) with separation (since that is the behavior of the overlap of the two wave functions).[11] In this Rapid Communication we study a classically chaotic problem whose quantum version corresponds to this class of random matrices.

A second generalization is also useful. Some of the most interesting problems in Hamiltonian chaos are those of periodically driven systems. Then the statistics of quasi-energies rather than energies is appropriate. The quasi-energies are eigenvalues (mod2π) of the operator H_q, where the time-evolution operator for one period t_0 is $T(t_0)=e^{-iH_q t_0}$.

Here we analyze the quasi-energy spacings for the periodically kicked quantum rotor,[12,13] with moment of inertia I and unit period. It was recently argued and supported by numerical evidence that states of fixed quasi-energy are exponentially localized in (angular) momentum (or energy) space. The localization mechanism for this problem is similar to Anderson localization[9] of an electron in a one-dimensional random potential.

The one-period evolution operator is $T=\exp(-ik\cos\theta)\times\exp(-i\tau p^2/2)$. (The first factor describes the kick of dimensionless strength k; the second describes free propagation between kicks, where $\tau=\hbar/I$, $p=-i\partial/\partial\theta$.) The states u_m^ω in the preferred- (angular-) momentum representation satisfy $\sum T_{nm}u_m^\omega=\Lambda_\omega u_n^\omega$ with $\Lambda_\omega=e^{-i\omega}$ and

$$T_{nm}=(-i)^{m-n}J_{m-n}(k)\exp(-i\tau m^2/2)\ .$$

The J_m are the Bessel functions of the first kind. We take $\tau=1$ so τ/π is irrational, so that the phase in T_{mn} is pseudorandom.[12] The important elements of T lie in a band around the diagonal. This property is probably the underlying reason for Anderson localization. It also enables us to diagonalize well-defined large blocks of this infinite matrix, corresponding to the use of finite-size samples in the random medium problem. We find that each of the eigenstates is exponentially localized around some site n_ω, namely, $|u_n^\omega|\sim e^{-\gamma|n-n_\omega|}$. The quasi-energy separations within each

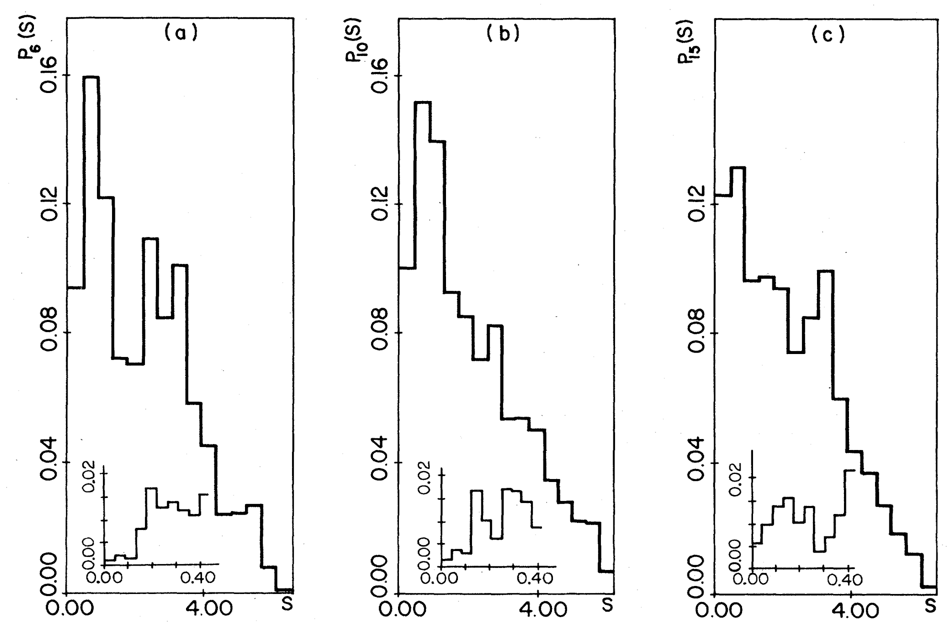

FIG. 1. The nearest-neighbor quasi-energy spacings distribution $P_L(S)$ for states separated a distance (a) $L=6$, (b) $L=10$, and (c) $L=15$, in momentum space, while $\tau=1$ and $k=5$. The distribution within the first bin is depicted in the insert.

large block *numerically exhibit a Poisson distribution,* as expected from the above discussion. We will present this clearcut numerical evidence elsewhere.

We turn now to the distribution $P_L(S)$ (Fig. 1) of quasi-energy separations S between states localized at *fixed separation* L in momentum space.[11] We define the center of a given quasi-energy state as that angular momentum n_ω, such that $u^{\omega}_{n_\omega} = \max_j(|u_j^{\omega}|)$. (The center is necessarily an approximate concept, but we have checked that any errors introduced are not appreciable.) It is clear that the probability for small separation is small. Moreover, it is expected that $P_L(S)$ will sharply decrease for $S \leq e^{-\gamma L}$. This is qualitatively true, but our data do not establish the functional dependence of $P_L(S)$ on L.

To improve the statistics, we study families of quasi-energies rather than just pairs. These families are similar to blocks of the Hamiltonian usually studied for time-independent chaotic systems. We specifically study families F_m of quasi-energies localized on a fixed number m $(=7)$ of momenta on an interval of length N with equal separation N/m. Each family is ordered in quasi-energy and the sequence of quasi-energy separations is calculated and accumulated for various families. The resulting distribution of nearest-neighbor quasi-energies $P(S)$ is plotted in Fig. 2 for $N=35$ and $k=1,3,5$, and in Fig. 3 for $k=3$, $N=7,21,91$. It is obvious that the repulsion *increases* with k, but *de-*

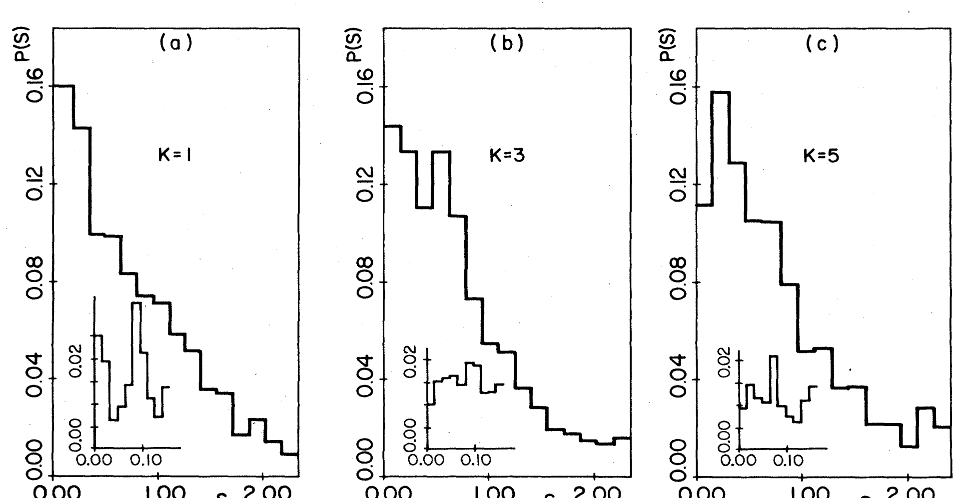

FIG. 2. The nearest-neighbor spacings distribution of quasi-energy states for families with $N=35$. The parameters of the Hamiltonian are $\tau=1$ while (a) $k=1$, (b) $k=3$, (c) $k=5$. The insert shows the distribution within the first bin.

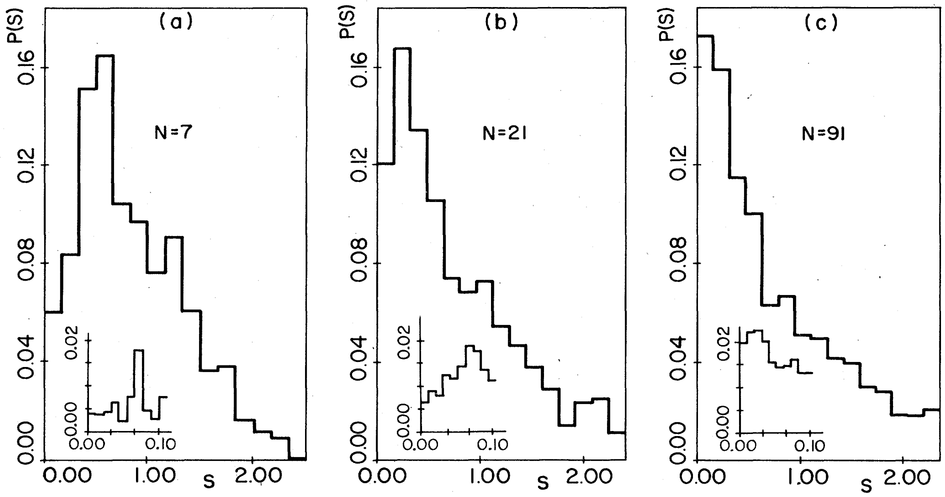

FIG. 3. Same as Fig. 2, but for $\tau = 1$, $k = 3$, with varying momentum-interval size (a) $N = 7$, (b) $N = 21$, (c) $N = 91$.

creases with N. This is explained by the localization picture, since $\xi = 1/\gamma$ increases with k ($\xi \cong 0.9, 4.6, 9.0$ for $k = 1, 3, 5$). Thus, for constant N, the wave function overlap *increases* with k, while for constant k (i.e., ξ), the overlap *decreases* with N. Other, seemingly natural explanations, for example, that an increase in repulsion corresponds to an increase in the underlying classical stochasticity with k (as in the time-independent models[8,14]) are ruled out by the N dependence.

We now give some details of the numerical procedure. The matrix T is truncated to 200×200 nonoverlapping blocks along the diagonal. Each block is diagonalized separately. States localized on edges of the blocks are affected by the truncation, while states localized in the interiors of the blocks are not. We can easily identify the unwanted states as those for which $||\Lambda_\omega| - 1| > \epsilon$, since the truncation destroys the unitarity of T. We have spot checked that these eigenvalues correspond to eigenvectors localized on the edges of blocks. The diagonalization was checked also, by applying it to T with m^2 in Fig. 3 replaced by m. For this case the exact solution is known,[15] and indeed for the values of $|\Lambda_\omega|$ close to unity the eigenvectors agreed with the exact solution. The size of the smallest bin in each histogram is $S_b = S_m/5$, where S_m is the mean separation. We have taken $\epsilon = S_b/30$, a third of the smallest bin in the insert. The results do not change when $\epsilon = S_b/60$. The number of blocks diagonalized was such that the number of differences used to obtain each of the histograms in Fig. 1 is 16000 while each histogram in Figs. 2 and 3 contains 7500 and 9500 differences, respectively.

In summary, this work confirms that the quasi-energy nearest-neighbor spacings distribution of the kicked rotor is Poissonian and therefore is similar to the corresponding distribution for energy levels spacing in the one-dimensional localization problem. This is further evidence for the similarity between these two problems. This distribution is very different from the ones that were obtained so far for stochastic problems with time-independent Hamiltonians. The relevant difference between the quantal behavior of these two classes of problems, both chaotic in their classical limit, is the extension or localization of the wave functions.

Note added. After submission of this paper we learned of work by F. M. Israilev (unpublished, in Russian) which is fully consistent with ours, although the emphasis is quite different. We thank M. V. Berry for communicating this work to us.

We thank H. G. Schuster for suggestions and encouragement at early stages of this work and M. V. Berry for comments on the results. Two of us (S.F. and D.R.G.) thank the Einstein Summer Institute for hospitality at the Weizmann Institute where this work was started. The work was supported in part by the U.S.–Israel Binational Science Foundation (BSF), by the Bat-Sheva de-Rotschild Fund for Advancement of Science and Technology, and by the NSF through Grants No. DMR-79-001172-A02 and No. DMR-79-08819.

[1]M. L. Mehta, *Random Matrices and Statistical Theory of Energy Levels* (Academic, New York, 1967); *Statistical Theories of Spectra; Fluctuations,* edited by C. E. Porter (Academic, New York, 1965); M. Carmeli, *Statistical Theory and Random Matrices* (Marcel Dekker, New York, 1983).

[2]For a review see, e.g., G. Zaslavsky, Phys. Rep. **80**, 158 (1981); M. V. Berry, Lecture Notes of the Nordita School on Chaos, 1982 (unpublished).

[3]T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. H. Wong, Rev. Mod. Phys. **53**, 385 (1981).

[4]P. Pechukas, Phys. Rev. Lett. **51**, 943 (1983).

[5]M. V. Berry, Ann. Phys. (N.Y.) **131**, 163 (1981).

[6]M. V. Berry and M. Tabor, Proc. R. Soc. London, Ser. A **356**, 375 (1977).

[7]O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984); S. W. McDonald and A. N. Kaufman, *ibid.* **42**, 1189 (1979); G. Casati, F. Valz-Gris, and I. Guarneri, Nuovo Cimento Lett. **28**, 279 (1980).

[8]E. Haller, H. Köppel, and L. S. Cederbaum, Phys. Rev. Lett. **52**, 1665 (1984); T. H. Seligman, J. J. M. Verbaarschot, and M. R. Zirnbauer (unpublished).

[9]For a review, see, e.g., D. J. Thouless, in *Ill-Condensed Matter*, Proceedings of Les Houches Summer School, Session XXXI, edited by R. Balian, R. Maynard, and G. Toulouse (North-Holland, Amsterdam, 1979), Vol. 31.

[10]S. A. Molčanov, Commun. Math. Phys. **78**, 429 (1981).

[11]N. F. Mott and E. A. Davis, *Electronic Processes in Non-Crystalline Materials* (Clarendon, Oxford, 1971).

[12]S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. **49**, 509 (1982); D. R. Grempel, R. E. Prange, and S. Fishman, Phys. Rev. A **29**, 1639 (1984), and references therein.

[13]B. V. Chirikov, F. M. Izrailev, and D. L. Shepelyansky, in *Soviet Scientific Reviews, Vol. 2 Section C,* edited by S. P. Novikov (Harwood, New York, 1981), p. 209; T. Hogg and B. A. Huberman, Phys. Rev. Lett. **48**, 711 (1982); Phys. Rev. A **28**, 22 (1983).

[14]M. V. Berry and M. Robnick, J. Phys. A (to be published).

[15]D. R. Grempel, S. Fishman, and R. E. Prange, Phys. Rev. Lett. **49**, 833 (1982); R. E. Prange, D. R. Grempel, and S. Fishman, Phys. Rev. B **29**, 6500 (1984).