CLASSICAL LIMIT OF QUANTUM CHAOS

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When a quantum system has a classically chaotic analog, the expectation value $\langle E|\hat{A}|E\rangle$ of any dynamical variable $\hat{A}$, for an energy eigenstate $|E\rangle$, tends to the classical microcanonical average of the analog variable A, at the same energy E Some numerical examples are discussed.

Although there is no consensus on the definition or nature of quantum chaos, nor even on its existence, there appears to be a growing body of theoretical [1–7] and numerical [8–12] evidence in support of the following property: If a quantum system has a classically chaotic analog, then, in the semiclassical limit $\hbar \to 0$, the energy eigenfunctions fill the entire accessible phase space and, moreover, their Wigner distributions [13] fluctuate around the classical microcanonical phase space density.

The purpose of this Letter is to give additional support to this conjecture and to base upon it a simple test for quantum chaos. We also discuss what happens when regular and chaotic orbits coexist at the same energy in the classical system.

If a classical system is ergodic (so that its only constant of motion is the energy) the time average $\bar{A}$ of a dynamical variable A is equal to the microcanonical phase space average of A at that energy.

$$\bar{A} = \frac{\int A(p,q)\,\delta[E - H(p,q)]\,\mathrm{d}p\,\mathrm{d}q}{\int \delta[E - H(p,q)]\,\mathrm{d}p\,\mathrm{d}q}\,, \qquad (1)$$

where H is the Hamiltonian, and p and q denote collectively all the canonical variables.

By virtue of the correspondence principle, if the analog quantum system is in a stationary state $|E\rangle$, the expectation value $\langle E|\hat{A}|E\rangle$ of the operator $\hat{A}$ should be close to $\bar{A}$ [14–16]. This ought to be a good approximation in the limit $\hbar \to 0$ or, equivalently, for large quantum numbers. Notice that $\langle E|\hat{A}|E\rangle$ is constant, even if $[\hat{A},\hat{H}] = 0$.

This conjecture was formally proved by Shnirel'man [17]* in the special case of the spectrum of the Laplace operator on a compact Riemannian manifold. Here, we shall test it with two physical models known to be classically chaotic. In the first model, we take as the Hamiltonian [18,19]

$$H = \tfrac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2) + 0.05x^2y^2\,, \qquad (2)$$

which is known to have a mixed behavior. For low E, most classical orbits are regular; as the energy increases, there is a growing proportion of chaotic orbits; and most orbits are chaotic for $E > 50$, say. There are, however, islands of stability even for large E (see fig. 2 of ref. [18]) whose importance will soon be apparent

The Hamiltonian (2) has several discrete symmetries ($x \leftrightarrow -x$, $y \leftrightarrow -y$ and $x \leftrightarrow y$). Here, we considered only the fully symmetric eigenfunctions of $\hat{H}$. As the basis, we took symmetrized products of oscillator wavefunctions, namely $u_m(x)u_n(y) + u_n(x)u_m(y)$, with m and n even and $m + n \leqslant 100$ (there are 676 functions in that basis). Taking $\hbar = 1$, we obtained 230 well converged energy levels and eigenfunctions (up to $E < 70.4$).

As the operator $\hat{A}$ we took $(xp_y - yp_x)^2$ which has no matrix element connecting the above symmetry class with other ones. Fig. 1 is a scatter plot of

$$J = [\langle E|(xp_y - yp_x)^2|E\rangle]^{1/2} \qquad (3)$$

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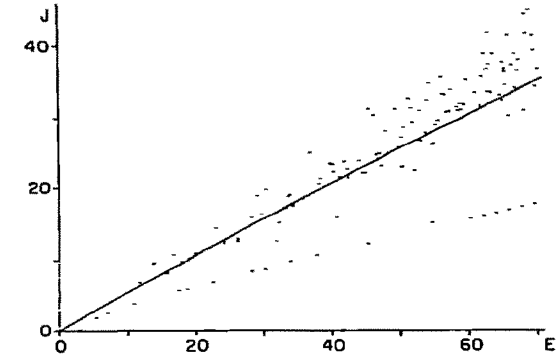

Fig. 1. The values of J, from eq. (3), versus those of E, for the 230 lowest energy levels. The solid line is the classical limit given by eq. (1).

versus E, for the 230 lowest levels. The corresponding classical result, obtained from eq. (1), is shown as a solid line. (We plotted J, rather than J^2, because the result is visually nicer. In particular, the classical result is very close to a straight line.)

Two features are apparent. Obviously, the quantum mechanical results are clustered around the classical microcanonical average. The correlation is even better if we lump together neighboring quantum levels, as shown in fig. 2. This should be contrasted with the situation for a regular quantum system (one with selection rules, i.e. with "good" quantum numbers other than the energy). In that case, a scatter plot such as fig. 1 would display a regular two-dimensional pattern of points [20].

The other remarkable feature of fig 1 is a sequence

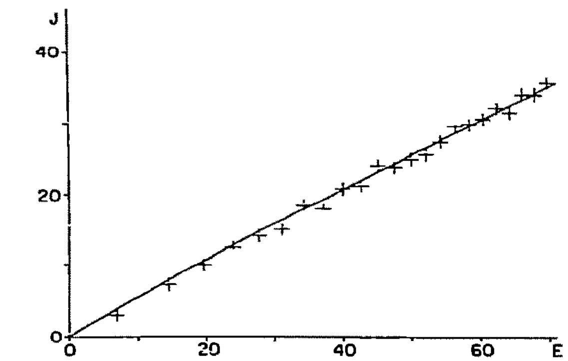

Fig. 2. Same as fig. 1, but with the *average* values of E and J in 23 sets of 10 consecutive levels. (This average *includes* the levels with anomalously low J.)

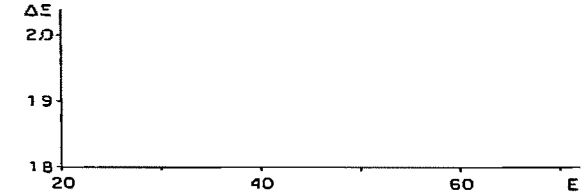

Fig. 3 The regular levels, with anomalously low J, have nearly constant energy spacings. (Notice that the scales on the axes do not start at zero.)

of levels having anomalously low J. These levels are *nearly equidistant* (see fig. 3), a clear indication of their regular nature [10,15,21–23]. They correspond to the classical islands of stability mentioned above, namely to classical orbits very close to the x or y axes. We infer this from the fact that the corresponding eigenvectors involve mostly very low m or n. This coexistence of regular and chaotic behavior would be difficult to detect by investigating the energy spectrum alone [19], without correlating it to another variable.

To be sure, the discovery of this regular set of levels hinges on our choice $\hat{A} = (xp_y - yp_x)^2$. Had we chosen a different $\hat{A}$, these levels could have remained hidden among the much more numerous chaotic ones. Conversely, it is not impossible that there are other sequences of regular levels, but that these sequences do not stand out in the chaotic crowd with the $\hat{A}$ which we used.

The second model is based on the Hamiltonian [20]

$$H = p_x + p_x + (L^2 - p_x^2)^{1/2}(M^2 - p_x^2)^{1/2} \cos x \cos y \, ; \tag{4}$$

where L and M are constants. It is convenient to define new variables $L_1 = (L^2 - p_x^2)^{1/2} \cos x$, $L_2 = (L^2 - p_x^2)^{1/2} \sin x$, and $L_3 = p_x$, which satisfy the same Poisson brackets as components of angular momentum. Likewise, we define $M_1 = (M^2 - p_y^2)^{1/2} \cos y$, etc. The Hamiltonian (4) thus becomes

$$H = L_3 + M_3 + L_1 M_1 \, , \tag{5}$$

and can be interpreted as representing a pair of nonlinearly coupled rotators.

In quantum theory, the constants L and M must satisfy $L^2 = \hbar^2 l(l+1)$ and $M^2 = \hbar^2 m(m+1)$ and the Hamiltonian (5) is a *finite* matrix of order $(2l+1) \times (2m+1)$. As in our preceding work [12,20,24,25] we

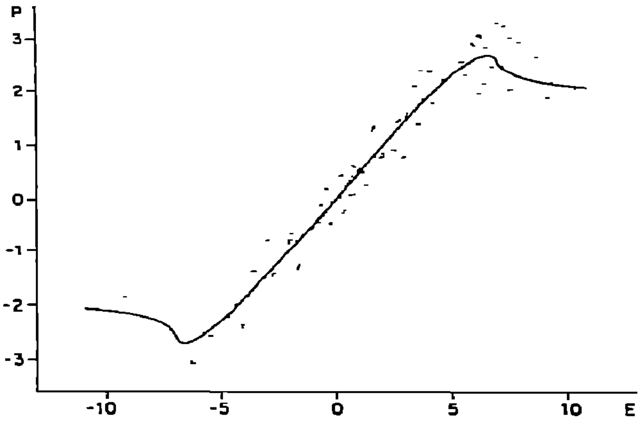

Fig. 4. The values of $P = \langle E|p_x + p_y|E\rangle$ versus those of E for 439 energy levels of the Hamiltonian (4). Two additional points are outside the limits of the figures $E = \pm 6.80, P = \pm 4.30$. The solid line is the classical limit given by eq. (1).

took $l = m = 20$ and $\hbar = 0.1707825$ (this corresponds to $L = M = 3.5$). We considered only the invariant subspace of Hilbert space which is even under $x \leftrightarrow y$ and where $(p_x + p_y)/\hbar$ is an even integer. The dimension of this subspace is $21^2 = 441$. The energy spectrum is non-degenerate and symmetric with respect to $E = 0$.

Fig. 4 is a scatter plot of $P = \langle E|p_x + p_y|E\rangle$ versus E. The corresponding classical result, obtained from eq. (1), is shown as a solid line. (The integrations to obtain it are somewhat tricky. It is best to first integrate over x, then over p_x, then over p_y and finally one has to perform a numerical integration over y. The limits of the various integrals are obtained from the requirement that expressions under square roots must be non-negative.) Here again, the quantum mechanical results are clustered around the classical microcanonical average ‡. There is also a small set of regularly spaced points for $|E| \gtrsim 9$, which cover a regular region of the classical phase space [12,20,27].

In summary, we have proposed here a *positive test* for quantum chaos Its characteristic feature is not the *absence* of order, but is a *new order*, which is close to the classical limit: "Though this be madness, yet there is method in't" [28].

‡ This result was predicted by Berry [26]. An interesting problem, which we did not solve, is to estimate the deviation of the quantum results from the classical ones. It probably behaves as a power of $\hbar$.

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