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Topological aspects of quantum chaos

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Quantized classically chaotic maps on a toroidal two-dimensional phase space are studied. A discrete, topological criterion for phase-space localization is presented. To each eigenfunction is associated an integer, analogous to a quantized Hall conductivity, which tests the way the eigenfunction explores the phase space as some boundary conditions are changed. The correspondence between delocalization and chaotic classical dynamics is discussed, as well as the role of degeneracies of the eigenspectrum in the transition from localized to delocalized states. The general results are illustrated with a particular model.

I. INTRODUCTION

In classical mechanics, when an integrable system (i.e., having as many global independent constants of the motion as degrees of freedom) is generically perturbed, some invariant tori are broken and replaced by small "stochastic" layers where chaotic motion takes place. This process is amplified as the perturbation becomes stronger. The way this transition occurs is described by the KAM theorem.¹ For a moderate perturbation, these two kinds of motion coexist forming an intricate mixed structure in phase space. For a sufficiently large perturbation, most invariant tori disappear and chaotic motion prevails.

Quantum mechanically the problem is less understood. In the integrable regime the semiclassical eigenstates (when represented in phase space) concentrate exponentially around the quantized EBK classical invariant tori, where the classical trajectories lie, and are small elsewhere. When those tori are broken by a perturbation, we expect the quantum eigenstates to "delocalize" and to cover, in a more or less uniform way, a fraction or the whole available phase space. The purpose of this contribution is to provide a quantum mechanical description of such a transition (see also Ref. 2).

Considering the special case of quantized, area-preserving maps of a phase space having a two-dimensional toroidal geometry, we will show how it is possible to associate to each eigenstate ψ_α of the quantum evolution operator an integer C_α , the Chern index. This integer characterizes the eigenstates as belonging to two different groups: those having $C_\alpha = 0$, which we shall see are localized states and those with $C_\alpha \neq 0$, the delocalized ones. The integer tests the way the eigenfunction explores the phase space as some boundary conditions (of a purely quantum mechan-

ical nature) of the system are varied. We correlate the spectrum of these integers with the associated regular-to-chaotic classical transition, and show that in the course of the transition most states change their character from localized to delocalized. Degeneracies of the spectrum are shown to play a fundamental role in the organization of this quantum-mechanical transformation. Comparison with classical invariant phase space structures suggests that these integers also reflect localization of eigenfunctions by unstable periodic orbits.

II. QUANTUM MECHANICS ON A TWO-DIMENSIONAL TORUS AS PHASE SPACE

A. Kinematics

The two-dimensional toroidal phase space (denoted by T_x^2) is a periodically repeated cell having sides (Q, P) in suitable (q, p) coordinates. Classically the dynamics of the system is assumed to be invariant under translations by the elementary cell $(nQ + mP)$, where (n, m) are arbitrary integers. Quantum mechanically, the states of the Hilbert space are required to be—under those translations—periodic functions (up to a phase)

$$\begin{aligned} T_1|\psi\rangle &= e^{i\theta_1}|\psi\rangle, \\ T_2|\psi\rangle &= e^{i\theta_2}|\psi\rangle, \end{aligned} \quad (1)$$

where $T_1 = \exp(iQ\hat{p}/\hbar)$, $T_2 = \exp(iP\hat{q}/\hbar)$, and (θ_1, θ_2) are two arbitrary independent phases ranging from 0 to 2π . In order to satisfy Eqs. (1), T_1 and T_2 must commute. This imposes that the area of T_x^2 measured in units of Planck's constant must be an integer

$$QP/2\pi\hbar = N.$$

Because the dynamics is assumed to be invariant under the action of T_1 and T_2 [cf. Eq. (9) below], $\theta = (\theta_1, \theta_2)$ are good quantum numbers preserved by the dynamics. The

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Hilbert space $\mathcal{H}_N(\theta)$ hence breaks down into subspaces parametrized by θ , and $\dim \mathcal{H}_N(\theta) = N, \forall \theta$. For a fixed area QP , the classical limit $\hbar \rightarrow 0$ is equivalent to $N \rightarrow \infty$ and corresponds to an increasing number of states supported by T_x^2 .

For each $\mathcal{H}_N(\theta)$ one can define a normalizable basis in the q representation through the periodicized sum

$$|n, \theta\rangle = \sum_{\nu=-\infty}^{\infty} e^{i(\nu + n/N)\theta_1} \left| \left[\frac{1}{N} \left(n + \frac{\theta_2}{2\pi} \right) + \nu \right] Q \right\rangle, \quad n = 0, \dots, N-1. \quad (2)$$

The kets appearing in the rhs of (2) are the standardized eigenvectors of the $\hat{q}$ operator. The states $|n, \theta\rangle$ satisfy the boundary conditions (1); they are the quantum-mechanical discretization of the continuous finite interval $q: [0, Q]$ —the arbitrariness in the boundary conditions can be viewed as an arbitrariness under shifts of $\mathcal{O}(1/N)$ of the position of the “comb” of δ 's in that interval of q . An arbitrary state of $\mathcal{H}_N(\theta)$ will be characterized by N complex numbers $|\psi(\theta)\rangle = \sum_{n=0}^{N-1} \psi_n(\theta) |n, \theta\rangle$ satisfying $\sum |\psi_n|^2 = 1$. An alternative representation of $\mathcal{H}_N(\theta)$ is in terms of coherent-states:^{3,4} $\psi(z, \theta) = \sum_{n=0}^{N-1} \langle z | n, \theta \rangle \psi_n(\theta)$, where

$$\begin{aligned} \langle z | n, \theta \rangle &= (\pi \hbar)^{-1/4} e^{in\theta_1/N} \exp \left\{ \frac{-2\pi N}{QP} \right. \\ &\times \left[\frac{z^2 + (n/N + \theta_2/2\pi N)^2 Q^2}{2} - \sqrt{2} z \left(\frac{n}{N} + \frac{\theta_2}{2\pi N} \right) Q \right. \\ &\left. \left. + \frac{\theta_2}{2\pi N} \right) Q \right] \left. + i\pi N \left[\left(\frac{n}{N} + \frac{\theta_2}{2\pi N} \right) \frac{Q}{P} \right. \right. \right. \\ &\left. \left. - i \frac{\theta_1}{2\pi N} - \frac{\sqrt{2} z}{P} \right] \right\} \left[i \frac{Q}{P} N \right], \quad (3) \end{aligned}$$

$z = (q - ip)/\sqrt{2}$ is a complex variable spanning T_x^2 and $\vartheta_3(v|\tau)$ is the Jacobi theta function.⁵ Here, $\psi(z, \theta)$ is an analytic function on the fundamental domain $[0, Q/\sqrt{2}] \times [0, P/\sqrt{2}]$ with its boundary Γ included and satisfying the quasiperiodic conditions

$$\psi(z + Q/\sqrt{2}, \theta) = e^{i\theta_1} e^{\pi N(Q/2P + \sqrt{2}z/P)} \psi(z, \theta),$$

$$\psi(z + iP/\sqrt{2}, \theta) = e^{i\theta_2} e^{\pi N(P/2Q - i\sqrt{2}z/Q)} \psi(z, \theta).$$

From these relations it follows that

$$(2\pi i)^{-1} \oint_{\Gamma} \frac{\psi'}{\psi} dz = N, \quad (4)$$

i.e., every state $\psi(z, \theta)$ of $\mathcal{H}_N(\theta)$ has exactly N zeros in the elementary cell T_x^2 . From multiplication formula of elliptic function theory, it is possible to reconstruct from its zeros the analytic function $\psi(z, \theta)$. Therefore, the knowledge of the position of the N zeros of $\psi(z, \theta)$ on T_x^2 , $\{z_k(\theta)\}_{k=1, \dots, N}$, completely determines the state of the system. Moreover, the positive periodic real function

$$W(q, p, \theta) = e^{-\bar{z}z/\hbar} |\psi(z, \theta)|^2 \quad (5)$$

is a quasiprobability distribution function on T_x^2 and has the same zeros as $\psi(z, \theta)$ (Ref. 4).

B. Dynamics

The classical dynamics of a point particle on T_x^2 will in general be defined by an area preserving map of T_x^2 onto itself

$$x_{n+1} = M(x_n, \gamma), \quad x = \begin{pmatrix} q \\ p \end{pmatrix} \bmod \begin{pmatrix} Q \\ P \end{pmatrix}, \quad \left| \frac{\partial x_{n+1}}{\partial x_n} \right| = 1, \quad (6)$$

which depends on some parameter γ that typically controls a regular-to-chaotic transition of the system. The quantum-mechanical analog of the classical one-step map (6) is provided by a one-step unitary matrix $U(\hat{p}, \hat{q}, \gamma)$ (Ref. 6)

$$|\psi_{n+1}(\theta)\rangle = U |\psi_n(\theta)\rangle. \quad (7)$$

Eigenstates are obtained through the stationary equation

$$U |\psi_\alpha(\theta)\rangle = e^{i\omega_\alpha(\theta)} |\psi_\alpha(\theta)\rangle, \quad (8)$$

while the evolution operator $U(\hat{q}, \hat{p}, \gamma)$ satisfies

$$[U, T_1] = [U, T_2] = 0, \quad (9)$$

insuring the invariance of the dynamics under translations by the elementary cell.

It is convenient to employ a unitary transformation

$$R = e^{-i\theta_2 \hat{p}/P} e^{i\theta_1 \hat{q}/Q}, \quad (10)$$

which transfers the θ dependence from the state vectors to U itself. Since the action of R over the operators $\hat{q}$ and $\hat{p}$ is $R^\dagger \hat{q} R = \hat{q} + (\theta_2/2\pi N) Q$ and $R^\dagger \hat{p} R = \hat{p} + (\theta_1/2\pi N) P$, then

$$\begin{aligned} \tilde{U}(\hat{q}, \hat{p}, \gamma, \theta) &= R^\dagger U R \\ &= U \left(\hat{q} + \frac{\theta_2}{2\pi N} Q, \hat{p} + \frac{\theta_1}{2\pi N} P, \gamma \right). \quad (11) \end{aligned}$$

The spectrum of eigenvalues of $\tilde{U}$ is invariant under $\theta_i \rightarrow \theta_i + 2\pi n_i$. Thus, the space of boundary conditions $\theta = (\theta_1, \theta_2)$ is also a torus, which we denote T_θ^2 . Note also that the classical limit $N \rightarrow \infty$ does not depend on θ , as expected.

III. THE CHERN INDEX

As shown in the previous section, quantum mechanically the system will have N eigenstates $\{|\psi_\alpha(\theta)\rangle\}_{\alpha=1, \dots, N}$ parameterized by the boundary conditions θ . Generally speaking, there is nothing which sanctifies a given set of boundary conditions; in principle, they are all equivalent. For that reason, in the following we are going to consider *averaged* properties of eigenstates over the whole set of boundary conditions T_θ^2 . We shall call that ensemble a *band*.

Each band α can be characterized by a topological invariant C_α

$$C_\alpha = \frac{i}{2\pi} \oint_{\mathcal{C}} \langle \psi_\alpha(\theta) | \partial_\theta \psi_\alpha(\theta) \rangle \cdot d\theta. \quad (12)$$

The contour integral is taken along a closed path $\mathcal{C}$ encircling the elementary cell T_θ^2 of boundary conditions. Here, C_α is known as the Chern number of the band⁷ connected

to the mapping of the torus of boundary conditions T_θ^2 onto the complex projective Hilbert space. It corresponds to the phase acquired by the eigenstate $|\psi_\alpha(\theta)\rangle$ as it is parallel-transported around the closed loop $\mathcal{C}$ on T_θ^2 [geometrical phases like Eq. (12) have been extensively studied in recent years; see Refs. 8 and 9]. In what follows, we will sketch some properties of C_α and show why it is a useful object when considering the structure of eigenstates of quantized classically integrable and chaotic systems.

The Chern number C_α as given by Eq. (12), is an integer. This can be seen as follows. Since the system is periodic in T_θ^2 , $|\psi_\alpha(\theta)\rangle$ can only change by an overall phase factor when $\theta_i \rightarrow \theta_i + 2\pi$,

$$\begin{aligned} |\psi_\alpha(\theta_1, 2\pi)\rangle &= e^{i\Phi(\theta_1)} |\psi_\alpha(\theta_1, 0)\rangle, \\ |\psi_\alpha(2\pi, \theta_2)\rangle &= e^{i\chi(\theta_2)} |\psi_\alpha(0, \theta_2)\rangle, \end{aligned} \quad (13)$$

where Φ and χ are uniform functions of their arguments. Using Eqs. (13), C_α can be written

$$C_\alpha = \frac{1}{2\pi} \left(\int_0^{2\pi} \Phi' d\theta_1 - \int_0^{2\pi} \chi' d\theta_2 \right), \quad (14)$$

where the prime indicates derivative. Since $\Phi(2\pi) = \Phi(0) \bmod 2\pi$, $\int_0^{2\pi} \Phi' d\theta_1$ measures the number of times $e^{i\Phi(\theta_1)}$ winds around the unit circle on the complex plane as θ_1 changes from 0 to 2π , say n_Φ times. A similar argument holds for χ , leading to $C_\alpha = n_\alpha$, $n_\alpha = n_\Phi - n_\chi$ being an arbitrary positive or negative integer.

The only necessary assumptions to get this result are Eqs. (13), which hold as long as there are no spectral crossings between the band α and the neighboring bands as we move on the parameter space T_θ^2 : the existence of a degeneracy will allow the eigenstate not to come back to itself up to a phase, as assumed in (13), but to shift, through the degeneracy, into a different state. But according to a theorem of von Neumann and Wigner¹⁰ degeneracies have generically codimension three, and therefore they will typically not be encountered in the two-dimensional torus T_θ^2 , and C_α will generically be an integer.

In terms of the transformed eigenstates $|\tilde{\psi}_\alpha(\theta)\rangle = R^+ |\psi_\alpha(\theta)\rangle$, the Chern index (12) can be written

$$C_\alpha = \frac{i}{2\pi} \oint_{\mathcal{C}} \langle \tilde{\psi}_\alpha(\theta) | \partial_\theta \tilde{\psi}_\alpha(\theta) \rangle \cdot d\theta + \frac{1}{N} \quad (15)$$

or, using Stoke's theorem

$$\begin{aligned} C_\alpha &= \frac{i}{2\pi} \int \int_{T_\theta^2} d^2\theta \\ &\times \sum_{\substack{\beta \\ (\beta \neq \alpha)}} \epsilon_{jk} \frac{\langle \tilde{\psi}_\alpha | \partial \tilde{U} / \partial \theta_j | \tilde{\psi}_\beta \rangle \langle \tilde{\psi}_\beta | \partial \tilde{U} / \partial \theta_k | \tilde{\psi}_\alpha \rangle}{|e^{i\omega_\alpha} - e^{i\omega_\beta}|^2} \\ &+ \frac{1}{N} = \sigma_\alpha + \frac{1}{N}. \end{aligned} \quad (16)$$

It follows from the antisymmetry of the tensor ϵ_{jk} and the fact that $(\partial \tilde{U}^+ / \partial \theta_i) = -\tilde{U}^+ (\partial \tilde{U} / \partial \theta_i) \tilde{U}^+$ that $\Sigma \sigma_\alpha = 0$, and therefore

$$\sum_{\alpha=1}^N C_\alpha = 1. \quad (17)$$

In the context of the quantum Hall effect, quantities like C_α have been introduced by Thouless *et al.*¹¹ in their study of the conductance of electrons in two-dimensional periodic potentials and strong magnetic field. There, C_α is known as the TKNN² integer of the band, its quantization being associated to the fact that each filled (sub)band contributes an integer to the electrical conductance. States having nonzero Chern number are associated to conducting states, as opposed to localized states with $C_\alpha = 0$ (which do not contribute to the Hall conductance).

A different interpretation of this delocalization was given^{12,13} in terms of the coherent state wavefunctions $\psi_\alpha(z, \theta)$. As explained in the previous section, $\psi_\alpha(z, \theta)$ is an analytic function having N zeros on T_x^2 . These zeros respond to changes in the boundary angles θ , $\{z_k^\alpha(\theta)\}_{k=1, \dots, N}$. For states having $C_\alpha \neq 0$, the zeros completely cover the phase space as one spans the θ torus. The Chern number measures how many times the surface spanned by the zeros winds around the phase space torus: a large C_α implies a high mobility of the zeros and consequently a pronounced sensitivity of the wave function to changes in the boundary conditions. If, on the contrary, as we change the boundary conditions there is a point z_0 for which $\psi_\alpha(z_0, \theta)$ never vanishes, the Chern number C_α is zero—the set of points satisfying this condition being interpreted as the localization domain T_x^2 . This follows from the fact that in the latter case the existence of points such as z_0 guarantees that a global choice of phase over T_θ^2 can be made. If that global choice is possible, then we can set $\Phi(\theta_1) = 0 \forall \theta_1$, $\chi(\theta_2) = 0 \forall \theta_2$ in Eq. (13) and, by Eq. (14), we get $C_\alpha = 0$.

This interpretation of delocalization introduces a notion of “ergodicity” in Hilbert space: C_α is different from zero if, as we explore the whole parameter space, $|\psi_\alpha(\theta)\rangle$ becomes orthogonal to all the rays in the projective Hilbert space. This insures that a global and uniform choice of phase cannot be made, leading to $C_\alpha \neq 0$.

Having in mind the above interpretation of the Chern index in terms of the covering of phase space by the zeros, let us now concentrate on the eigenstates of classically integrable and chaotic systems. In the former case, the quantum phase space distributions $W_\alpha(q, p, \theta)$ $\alpha = 1, \dots, N$ associated to the eigenstates are, in the semiclassical limit, exponentially concentrated around the quantized EBK classical invariant tori¹⁴ (for one-dimensional conservative systems the classical invariant tori are just labeled by the energy),

$$W_\alpha(q, p, \theta) = \frac{\mathcal{N}}{v(q, p)} \exp \left[\frac{2}{N} \left(\frac{H(q, p) - E_\alpha}{v(q, p)} \right)^2 \right], \quad (18)$$

where E_α , $v(q, p)$, and $H(q, p)$ are the energy of the quantum eigenstate α , the phase space velocity of the classical trajectory having energy E_α , and the classical Hamiltonian, respectively. Such states have an exponentially small sensitivity to changes in the boundary conditions, and Eq. (18) defines a region surrounding the quantized classical

invariant torus where the zeros—in their motion as a function of the boundary conditions—cannot enter for any θ . Thus *eigenstates of quantized classically integrable systems will generically have zero Chern index*.¹⁵ As we will show in the next section, for integrable systems the sum rule (17) will be fulfilled by eigenstates that are not associated to invariant tori but to separatrices.

Quite the opposite happens with the completely chaotic systems, i.e., those for which a typical orbit completely fills the entire phase space T_x^2 . The correspondence principle suggests in this case that $W_\alpha(q, p, \theta)$ must tend, in the semiclassical limit, to the microcanonical uniform distribution on T_x^2 . However, the mere presence of the zeros prevents that from being the case.⁴ But we do not expect distribution of zeros in T_x^2 for a particular θ to be privileged over others, and compatibility with the uniform microcanonical measure suggests a covering of T_x^2 as the whole T_θ^2 is explored, recovering in this way the uniform measure in an averaged sense. As opposed to the result obtained in the integrable case, we thus expect *eigenstates with support in chaotic regions of phase space to have a nonzero Chern index*.

We shall now focus on the transition between these two kinds of eigenstates. Before that, we introduce an alternative interpretation of the Chern index and of its quantization which will make the following discussion easier. Using Stokes's theorem, the Chern index was written in (16) as the flux of a certain vector field over the closed surface T_θ^2 , $\iint T_\theta^2 V_\alpha \cdot dS$. M. V. Berry, in his study of the adiabatic quantum phases,⁸ has demonstrated that away from degeneracies the vector field V_α has zero divergence. By Gauss's theorem, the integral over T_θ^2 can be reduced, interpolating T_θ^2 to a three-dimensional "filled torus," to a sum of integrals over small spheres enclosing the degeneracies contained inside T_θ^2 . Each degeneracy has an integer "charge," since the integral over the sphere counts the number of times the phase acquired by the eigenfunction over a closed path in parameter space winds around the unit circle as the path is deformed from one point to another tracing out the sphere in between.¹⁶ The Chern index is then the total charge enclosed by T_θ^2 .

If we take into account the parameter γ controlling the dynamics, the system depends on three parameters $(\theta_1, \theta_2, \gamma)$ [cf. Eq. (11)]. Then T_θ^2 is a two-dimensional torus embedded in that three-dimensional space. According to the theorem of von Neumann and Wigner mentioned above, degeneracies are isolated points in that parameter space.

Let us assume that for a certain γ_0 the system is classically integrable. Eigenstates associated to quantized invariant tori have $C_\alpha = 0$. As γ is varied, T_θ^2 "expands" in the full parameter space. As long as we do not encounter degeneracies, the charge inside the torus T_θ^2 remains constant. As opposed to classical mechanics where as soon as we perturbed an integrable system we have small chaotic layers whose volume increases as the perturbation increases, the perturbation of a quantized classically integrable system causes no immediate dramatic change in the eigenfunctions (at least as far as the Chern indices are

concerned), and C_α remains zero. This can be understood from semiclassical arguments: since each state occupies a volume $2\pi\hbar$ in phase space, we do not expect a variation of the Chern index as long as the volume of the chaotic layers do not reach that size.

Changes in C_α only happen when a degeneracy with another band β is met as the torus T_θ^2 expands in the γ direction. Then the Chern index of both bands changes according to the charge of the degeneracy encountered. Equation (17) implies that if C_α changes by ΔC , C_β must change by $-\Delta C$, ΔC being the integer charge associated to the degeneracy. A generic conical intersection has an associated charge of $\Delta C = \pm 1$ (Ref. 8). Higher-order dependences on the parameters (like quadratic glancing intersections) can however lead to higher-order charges for the degeneracies.

The quantum transition from a localized regime to a delocalized one is thus organized by the location and charge of the degeneracies of the spectrum. As γ is changed and the classically chaotic layers increase their size, we expect more and more states to become delocalized, and to arrive eventually at a certain critical γ for which most states have $C_\alpha \neq 0$. In this way the classical continuous transition is, at the quantum level, discretized, the fraction of quantum states having $C_\alpha \neq 0$ increasing by small steps from zero to one.

IV. THE KICKED HARPER MODEL

Some of the points made in the previous section may now be illustrated by a time-dependent version of the Harper Hamiltonian,

$$H(p, q; t) = -V_2 \cos(2\pi p/P) - V_1 \cos(2\pi q/Q)K(t),$$

$$K(t) = \tau \sum_n \delta(t - n\tau),$$
(19)

describing a particle subjected to a periodic impulse whose amplitude depends on the particle position itself. In terms of the dimensionless coordinates $x = q/Q$ and $y = p/P$ and parameters $\gamma_i = 2\pi V_i \tau / PQ$, the classical map obtained by integrating the equations of motion between successive kicks is

$$x_{n+1} = x_n + \gamma_2 \sin(2\pi y_{n+1})$$

$$y_{n+1} = y_n - \gamma_1 \sin(2\pi x_n).$$
(20)

In the limit $\tau \rightarrow 0$ we recover the continuous time evolution for the integrable Harper Hamiltonian [i.e., $K(t) = 1$], while for finite τ the motion is chaotic. This transition is shown in Fig. 1 for differing values of γ (in the following we consider the special case $\gamma_1 = \gamma_2 = \gamma$). For $\gamma \gtrsim 0.63$ the dynamics is dominated by chaotic orbits. The map (20) has at least four simple periodic orbits of period one, existing $\forall \gamma$: $(x, y) = (0, 0), (\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2})$, and $(\frac{1}{2}, 0)$. The first two orbits are stable for $\gamma < 1/\pi \approx 0.318$ and unstable otherwise, while the other two orbits are always unstable.

The quantized system is described by the time evolution operator

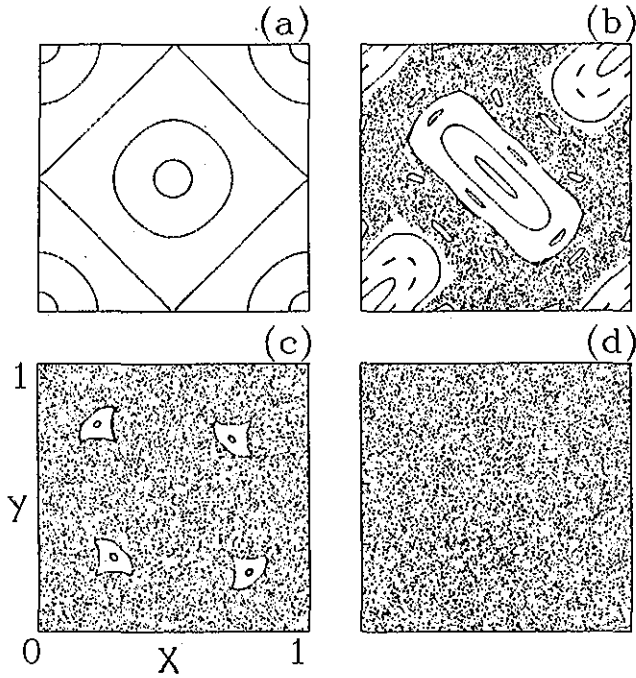


FIG. 1. The map of Eq. (20) for (a) $\gamma = 0.00063$, (b) $\gamma = 0.31$, (c) $\gamma = 0.565$, and (d) $\gamma = 0.63$.

$$U = \exp[iN\gamma_1 \cos(2\pi\hat{x})] \exp[iN\gamma_2 \cos(2\pi\hat{y})], \quad (21)$$

($\hat{x} = \hat{q}/Q$, $\hat{y} = \hat{p}/P$) which we diagonalize numerically to obtain the eigenstates $|\psi_\alpha(\theta)\rangle$ and eigenangles $\omega_\alpha(\theta)$ [cf. Eq. (8)]. This evolution operator has certain nongeneric symmetries. For N even, the spectrum is complex-conjugation invariant; for N odd, the spectrum is invariant under the combination of complex conjugation and $(\theta_1, \theta_2) \rightarrow (\theta_1 + \pi, \theta_2 + \pi)$. When $\gamma_1 = \gamma_2$, additional symmetries are present. These symmetries guarantee that $C_{N+1-\alpha} = C_\alpha$.

In Fig. 2 we display the evolution of the $\{C_\alpha\}_{\alpha=1,\dots,5}$ for $N=9$ as a function of γ . For vanishingly small γ , we recover the usual integrable Harper Hamiltonian. In that limit, the Chern indices were studied by Thouless *et al.*¹¹ in the context of the quantum Hall effect. Classically the phase space is filled by invariant tori (except for the separatrix) and eigenstates associated with them have zero Chern index. The only classical orbit that winds around the phase space torus is the separatrix. The symmetry of

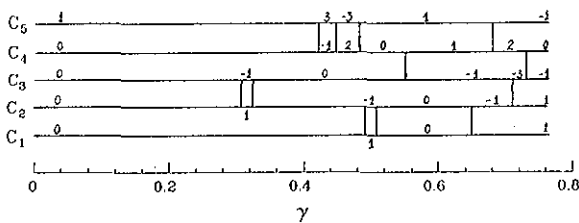


FIG. 2. Evolution of the Chern numbers for $N=9$ as a function of γ . Horizontal lines correspond to bands; vertical lines indicate the existence of a degeneracy in the θ plane for that value of γ . The Chern numbers for $\alpha > 5$ are symmetric with respect to $\alpha = 5$.

the model guarantees that for N odd the central band will be on the separatrix (cf. Fig. 2). The band related to that orbit is a conducting one, having $C_\alpha = 1$ because of (17).¹⁷

As γ increases and as the classical invariant tori disappear, we encounter degeneracies and the spectrum of Chern numbers changes. Because of the symmetry, the central state crosses with two adjacent levels and its Chern number changes in multiples of two. For $N=9$, additional nonzero Chern numbers appear in the interval $0.3 \lesssim \gamma \lesssim 0.6$. This transition is associated with the breakdown of the classical separatrix and emergence of stochastic domains. As expected from semiclassical arguments, the value of γ at which this first occurs decreases with increasing N . The fraction of states with nonzero C_α and the fluctuations in the C_α themselves increase as γ increases. [For example, for $N=3$ and $\gamma=20$ we find $C_1 = -16$ and $C_2 = 33$; for $N=9$ and $\gamma=2$ we find $C_1 = -4$, $C_2 = 2$, $C_3 = 3$, $C_4 = 4$, and $C_5 = -9$.] At $\gamma \approx 0.65$, all states have $C_\alpha \neq 0$.

States having $C_\alpha = 0$ for $\gamma \gtrsim 0.65$ are localized states lying in chaotic regions of phase space. The analysis of the quasiprobability distributions $W_\alpha(x, y, \theta)$ shows that their localization domain (i.e., the region of T_x^2 where the zeros never enter as the boundary conditions are changed) is located around unstable periodic orbits. This makes the connection between localized eigenstates in fully classically chaotic regions and what is often referred to as *scars*.

We stress that the large γ limit of our problem is quite different from the generic case of the quantum Hall effect in random potential, whose classical limit is not chaotic. In the Hall effect problem,¹³ subbands with energies in the tails of the Landau levels always have zero Chern number, for they correspond to semiclassically localized states living on the peaks or in the valleys of the random potential. The fraction of states which are extended is believed to be vanishingly small in the thermodynamic limit. By contrast, in our model, when γ becomes large, all the states are delocalized.

In order to make more clear the connection between the Chern numbers and the classical invariant sets, we show some phase space quasiprobability distributions $W_\alpha(x, y, \theta)$ for $\gamma = 0.565$, $N=9$, at the special point $\theta = (0, 0)$. States $\alpha = 1, 2, 8, 9$ all have zero Chern number, while states 3, 4, 5, 6, 7 give $C_\alpha = -1, 1, 1, 1, -1$, respectively (see Fig. 2). In Figs. 3(a) and 3(b) we plot $W_\alpha(x, y)$ for states 8 and 9, which are both localized. The plots use a linear density scale stressing in black the peaks of the distribution. While state 8 is located in a regular region of phase space (a “regular” state with support on a classical invariant torus), state 9 lies in a chaotic region of phase space, and localizes about the *unstable* period 1 orbit at $(x, y) = (\frac{1}{2}, \frac{1}{2})$. These distributions are relatively insensitive to changes in the boundary conditions and both states possess a localization domain as defined above. Similar results hold for states 1 and 2.

Some states with $C \neq 0$ were seen to be highly concentrated, for certain values of θ , about an unstable periodic orbit, but as θ was varied they spreaded out over the chaotic domain. This is in particular the case of the state $\alpha = 5$

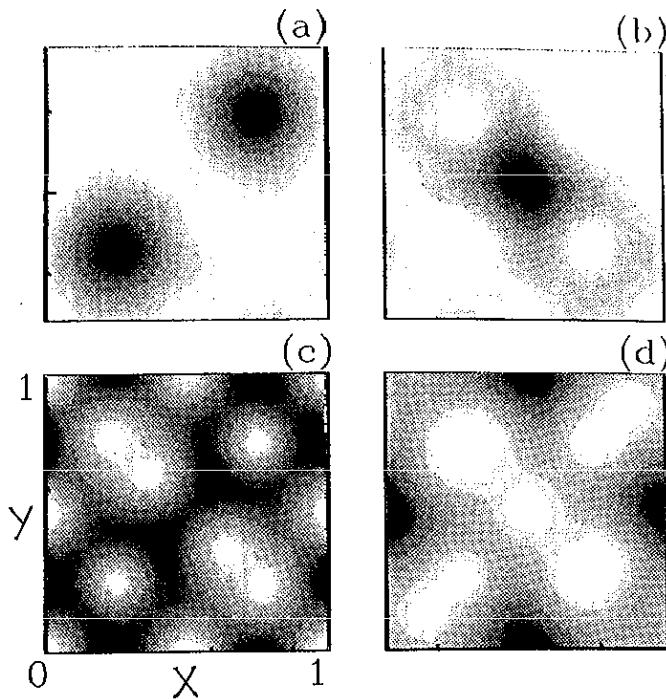


FIG. 3. Phase space distributions $W_\alpha(x, y, \theta=0)$ for $N=9$ and $\gamma=0.565$. (a) $\alpha=8$ (localized on a torus), (b) $\alpha=9$ [localized about an unstable periodic orbit, cf. Fig. 1(c)], (c) $\alpha=4$ (delocalized), (d) $\alpha=5$ (delocalized).

shown in Fig. 3(d). This emphasizes the fact that an examination of any state at a *particular* value of θ may not necessarily be illuminating; generally there is nothing which sanctifies a given set of boundary conditions, and it is the *average* over *all* boundary conditions, embodied by the formula of Eq. (12), which is relevant. In Figs. 3(c) and 3(d), we plot the Husimi distributions for $\alpha=4$ and $\alpha=5$, two delocalized states.

V. CONCLUSION

In conclusion, we have identified a criterion for phase space localization in two-dimensional quantized maps. To each state—more precisely to each *band* of states parametrized by two phase space boundary angles—we associate an integer topological invariant, in precise analogy with the TKN² analysis of the quantum Hall effect.¹¹ Unlike the

integrable case, quantum systems whose classical limit is dominated by chaos are distinguished by a large spectral fraction of nonzero Chern numbers. In analogy to the classical route to chaos through overlapping of resonances, the quantum path was shown to be intimately related to the degeneracies of the eigenspectrum. Although we have emphasized the connection with classical mechanics, the Chern index is a fully quantum object reflecting some structural properties of eigenstates. It will be useful to generalize this concept to arbitrary systems not initially formulated in a toroidal geometry.

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- ¹⁵Multiple-well type potentials where exponentially small spectral degeneracies occur, are examples of integrable systems where all the quantum states have a nonzero Chern number; this phenomenon is due to delicate resonances in which the particle tunnels from one well to the other as the boundary conditions are changed, and Eq. (18) is not valid in this case (see Ref. 11 for a more detailed discussion). An arbitrary perturbation, as for example the kicking studied in Sec. IV, will destroy the phenomenon.
- ¹⁶See A. J. Stone, *Proc. R. Soc. London Ser. A* **351**, 141 (1976) for a more detailed explanation of this construction.
- ¹⁷For even N there are two states associated with the classical separatrix which become degenerate at $\theta=(0,0)$; the Chern number of this pair of bands remains well defined and is $C=1$.