

Phase locking, period doubling, and chaotic phenomena in externally driven excitable systems

Mario Feingold

The James Franck Institute, The University of Chicago, Chicago, Illinois 60637

Diego L. Gonzalez

Departamento de Física, Universidad Nacional de La Plata, Casilla de Correo 67, 1900 La Plata, Buenos Aires, Republica Argentina

Oreste Piro*

The James Franck Institute, The University of Chicago, Chicago, Illinois 60637

Hector Viturro

Departamento de Física, Universidad Nacional de La Plata, Casilla de Correo 67, 1900 La Plata, Buenos Aires, Republica Argentina

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The behavior of excitable systems periodically stimulated by pulses is investigated. The non-self-oscillating regime is studied in the simplest realistic model. The global bifurcation structure is similar in several aspects to the one corresponding to driven oscillators. It includes phase locking, quasiperiodicity, period doubling, and chaos. The unexpected persistence of this structure is explained by analytical arguments and its scaling properties are numerically determined.

Excitable systems are characterized by their behavior under external stimulation. (i) There is an amplitude threshold for the stimuli which separates two types of responses. Subthreshold perturbations leave the system close to the quiescent state while upper-threshold amplitudes generate transitions to a state with very different properties and values of its characteristic variables. (ii) This *excited* state decays to the original quiescent state after a characteristic lifetime which is insensitive to the amplitude of the stimulus. Nature provides many examples where excitability plays a fundamental role. This is, for instance, the main subject of neural and cardiac electrophysiological research.¹⁻⁴ Excitable behavior is present in many effects in nonequilibrium physics, chemistry, and biology^{4,5} as well as in many electronic devices.⁶ Frequently, there is also an external periodic driving agent in the problem. Hence, an understanding of the behavior of those systems under periodic stimulation becomes of general interest.

Excitable systems become oscillators for some values of their parameters.⁶ The dynamics of externally driven oscillators has been extensively studied.⁷⁻¹³ This problem is equivalent to the study of iterated circle maps which provides much qualitative^{9,10,14} and quantitative¹⁵⁻²¹ universal information about typical phenomena like phase locking, quasiperiodic transition to chaos, etc. On the contrary, the non-self-excited regime has never been studied in detail, either theoretically or numerically. Our aim is to show that the global bifurcation structure of this regime has similar form and scaling properties as in oscillators and circle maps. The similarity includes both the phase locking, quasiperiodicity and chaotic phenomena. This result is surprising because while in nonlinear driven oscillators the phase-locking structure arises from the competition of two different frequencies, in non-self-

excited excitable systems, one of those frequencies is lacking. An extensive numerical study of the paradigmatic van der Pol-Fitzhugh-Nagumo model^{22,23} with the simplest form of external stimulation is presented. An analytical argument is also given to explain the origin of a second effective frequency responsible for the phase-locking structure.

Let us consider the following equation:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x + a = 0, \tag{1}$$

which can be written as

$$\dot{x} = \mu \left[y - \frac{x^3}{3} + x \right]; \quad \dot{y} = - \frac{(x+a)}{\mu}. \tag{2}$$

It is instructive to think of this equation as a qualitative description of the circuit in Fig. 1(a). There, the nonlinear device which shunts the capacitor *C* can be any one having a three-valued *i-v* characteristic curve as shown in Fig. 1(b) (e.g., a neon bulb). From the Kirckhoff laws we can see that this curve is mimicked in Eqs. (2) by the cubic function $y = f(x) = x^3/3 - x$. Also, the line $y = a$ acts

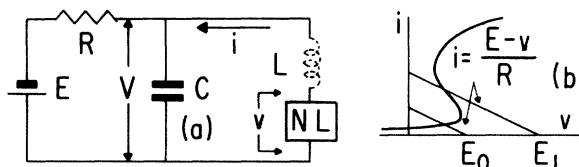


FIG. 1. (a) Electronic circuit modeling an excitable system. (b) The *i-v* curve of the nonlinear device (NL). For $E = E_1$ the capacitor *C* is periodically discharged through NL while for $E = E_0$ the charge and the voltage on *C* reach static equilibrium values.

as the load line $i = (E - v)/R$ in the circuit equations, μ measures the ratio RC/L , and the time scale is RC . In the limit $\mu \rightarrow \infty$ corresponding to negligible inductances L , the first of the Eqs. (2) becomes algebraic instead of differential. It just represents a constraint on the evolution of y (or the voltage v) and x (or i). In fact, the existence of a small parameter $1/\mu$ (or L) allows one to consider independently *slow* ($\dot{x} \approx 1/\mu$) motions taking place on the curve $y = f(x)$ and *fast* ($\dot{x} \approx \mu$) motions which run essentially perpendicularly to the y axis.

The intersection of the "nullclines"²⁴ $x = -a$ and $y = x^3/3 - x$ is a fixed point. The fixed point is stable if $|a| > 1$ and unstable for $|a| < 1$. In the latter case, we also have an attracting limit cycle which can be visualized by considering the motion in the limit $\mu \rightarrow \infty$. In this limit x can be expressed as a multivalued function $f^{-1}(y)$. From (2) we see that if $x < (>) -a$ then $\dot{y} > (<) 0$. This is stressed in Fig. 2 by the arrows indicating the flow direction on the slow manifold. Now, when the system reaches the critical points $y = \pm \frac{2}{3}$ it is pushed by the flow out of the slow manifold. \dot{x} becomes then extremely large and the system jumps almost instantaneously to the other branch of $f^{-1}(y)$. It is clear from Fig. 2(a) that for $\mu \rightarrow \infty$ and $|a| < 1$ the system evolves (after a transitory motion) on a limit cycle with a well-defined natural frequency. These arguments can be extended for finite μ .²⁵

On the contrary, for $|a| > 1$ [Fig. 2(b)] there is just one stable fixed point with no finite natural frequency. The electronic example illustrates the two cases: For some ranges of the circuit parameters (say E) the neon bulb spontaneously switches periodically on and off while for other ranges the asymptotic state is just off. Here we are interested in the latter case which we call *excitable*.

These systems can be excited (i.e., forced to jump temporarily to the branch which does not contain the fixed point) by modifying a during a short time interval. This is like adding a short voltage pulse to E which favors the neon-bulb ionization. During the pulse, the fixed point is shifted to the unstable branch and the system can eventually reach the point where jumps take place. There is a threshold value for the pulse height h below which excitations cannot occur for any stimulus duration D . Moreover, for each amplitude above this threshold there is still a minimum value of D which produces excitation. This is the time it takes the system to go from the fixed point to the switching point $y = +(-)\frac{2}{3}$. Therefore, there is an amplitude-duration threshold curve for excitatory pulses. However, in the limit $D \rightarrow 0$, $h \rightarrow \infty$ and $hD = V_E$ there is only a threshold for the pulse area V_E . If μ is finite, this

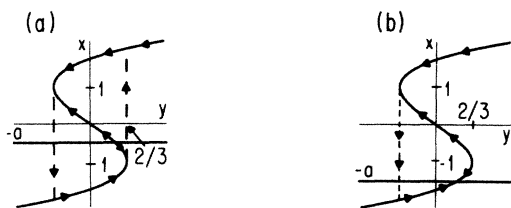


FIG. 2. The flow on the slow manifold described by Eqs. (1) and (2). (a) $|a| < 1$ (self-excited); (b) $|a| > 1$.

limiting case is well described by a δ function on the right-hand side of Eq. (1). Each δ produce an instantaneous displacement of magnitude V_E in \dot{x} and y . The V_E threshold is the minimal distance that the system should be pushed away from the fixed point $(-a, a - a^3/3)$ in order to make it jump to the opposite branch. If μ is large this value is close to $\frac{2}{3} - |a - a^3/3|$. Figure 3 shows examples of both subthreshold and upper-threshold responses. Notice that the excited system relaxes to the initial state after traveling on the upper branch and jumping down at $y = -\frac{2}{3}$. Hence, a second upper-threshold pulse arriving when the system is still far from the fixed point might be unable to excite the system again. This *refractoriness* is commonly observed in real systems.

For studying the behavior under periodic stimulation we introduce the forcing term

$$F(t) = V_E \sum_{n=-\infty}^{\infty} \delta(t - nT_E) \tag{3}$$

in the right-hand side of Eq. (1). Here T_E is the period of the stimulation. An extensive simulation for the forced model is performed by numerically integrating Eq. (2) and adjusting the y variable after each pulse. For given V_E and T_E we compute the stable periodic solutions by starting from an arbitrary initial condition and letting the transient motions decay. Such solutions repeat themselves after an integer number of pulses. During the cycle some pulses excite the system and some do not.

In Fig. 4 we show the regions of the parameter plane where orbits of a given period are stable. The period, measured in units of T_E , is indicated by the number b of the symbol $a:b$. Remarkably, this picture is similar to the well-known phase-locking structure observed in the dynamics of driven self-oscillators and circle maps.^{9-11,19,20} In those cases locking is the result of the competition between two independent finite frequencies: The oscillator frequency and the external one. However, since in our problem there is only one finite frequency, the persistence of this structure is surprising. There is no natural angular variable in the problem which could be considered a locked phase. We cannot even define the rotation number in the standard way. However, if the number of excited responses indicated by a in the symbol $a:b$ is considered, the Farey organization of the usual phase locking is observed here as well. As we will show later, the scaling of the locked regions at large enough T_E (and also at small

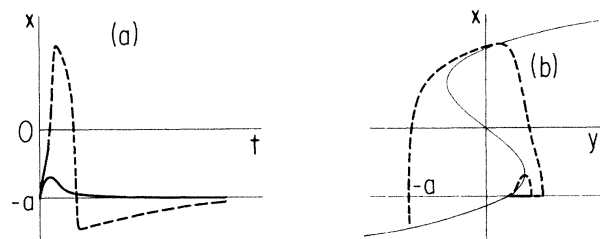


FIG. 3. Response to an upper-threshold pulse (dashed line) and to a subthreshold one (solid line). (a) Time dependence of x . (b) Phase trajectory. Notice that only the first trajectory crosses the upper branch of f^{-1} .

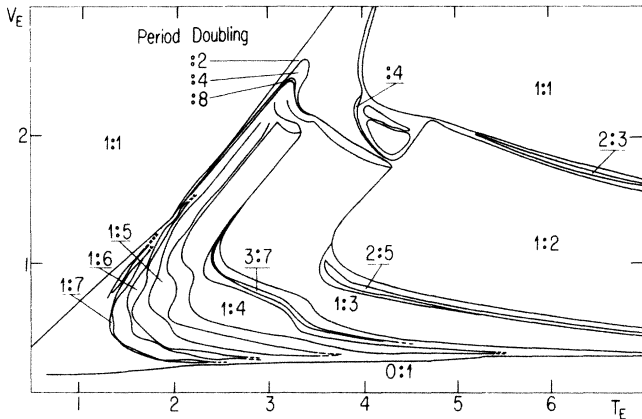


FIG. 4. Regions of the (V_E, T_E) plane where the periodic solutions of Eq. (1) with the forcing term (3) are stable. The corresponding period nT_E is indicated by n in the symbol $m:n$. m indicates the number of excitations occurring in a given cycle of the subcritical regime. Since there are supercritical bifurcations (not discussed here) which change this number, the labeling of the period doubling sector is incomplete. Here $\mu=3.5$ and $a=1.35$. V_E and T_E are given in arbitrary units.

enough V_E) is consistent with the subcritical behavior of circle maps. This implies the existence of a finite measure of parameter space for which the motion is quasiperiodic. Furthermore, the similitude between the two diagrams includes also the *upper-critical* behavior. There are places where the main stability regions bifurcate in period doubling sequences which end in a zone where the motion is chaotic.^{11,19,26} Period doubling and chaos appear coincidentally with overlapping of stability regions in the same way it happens for driven oscillators and circle maps.

There are, of course, some differences between our stability diagram and the standard mode locking. In the latter case for example, the tip of the *Arnold tongues* touches the T_E axis at rational T_E/T_0 ratios. The equivalent objects in our diagram are bent such that they never reach the horizontal axis except possibly at infinity. This is a consequence of the infinite *natural frequency*. The following arguments give some intuition on the origin and structure of our stability diagram. We first rewrite $F(t) = V_E/T_E + G(t)$ where $G(t)$ is a zero-mean oscillating term. Then the average force V_E/T_E renormalizes the parameter a . The dressed parameter, $a' = a - V_E/T_E$, can

assume values within the range corresponding to self-oscillations supplying in this way the missing frequency. We can derive the period of this effective oscillator in the limit $\mu \rightarrow \infty$. In fact, if the motion takes place on the slow manifold we have from Eqs. (2)

$$dt = -\mu \frac{dy}{x+a'} = -\mu \frac{f'(x)}{x+a'} dx \quad (4)$$

Neglecting the $O(\mu^{-1})$ transit time from one branch to the other, we obtain the effective period

$$T_0^{\text{eff}} = -\mu \int_{-1}^{-2} \frac{x^2-1}{x+a'} dx - \mu \int_2^1 \frac{x^2-1}{x+a'} dx \\ = \mu \left[3 + (a'^2-1) \ln \frac{4-a'^2}{1-a'^2} \right] \quad (5)$$

Notice that T_0^{eff} is bounded between $(3-\ln 4)\mu$ for $a'=0$ and 3μ for $a'=1$. By using Eq. (5) we find the curves in the (V_E, T_E) space where the ratio $T_E/T_0^{\text{eff}} = \rho$ is constant. As is shown in Fig. 5 these curves give a good description of the geometry of the phase diagram of the Fig. 4.

We have studied the scaling properties of this diagram in the subcritical regime. Since the results are the same for all regions in this regime, we illustratively show the scaling of the locked intervals on the $T_E=7$ line in the parameter space. The fine structure between 1:1 and 1:2 regions is displayed in Fig. 6(a). Since we plot the period rather than the rotation number, this picture is an unfolding of the *devil's staircase* usually considered in the study of the mode locking.²⁰ We can identify sequences of intervals of period $rn+s$ (with $n=1,2,3, \dots$ and r,s constant integers) often called *period adding* sequences.²⁷ These sequences are composed of the intervals with symbols $(p+q):(rn+s)$ where $|ps-qr|=1$. As $n \rightarrow \infty$ those intervals accumulate at the border of the one labeled $(p:r)$. For circle maps the convergence is of the form n^{-2} while the width of the intervals vanishes as n^{-3} . This is a consequence of the fact that the $p:q$ orbit appears after a tangent bifurcation.²⁶ In our system we find the same scaling behavior which indicates the presence of those bifurcations at the border of the stability regions. Figure 6(b), for instance, illustrates the power law n^{-2} showing the intervals on a logarithmic scale with the singularity at the end of the 1:1 region. Similarly, the interval widths scale like n^{-3} . Our diagram is also consistent with the

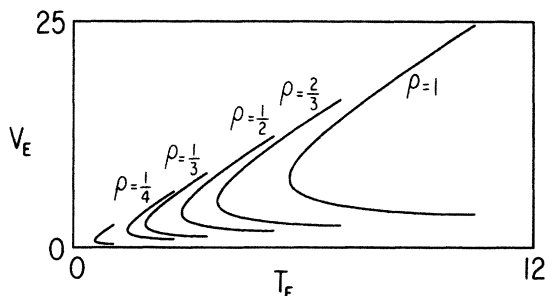


FIG. 5. Curves of constant $\rho = T_0^{\text{eff}}/T_E$ [see Eq. (1)]. The values of μ and a are the same as in Fig. 4. V_E and T_E are given in arbitrary units.

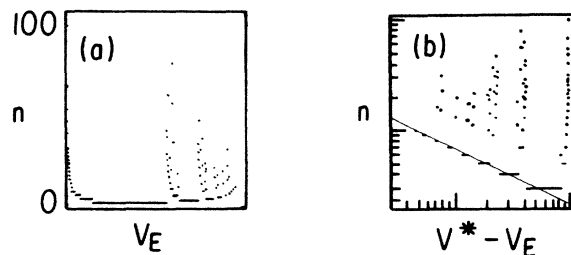


FIG. 6. (a) Period of the stable solutions as a function of V_E on the line $T_E=7$ (see Fig. 4). The range of V_E is (1.53, 1.64). (b) As in (a) but with $V_E \rightarrow V^* - V_E$ and on logarithmic scale. V^* is the border of stability of the 1:1 region. The straight line indicates the n^{-2} law.

geometrical convergence of the intervals around the golden mean found for circle maps.^{15,17}

In summary, the global bifurcation structure of driven excitable systems is qualitatively similar and has the same scaling properties as the usual mode locking. Both the analytical and numerical results suggest that those systems can be described by circle maps although the connection between them is not so transparent as in the oscillatory case. The presence of chaotic behavior should be also remarked as a possible origin (often not considered) of disorder in excitable media. We finally stress that all the

phenomena described in this paper have been observed in experiments both in electronics and electrophysiology.^{28,29}

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*Also at Departamento de Física, Universidad Nacional de La Plata, Casilla de Correo 67, 1900 La Plata, Buenos Aires, República Argentina.

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