The Non Equilibrium Steady State of Sparse Systems with Non Trivial Topology

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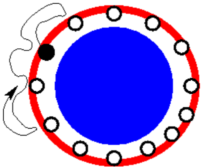
$$\mathcal{H}_{\text{total}} = \text{diag}\{E_n\} - f(t)\{V_{nm}\} + F\{W_{nm}\} + \mathcal{H}_{Bath}$$

- Doron Cohen (BGU) [1,2]
- Saar Rahav (Technion)[2]

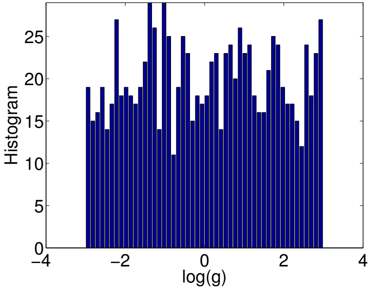


1. D. Hurowitz and D. Cohen, Europhysics Letters 93, 60002 (2011)
2. D. Hurowitz, S. Rahav and D. Cohen, Europhysics Letters 98, 20002 (2012).

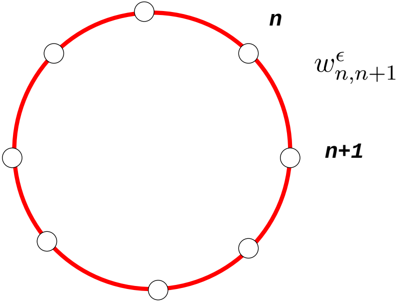
"Sparsity"

$$\mathcal{H}_{\text{total}} \;=\; \text{diag}\{E_n\} \;-\; f(t)\{V_{nm}\} \;+\; F(t)\{W_{nm}\}$$

Histogram of couplings



$\longleftarrow$ few decades $\longrightarrow$



$$w^{\epsilon}_{n+1,n} = g_n \epsilon^2$$

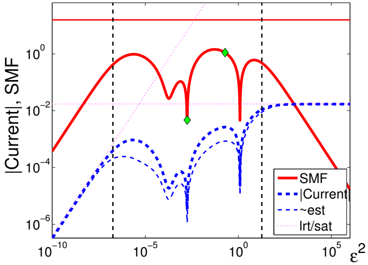
$$g_n \;=\; |V_{n,n+1}|^2 = \text{couplings}$$

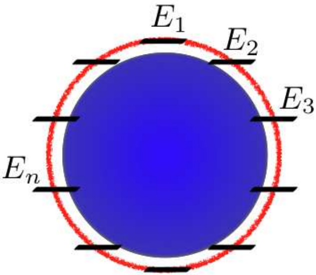
"sparsity" $=$ log wide distribution of couplings

Current vs. driving

Driving $\rightsquigarrow$ Stochastic Motive Force $\rightsquigarrow$ Current

Regimes: LRT regime, Sinai regime, Saturation regime





$$I \;\sim\; \frac{1}{N}\,\overline{w}\,\exp\left[-\frac{\mathcal{E}_{\cap}}{2}\right]\; 2\sinh\left(\frac{\mathcal{E}_{\circlearrowleft}}{2}\right)$$

Extent of the "Sinai regime" is determined by width of distribution of rates

Master equation description of dynamics

$$\mathcal{H}_{\text{total}} = \text{diag}\{E_n\} - f(t)\{V_{nm}\} + F(t)\{W_{nm}\} + \mathcal{H}_{Bath}$$

Quantum master equation for the reduced probability matrix:

$$\frac{d\rho}{dt} = -i[\mathcal{H}_0, \rho] - \frac{\epsilon^2}{2}[V,[V,\rho]] + \mathcal{W}^{\beta}\rho \equiv \mathcal{W}\rho$$

Stochastic rate equation:

$$\frac{dp_n}{dt} = \sum_m w_{nm}p_m - w_{mn}p_n$$

Steady state equation:

$$\dot{\rho} = \mathcal{W}\rho = 0$$

The transition rates:

$$w_{nm} = w^{\epsilon}_{nm} + w^{\beta}_{nm}$$

$$w^{\epsilon}_{nm} = w^{\epsilon}_{mn} = g_{nm}\epsilon^2$$

$$\frac{w^{\beta}_{nm}}{w^{\beta}_{mn}} = \exp\left[-\frac{E_n - E_m}{T_B}\right]$$

The Stochastic Motive Force (SMF)

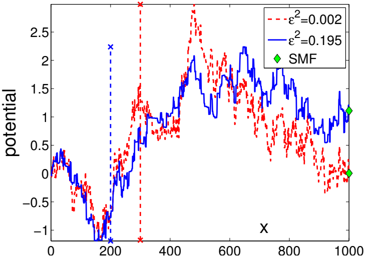
If we had only a bath

$$\frac{w_{nm}}{w_{mn}} = \exp\left[-\frac{E_n - E_m}{T_B}\right]$$

We define a "field"

$$\mathcal{E}(x) \equiv \ln\left[\frac{w_{nm}}{w_{mn}}\right]$$

and "potentials"



$$\mathcal{E}(x_1 \rightsquigarrow x_2) = \int_{x_1}^{x_2} \mathcal{E}(x)dx$$ [potential variation]

$$\mathcal{E}_{\cap} \equiv \text{ maximum}\left\{\left|\mathcal{E}(x_1 \rightsquigarrow x_2)\right|\right\}$$ [activation barrier]

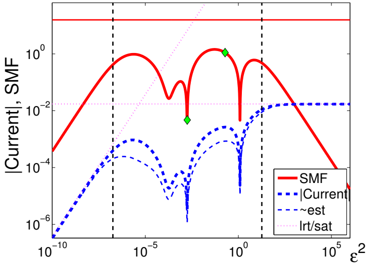
$$\mathcal{E}_{\circlearrowleft} \equiv \oint \mathcal{E}(x)dx$$ if no driving $= 0$ [SMF]

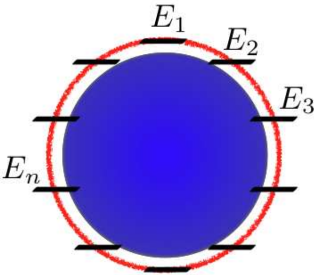
With driving, $\mathcal{E}_{\circlearrowleft} \neq 0$. This means $\prod_n w_{n,n+1} \neq \prod_n w_{n+1,n}$.

Current vs. driving

Driving $\rightsquigarrow$ Stochastic Motive Force $\rightsquigarrow$ Current

Regimes: LRT regime, Sinai regime, Saturation regime





$$I \sim \frac{1}{N}\, \overline{w}\, \exp\left[-\frac{\mathcal{E}_{\cap}}{2}\right] \; 2\sinh\left(\frac{\mathcal{E}_{\circlearrowleft}}{2}\right)$$

Extent of the "Sinai regime" is determined by width of distribution of rates

Emergence of the "Sinai regime"

Sinai [1982]: Transport in a chain with random transition rates.

Assume transition rates are uncorrelated.

$\rightsquigarrow$ build up of a potential barrier $\mathcal{E}_{\cap} \propto \sqrt{N}$.

$\rightsquigarrow$ exponentially small current.

But... we have telescopic correlations: $\mathcal{E}_{n,n+1} \sim \Delta_n \equiv (E_n - E_{n+1})$

Yet... we have sparsely distributed couplings: $w^{\epsilon}_{n,n+1} = g_n \epsilon^2$

$$\mathcal{E}_{\circlearrowleft} \approx -\sum_n \left[\frac{1}{1+g_n\epsilon^2}\right]\frac{\Delta_n}{T_B} \sim \frac{1}{T_B}\begin{cases} \epsilon^2, & \epsilon^2 < 1/g_{\max} \\ 1/\epsilon^2, & \epsilon^2 > 1/g_{\min} \\ [\pm]\sqrt{N}\Delta, & \text{otherwise} \end{cases}$$

Build up may occur if g_n are from a **log-wide** distribution.

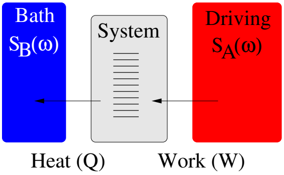
$$I \sim \frac{1}{N}\,\overline{w}\,\exp\left[-\frac{\mathcal{E}_{\cap}}{2}\right]\,2\sinh\left(\frac{\mathcal{E}_{\circlearrowleft}}{2}\right)$$

Generalized Fluctuation-Dissipation phenomenology

$$\mathcal{H}_{\text{total}} = E_n\delta_{nm} - f(t)V_{nm} + F(t)W_{nm}$$

$$\dot{\mathsf{W}} = \text{rate of heating} = \frac{D(\epsilon)}{T_{\text{system}}}$$

$$\dot{\mathsf{Q}} = \text{rate of cooling} = \frac{D_B}{T_B} - \frac{D_B}{T_{\text{system}}}$$



Hence at the NESS:

$$T_{\text{system}} = \left(1 + \frac{D(\epsilon)}{D_B}\right) T_B$$

$$\dot{Q} = \dot{W} = \frac{1/T_B}{D_B^{-1} + D(\epsilon)^{-1}}$$

Experimental way to extract response:

$$\dot{Q}(\infty) = \frac{D_B}{T_B}$$

$$D(\epsilon) = \frac{\dot{Q}(\epsilon)}{\dot{Q}(\infty) - \dot{Q}(\epsilon)} D_B$$

$D(\epsilon)$ exhibits LRT to SLRT crossover
SLRT requires resistor network calculation

$$D(\epsilon) = \left[\overline{\left(\frac{w_n}{w_\beta + w_n}\right)}\right] \left[\overline{\left(\frac{1}{w_\beta + w_n}\right)}\right]^{-1}$$

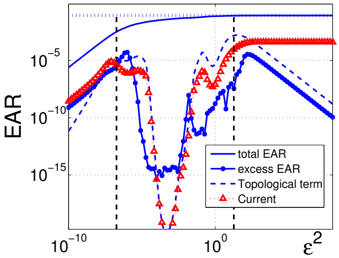
$$w_n = g_n \epsilon^2$$

$$D_{[\text{LRT}]} = \overline{g_n}\epsilon^2 \qquad \text{[weak driving]}$$

$$D_{[\text{SLRT}]} = [\overline{1/g_n}]^{-1}\epsilon^2 \qquad \text{[strong driving]}$$

Topological term in EAR formula

$$
\begin{aligned}
\dot{\mathsf{Q}} &= \sum_n \left[w^{\beta}_{\overleftarrow{n}} p_n - w^{\beta}_{\overrightarrow{n}} p_{n-1} \right] \Delta_n \\
&\approx \left[\frac{D_B}{T_B} - \frac{D_B}{T^{(0)}} \right] + T_B \mathcal{E}_{\circlearrowleft} \, I \\
&\approx \frac{D_B}{T_B} \left[\overline{(g_n \epsilon^2) - (g_n \epsilon^2)^2} + \mathsf{Var}(g) \epsilon^4 \right]
\end{aligned}
$$



The EAR is correlated with the current.

Digression - derivation of the cooling rate formula

$$\dot{\mathsf{Q}} \;=\; \text{cooling rate} \;=\; -\sum_{n,m}(E_n - E_m)\; w^{\beta}_{nm}\; p_m$$

$$p_n - p_m \;=\; \text{occupation imbalance} \;=\; \left[2\tanh\left(-\frac{E_n-E_m}{2T_{nm}}\right)\right]\bar{p}_{nm}$$

$$w^{\beta}_{nm} - w^{\beta}_{mn} \;=\; \text{up/down transitions imbalance} \;=\; \left[2\tanh\left(-\frac{E_n - E_m}{2T_B}\right)\right]\bar{w}^{\beta}_{nm}$$

$$\dot{\mathsf{Q}} \;=\; \frac{1}{2}\sum_{n,m}(E_n-E_m)^2\,\frac{\bar{w}^{\beta}_{nm}}{T_B}\bar{p}_{nm} - \frac{1}{2}\sum_{n,m}(E_n-E_m)^2\,\frac{\bar{w}^{\beta}_{nm}}{T_{nm}}\bar{p}_{nm} \;=\; \frac{D_B}{T_B} - \frac{D_B}{T_{\text{system}}}$$

definition of the diffusion coefficient: $\quad D_B \;\equiv\; \overline{\left[\frac{1}{2}\sum_n (E_n-E_m)^2\; w^{\beta}_{nm}\right]}$

definition of effective system temperature: $\quad \dfrac{1}{T_{\text{system}}} \;\equiv\; \overline{\left[\dfrac{1}{T_{nm}}\right]}$

The quantum mechanical steady state

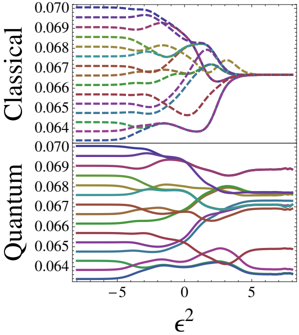
Stochastic

$$\frac{dp_n}{dt} = \sum_m w_{nm}p_m - w_{mn}p_n$$

$$I_{n\to m} = w_{mn}p_n - w_{nm}p_m \equiv tr(\hat{\mathcal{I}}_{n\to m}\rho)$$

$$\hat{\mathcal{I}}^{\epsilon}_{n\to m} = |n\rangle \color{red}{w^{\epsilon}_{mn}} \langle n| - |m\rangle \color{red}{w^{\epsilon}_{nm}} \langle m|$$

$$\hat{\mathcal{I}}^{\beta}_{n\to m} = |n\rangle \color{blue}{w^{\beta}_{mn}} \langle n| - |m\rangle \color{blue}{w^{\beta}_{mn}} \langle m|$$

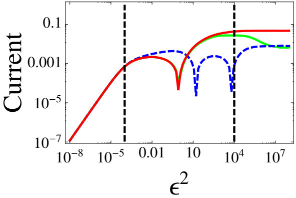


Quantum

$$\frac{d\rho}{dt} = -i[\mathcal{H}_0,\rho] - \frac{\color{red}{\epsilon^2}}{2}[V,[V,\rho]] + \color{blue}{\mathcal{W}^{\beta}}\rho$$

$$\hat{\mathcal{I}}^{\epsilon}_{n\to m} = i\color{red}{\epsilon^2}\left[\hat{\mathcal{J}}^{nm},\hat{V}\right]$$

$$\hat{\mathcal{J}}^{nm} = i\Big(|m\rangle V_{mn}\langle n| - |n\rangle V_{nm}\langle m|\Big)$$



Summary of main results

1. Due to the sparsity of the perturbation matrix, the NESS is of glassy nature [1].
2. An extension of the Fluctuation-Dissipation phenomenology has been proposed [1].
3. A log-wide distribution of couplings is required in order to have a Sinai regime.
4. The topological term in the EAR is correlated with the current but sub-linear in driving intensity.
5. Novel saturation effect in the quantum model.
6. The quantum current operator in the reduced description includes off diagonal elements of the probability matrix.

[1] D. Hurowitz and D. Cohen, Europhysics Letters 93, 60002 (2011).

References and Acknowledgements

1. D. Hurowitz and D. Cohen, Europhysics Letters 93, 60002 (2011).
2. D. Hurowitz, S. Rahav and D. Cohen, Europhysics Letters 98, 20002 (2012).

- Sparsity: Austin, Wilkinson, Prosen, Robnik, Alhassid, Levine, Fyodorov, Chubykalo, Izrailev, Casati
- Energy absorption by sparse systems: Cohen, Kottos, Schanz, Wilkinson, Mehlig
- Network theory: Schnakenberg, Zia, Hill
- Sinai physics: Sinai, Derrida, Pomeau, Burlatsky, Oshanin, Mogutov, Moreau, Bouchard

Acknowledgement: Bernard Derrida