Percolation, sliding, localization and relaxation in topologically closed circuits

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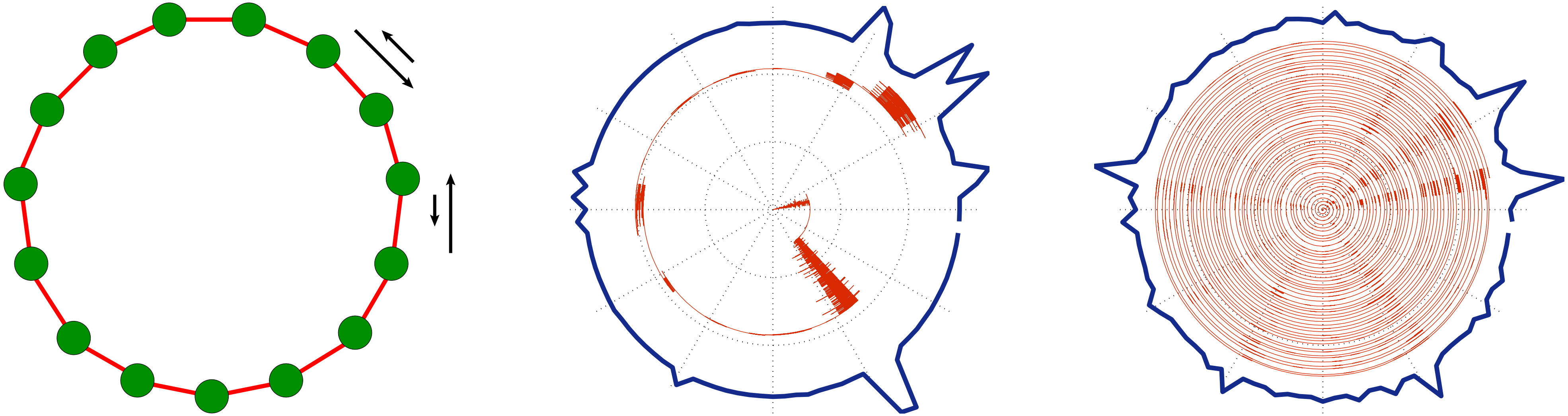


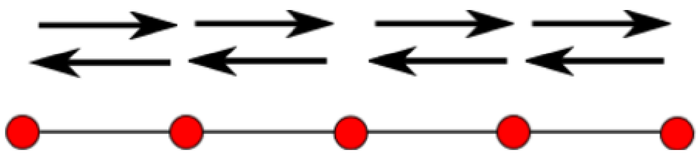
Disordered ring — Overdamped — Underdamped

[1] D. Hurowitz and D. Cohen, arXiv:1512.00258 (2015)

[2] D. Hurowitz and D. Cohen, Phys. Rev. E 90, 032129 (2014).

Types of random walk

Simple random walk, aka Brownian motion [Einstein]
Strictly periodic lattice ($a = 1$). All rates are equal (w)



$$D \quad = \quad w \qquad \text{(near-neighbor hopping)}$$

Random walk on a disordered lattice [1]
Random lattice. Symmetric transition rates w_n

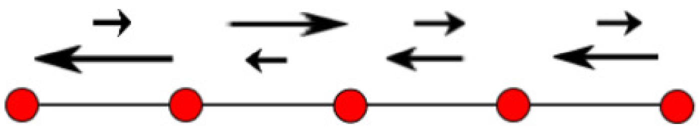
$$P(w) \quad \propto \quad w^{\alpha-1} \quad \text{(for small } w\text{)}$$

$\alpha \sim$ sparsity parameter
(resistor network calculation)

$$D \quad = \quad \left\langle \frac{1}{w} \right\rangle^{-1}$$

Non-percolating for $\alpha < 1$
Percolation-like transition

Random walk in random environment [2]
Rates allowed to be asymmetric: $\overleftarrow{w}_n \neq \overrightarrow{w}_n$
Sub-diffusion for low bias [Sinai, Derrida,...]
Sliding transition



[1] Alexander, Bernasconi, Schneider, Orbach, Rev. Mod. Phys. (1981).
[2] Bouchaud, Comtet, Georges, Le Doussal, Annals Phys. (1990).

Definition of the model

Conservative rate equation

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{W}\boldsymbol{p}$$

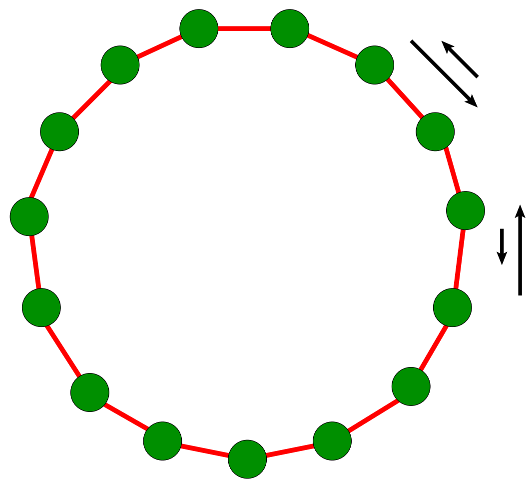
Rates allowed to be asymmetric $\overrightarrow{w}_n / \overleftarrow{w}_n = e^{\mathcal{E}_n}$

Affinity: $S_{\circlearrowleft} = \sum \mathcal{E}_n = Ns$

Stochastic field: $\mathcal{E}_n = s + \varsigma_n$ where $\varsigma_n \in [-\sigma, \sigma]$

Transition rates across n^{th} bond are $w_n \mathrm{e}^{\pm \mathcal{E}_n/2}$

Resistor network disorder: $P(w) \propto w^{\alpha-1}$



$$\boldsymbol{W} = \begin{bmatrix} -\gamma_1 & w_{1,2} & 0 & \dots \\ w_{2,1} & -\gamma_2 & w_{2,3} & \dots \\ 0 & w_{3,2} & -\gamma_3 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

Sum of elements in each column is zero

How do spectral properties of $\boldsymbol{W}$ depend on (α, σ, s)?

$\alpha \sim$ sparsity, $\sigma \sim$ field disorder, $s \sim$ affinity

Related models

Vortex depinning in type II superconductors (s = applied transverse magnetic field)

- Hatano, Nelson, PRL (1996), PRB (1997).
- Shnerb, Nelson, PRL (1998).
- Follow ups: Brouwer, Silvestrov, Beenakker, PRB (1997). Goldsheid, Khoruzhenko, PRL (1998). Feinberg, Zee, PRE (1999). Molinari, Linear Algebra and its Applications (2008).

Pulling pinned polymers, DNA denaturation (s = pulling force)

- Lubensky, Nelson, PRL (2000), PRE (2002).

Population biology (s = convective flow of bacteria relative to the nutrients)

- Nelson, Shnerb, PRE (1998).
- Dahmen, Nelson, Shnerb, Springer (1999).

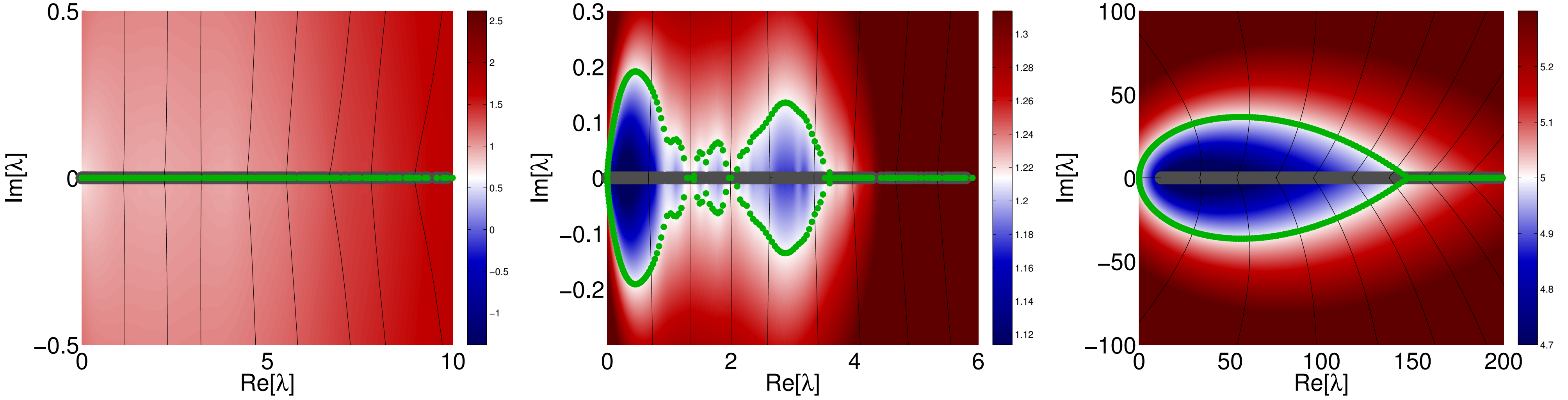
Molecular motors (s = affinity of chemical cycle)

- Fisher, Kolomeisky, PNAS (1999).
- Rief et al, PNAS (2000).
- Kafri, Lubensky, Nelson, Biophysical Journal (2004), PRE (2005).

None of the above concern relaxation modes of a **conservative** system!

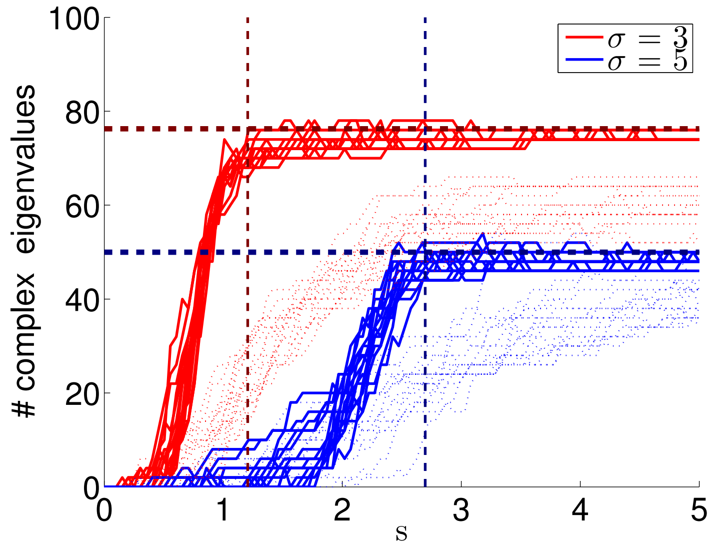
Implications of the percolation and sliding transitions on relaxation modes of the ring?

The spectrum of $\boldsymbol{W}$



$s < s_c$ $s_c < s < s_\infty$ $s > s_\infty$

- Due to conservativity $\lambda_0 = 0$
- The other eigenvalues are $\{-\lambda_k\}$
- Complex low-laying bubble for $s > s_c$
- Complexity saturation for $s \gg s_\infty$
- Implication of the percolation transition?
- Implication of the sliding transition?



The spectral equation

We are looking for the eigenvalues $\{-\lambda_k\}$ of the matrix $\boldsymbol{W}$.
The characteristic equation is:

$$\det(z+\boldsymbol{W}) \;=\; \det(z+\tilde{\boldsymbol{W}}) \;=\; \det(z+\boldsymbol{H}) \;-\; 2\left[\cosh\left(\frac{S_{\circlearrowleft}}{2}\right)-1\right](-\overline{w})^N \;=\; 0$$

$$\boldsymbol{W} \;=\; \text{diagonal}\Big\{-\gamma_n(s)\Big\}+\text{offdiagonal}\Big\{w_n \mathrm{e}^{\pm\frac{\mathcal{E}_n}{2}}\Big\} \qquad \mathcal{E}_n \;=\; s+\varsigma_n$$

$$\tilde{\boldsymbol{W}} \;=\; \text{diagonal}\Big\{-\gamma_n(s)\Big\}+\text{offdiagonal}\Big\{w_n \mathrm{e}^{\pm\frac{S_{\circlearrowleft}}{2N}}\Big\} \qquad S_{\circlearrowleft} \;=\; N\,s$$

$$\boldsymbol{H} \;=\; \text{diagonal}\Big\{-\gamma_n(s)\Big\}+\text{offdiagonal}\Big\{w_n\Big\} \quad\rightsquigarrow\quad \text{diagonal}\Big\{-\epsilon_k(s)\Big\}$$

Diagonal always depends on s

The characteristic equation:

$$\prod_k\left(\frac{z-\epsilon_k(s)}{(-\overline{w})}\right) \;=\; 2\left[\cosh\left(\frac{N s}{2}\right)-1\right]$$

The electrostatic version (RHS is $\Psi(0)$ because λ_0=0 is in the spectrum)

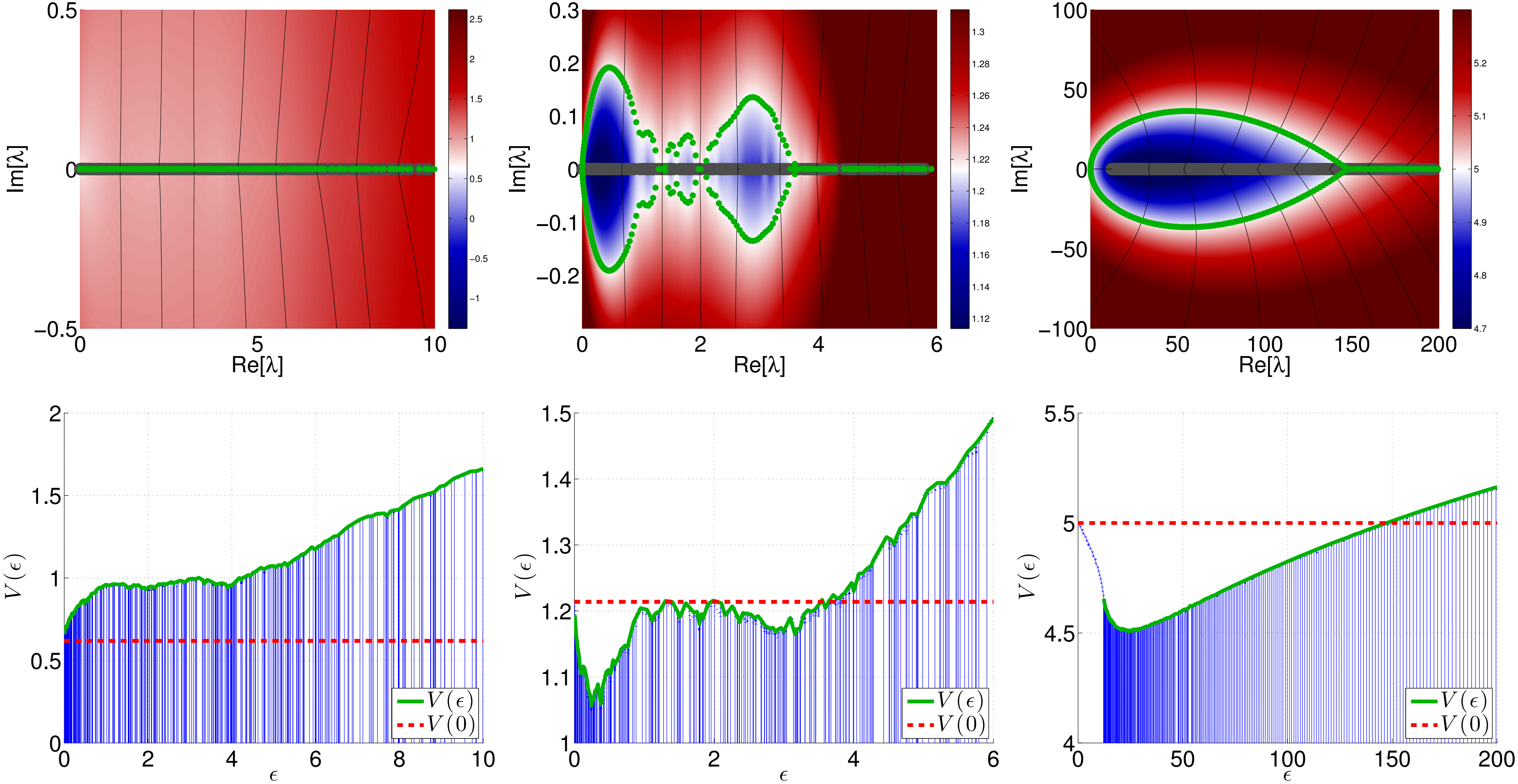
$$\Psi(z)=\Psi(0) \qquad\qquad \Psi(z)\equiv\sum_k \ln\left(\frac{z-\epsilon_k(s)}{(-\overline{w})}\right)$$

Below we work with units such that $\bar{w}=1$.

The electrostatic picture

The complex potential: $$\Psi(z) = \sum_k \ln(z-\epsilon_k) + \text{const} = V(x,y) + iA(x,y)$$

Characteristic equation: $$V(x,y) = V(0) \qquad A(x,y) = 2\pi * \text{integer}$$



$s < s_c$ $\qquad$ $s_c < s < s_\infty$ $\qquad$ $s > s_\infty$

The formation of a complex bubble

The λ spectrum is real if $V(\epsilon) > V(0)$.
The characteristic equation is $V(\epsilon) = V(0)$
leading to $\lambda_k \sim \epsilon_k$

In the continuum approximation

$$\rho(\epsilon) \quad = \quad \epsilon^{\mu-1} \quad \text{(for small } \epsilon) \qquad [\mu \text{ depends on } s]$$

$$V(\epsilon) \quad = \quad \int \ln\left(\left|\epsilon - \epsilon'\right|\right)\rho(\epsilon')d\epsilon' \quad \sim \quad \text{inverse localization length}$$

The threshold s_c is determined from the condition $V'(0) < 0$

$$V'(\epsilon \to 0) \quad \approx \quad \frac{\epsilon^{\mu-1}}{\epsilon_c^{\mu}} \pi\mu \cot(\pi\mu)$$

The derivative changes sign from positive to negative at $\mu = 1/2$.
We define $s_{1/2}$ as the value of s for which $\mu = 1/2$
For full disorder we make the identification $s_c = s_{1/2}$.

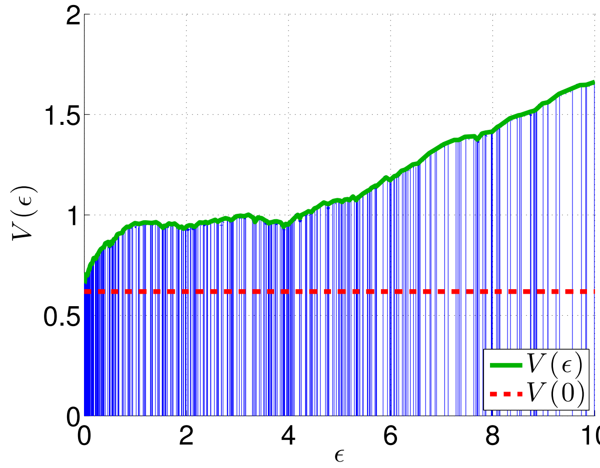
$D = 0$ for $s < s_{1/2}$, and $v = 0$ for $s < s_1$

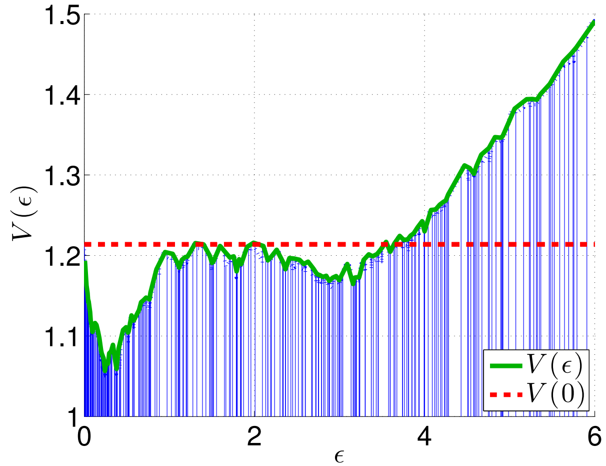
- For Anderson problem - $V(\epsilon)$ diverges at the band edge
- For Debye model - $V(\epsilon)$ goes to zero at the band edge
- A conservative $\boldsymbol{H}$ is formally like Debye model
- As the affinity is increased the conservativity of $\boldsymbol{H}$ is spoiled.

Charge density:

$$\rho(\epsilon) \quad \equiv \quad \sum_k \delta(\epsilon - \epsilon_k(s))$$

Electrostatic potential:





Digression - the determination of μ

The thresholds s_μ are defined from

$$\left\langle e^{-\mu\mathcal{E}}\right\rangle \;\equiv\; \mathrm{e}^{-(s-s_\mu)\mu} \;=\; 1$$

For an infinite chain:

$D = 0$ for $s < s_{1/2}$,

$v = 0$ for $s < s_1$.

For Gaussian disorder: $s_\mu \;=\; \frac{1}{2}\sigma^2\mu$

For Box disorder: $s_\mu \;=\; \frac{1}{\mu}\ln\left(\frac{\sinh(\sigma\mu)}{\sigma\mu}\right)$

In the latter case note that $s_\infty = \sigma$.

With a given s we associate μ such that $s = s_\mu$.

This μ is reflected in the time dependent spreading $x \sim t^\mu$

Correspondingly it is reflected in the density of eigenvalues:

$\rho(\epsilon) \quad\propto\quad \epsilon^{\mu-1} \qquad$ (for small ϵ)

For $s > s_\infty$ a gap is opened.

Resistor network disorder

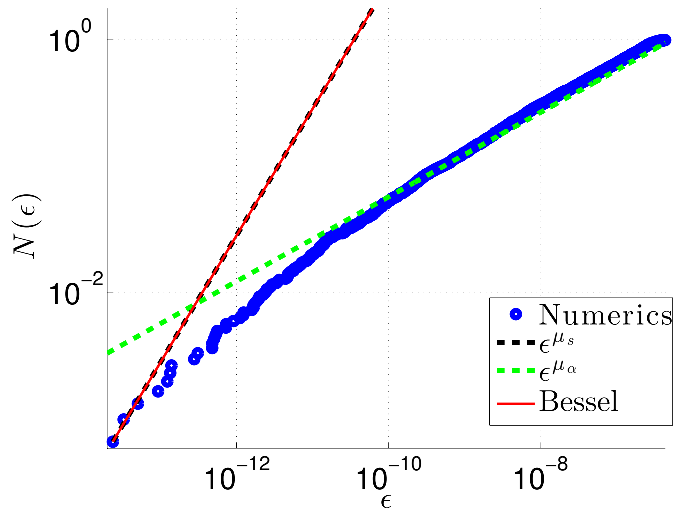
Without stochastic field:

$\mu = \frac{\alpha}{1+\alpha}$ for $\alpha<1$

$\mu = \frac{1}{2}$ for $\alpha>1$

$\mu = \alpha$ adding large s

With stochastic field disorder:



The determination of s_c - sparse disorder

Sparse disorder

Clean ring with a single defected bond

$$s_c \;\propto\; \frac{S_c}{N} \;\ll\; s_1$$

For $M \ll N$ defects

$$s_c \;\propto\; \frac{\sqrt{M}}{N}$$

Non-percolating ring

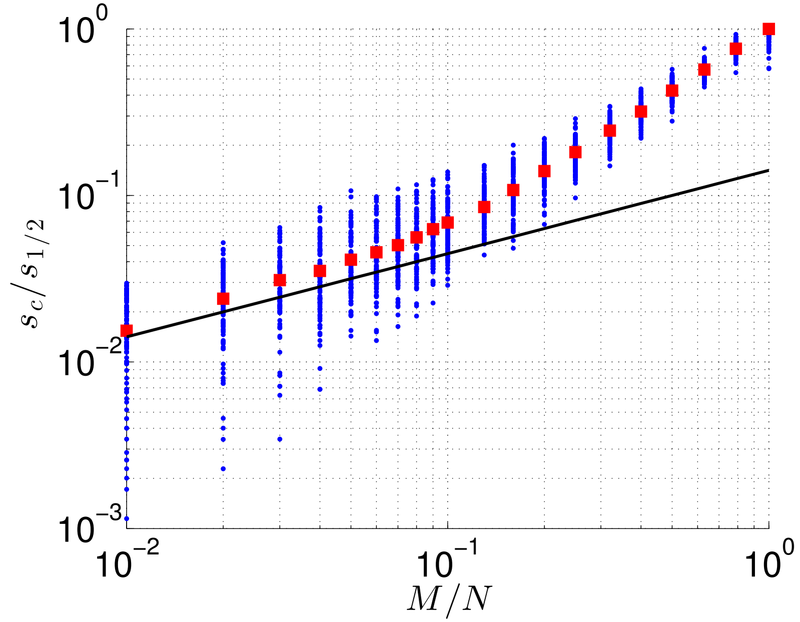
$$s_c \;=\; \infty \qquad \text{for } \alpha < 1/2$$

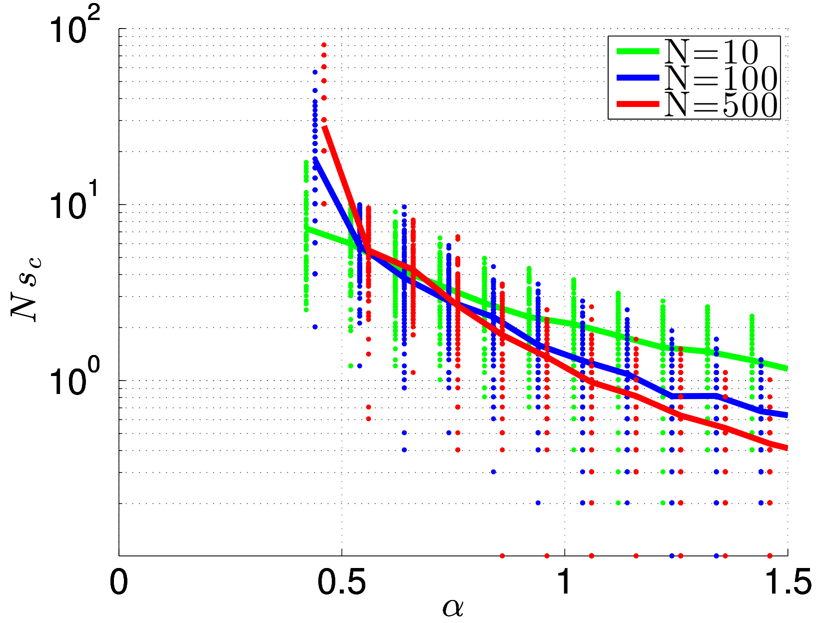
In this regime the spectrum remains real

Residual percolation

$$s_c \;\sim\; \frac{1}{N} \qquad \text{for } \alpha > 1/2$$

In this regime it is like sparse disorder





Complexity saturation

The cutoff frequency ϵ_c of the complex bubble is determined by the characteristic equation

$$\sum_k \ln|\epsilon - \epsilon_k(s)| \;=\; \ln\left\{2\left[\cosh\left(\frac{Ns}{2}\right) - 1\right]\right\}$$

We can solve it easily for $s \gg s_\infty$

$$\epsilon_k(s) \;\mapsto\; \gamma_n \;\approx\; w_n \mathrm{e}^{\mathcal{E}_n/2}$$

$$\epsilon_n(s) \;=\; w_n\, \mathrm{e}^{(s+\varsigma_n)/2} \qquad \varsigma_n \in [-\sigma, +\sigma]$$

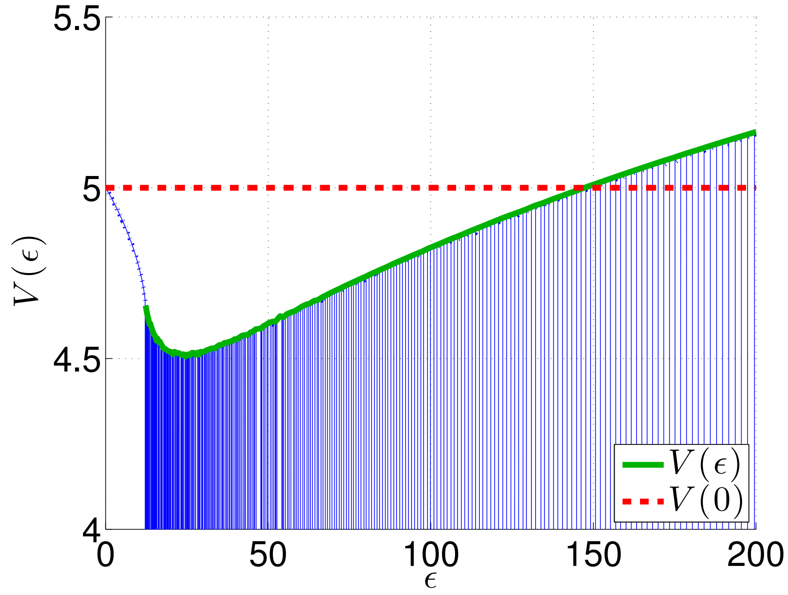
$$\epsilon_c(s) \;\equiv\; \bar{w}\;\; \mathrm{e}^{(s+\sigma_c)/2}$$

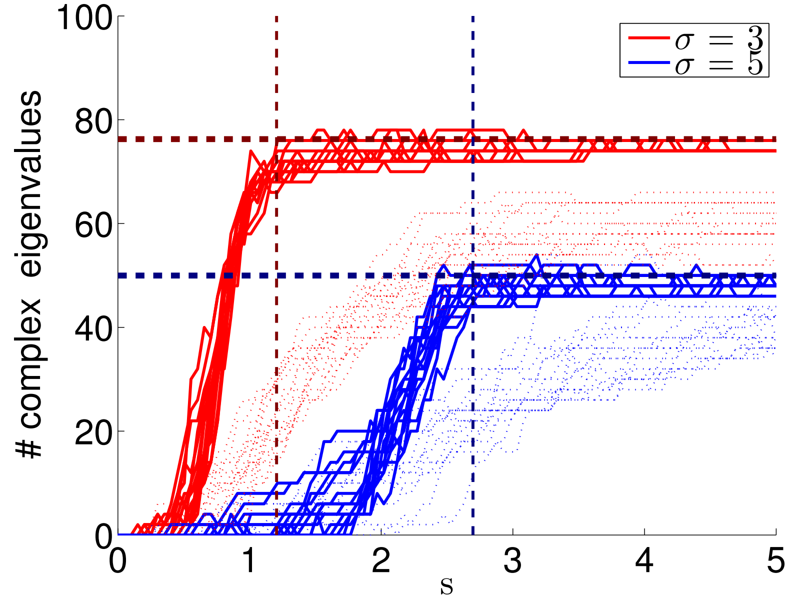
The characteristic equation takes the form

$$\frac{1}{N}\sum_n \ln\left|\bar{w}\mathrm{e}^{(s+\sigma_c)/2} - w_n \mathrm{e}^{(s+\varsigma_n)/2}\right| \;=\; \frac{s}{2}$$

We get an s independent equation for σ_c

$$\overline{\ln\left[\;\bar{w}\;\mathrm{e}^{\sigma_c/2} \;-\; w\;\mathrm{e}^{\varsigma/2}\;\right]} \;=\; 0$$





Summary

Relaxation properties of a closed circuit, whose dynamics is generated by a conservative rate-equation, is dramatically different from that of a biased non-hermitian Hamiltonian.

Type of disorder	Parameters	s_c	Remarks
No disorder	$\alpha{=}\infty,\ \sigma{=}0$	$s_c = 0$	diffusive system
Resistor-network disorder	$\alpha{<}\frac{1}{2},\ \sigma{=}0$	$s_c = \infty$	non-percolating
Resistor-network disorder	$\frac{1}{2}{<}\alpha{<}1,\ \sigma{=}0$	$s_c \sim (1/N)$	residual percolation
Sparse disorder	$(M/N) \ll 1$	$s_c \sim (1/N)$	both disorder types
Stochastic field disorder	$\alpha{>}1,\ \sigma{\neq}0$	$s_c \approx s_{1/2}$	percolating

- Relaxation becomes under-damped due to the appearance of a complex-bubble at the band floor.
- Sparse disorder - the threshold s_c diminishes as $1/N$
- Resistor network disorder - Transition to complexity happens **before** the percolation transition.
- Stochastic field disorder - Transition to complexity happens **before** the sliding transition.
- Increasing further the affinity - **“complexity saturation”** is observed.