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Covariant relativistic hydrodynamics of multi-species plasma and generalized Ohm's law

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Fully covariant hydrodynamical equations for multi-species relativistic plasma in the external electromagnetic field are derived. The derived multi-fluid description takes into account binary Coulomb collisions, annihilation, and interaction with the photon background in terms of the invariant collision cross-sections. Generalized Ohm's law is derived in a manifestly covariant form. Particular attention is devoted to the relativistic electron-positron plasma.

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Relativistic plasma attracts growing interest in connection with its possibly important role in active galactic nuclei, black hole magnetospheres, early Universe, relativistic jets, and other highly energetic astrophysical objects [1]. Among these plasmas the relativistic electron-positron plasma (sometimes contaminated with other species [2]) is of particular interest because it can be produced even in laboratory conditions [3], but mostly because of its well-established prominent role in the pulsar operation and interaction with the surrounding matter [4]. Complete kinetic description of the relativistic plasma is difficult and in many cases redundant, for example, when large scale bulk plasma motion is considered. In this case multi-fluid hydrodynamics is the most appropriate description. In the extreme case it is desirable to coarse the description to a one-fluid magnetohydrodynamics (MHD). MHD equations usually contain the particle number conservation law, the energy-momentum conservation law, and the evolution equation for the magnetic field [5]. These have to be completed by an appropriate Ohm's law. An attempt to derive such Ohm's law for weakly collisional pair plasma have been done recently [6]. However, the restrictions imposed and non-covariant consideration make this attempt not very useful. The goal of the present paper is to derive the manifestly-covariant multi-fluid hydrodynamics for collisional plasmas, and to find the covariant form of the Ohm's law.

We describe each species s by its distribution function which satisfies the 4-dimensional Vlasov-Boltzmann equation

$$p^\mu \frac{\partial f_s}{\partial x^\mu} + q_s p_\nu F^{\mu\nu} \frac{\partial f_s}{\partial p^\mu} = C(x,p,s), \tag{1}$$

where p^μ is 4-momentum, $F^{\mu\nu}$ is the electromagnetic field tensor, q_s and m_s are the species charge and rest mass, and we raise and lower indices with the metric tensor $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. The collision term in the right hand side of Eq. (1) includes binary collisions, annihilation (for pair plasmas), radiation, Compton scattering, and pair photoproduction (the last two processes are significant when there is a suitable photon background). The synchrotron radiation can be neglected if the magnetic field is not too high [6] (the primordial plasma, the pair plasma of pulsar winds) or if it is so high that the perpendicular momenta are radiated out and the distribution function is one-dimensional [7] (pulsar inner magnetospheres). If the radiative background is strong (as in the case of primordial plasma), annihilation can be balanced by photoproduction. However, in this case Compton scattering should be taken into account also. In the opposite case Compton scattering and pair photoproduction are negligible, but annihilation should be taken into account along with the binary collisions. The distribution function is defined on the mass shell $p \cdot p = p_\mu p^\mu = m_s^2$. The corresponding hydrodynamical equations are obtained by taking moments of Eq. (1) [8] as follows:

$$\partial_\mu N_s^\mu = -DN_s, \qquad \partial_\mu \equiv (\partial/\partial x^\mu), \tag{2}$$

$$\partial_\nu T_s^{\mu\nu} = q_s N_{s\nu} F^{\mu\nu} - DP_s^\mu, \tag{3}$$

where

$$N_s^\mu = \langle p^\mu \rangle, \quad T_s^{\mu\nu} = \langle p^\mu p^\nu \rangle, \tag{4}$$

and

$$\langle \ldots \rangle = \int (\ldots) f_s(p,x) d^4p\, \theta(p^0) \delta(p \cdot p - m_s^2). \tag{5}$$

The terms DN and DP denote the number and momentum losses due to collisions. The total current $j^\mu = \sum_s q_s N_s^\mu$ is conserved $\partial_\mu j^\mu = 0$. The rest frame number density and the hydrodynamical velocity of the species s are defined as follows

$$n_s = (N_s \cdot N_s)^{1/2}, \quad u_s^\mu = N_s^\mu / n_s. \tag{6}$$

Let us consider first the Coulomb collision term. It does not contribute to DN. To estimate the contribution to DP we represent the colliding species as beams with 4-velocities u_s^μ and particle momenta $p_s^\mu = m_s u_s^\mu$. According to Ref. 8 the collision rate $\Gamma_{ss'}$ can be written as $\Gamma_{ss'} = n_s n_{s'} \sigma_{ss'} F_{ss'}$, where n_s and $n_{s'}$ are the invariant rest frame densities, $\sigma_{ss'}$ is the invariant cross-section, and $F_{ss'} = \sqrt{(u_s \cdot u_{s'})^2 - 1}$ is the invariant flux. It is easy to show that the average momentum change per collision [8] can be written as $\Delta p_s^\mu = (m_s m_{s'}/(m_s + m_{s'}))\Pi_{ss'}^{\mu\nu}(u_{s\nu} - u_{s'\nu})$, where the projection operator $\Pi_{ss'}^{\mu\nu} = g^{\mu\nu} - P^\mu P^\nu/(P \cdot P)$, $P = p_s + p_{s'}$, reflects the fact that the energy of each colliding particle does not change in the center-of-mass frame. More generally, $\Gamma_{ss'}\Delta p_s^\mu$ should be averaged over the distributions of both species. We shall write the result of averaging in a model form as follows:

$$DP_s^{(c)\mu} = -\sum_{s' \neq s} n_s n_{s'} \frac{m_s m_{s'}}{m_s + m_{s'}} \sigma_{ss'}^{(c)} F_{ss'} (u_s^\mu - u_{s'}^\mu). \tag{7}$$

The quantities σ and F are determined by the invariant averaging and, therefore, are invariant also. The projection operator disappears due to the averaging over directions. Similarly, for the annihilation one can write

$$DP^{(a)\mu} = -\sum_{s'=-s} m_s n_s n_{s'} \sigma_{ss'}^{(a)} F_{ss'} u_s^\mu, \tag{8}$$

$$DN^{(a)} = -\sum_{s'=-s} n_s n_{s'} \sigma_{ss'}^{(a)} F_{ss'}, \tag{9}$$

where the summation is over particle-antiparticle pairs only. In the same way, the hydrodynamical friction term for the Compton scattering can be described as follows:

$$DP_{s\gamma}^\mu = -\sigma_{s\gamma} n_s n_\gamma m_s (u_s^\mu - u_\gamma^\mu), \tag{10}$$

where n_γ and u_γ are the density and hydrodynamical 4-velocity of the photon background. We shall not write the corresponding expression for photoproduction, assuming that either it balances annihilation, or that the photon background is negligible. Anomalous resistivity due to scattering on the turbulence is more difficult to estimate. It can be probably taken into account phenomenologically by introducing

$$DP^{(an)\mu} = -\nu_s^{(an)} (u_s^\mu - U_0^\mu), \tag{11}$$

where U_0^μ is the proper frame velocity (see below), ν_s is the corresponding collision frequency, and the momentum-energy conservation is ensured by the requirement $\sum_s DP_s^{(an)\mu} = 0$.

Mass loading also can be taken into account in the above equations by introducing appropriate source terms in DN and DP. For example, is the mass loading is due to the shock passage through a cold matter, the source terms can be written as follows (cf. Ref. 9):

$$DN_l = n_l \delta(V_{sh}^\mu x_\mu - C), \quad DP_l^\mu = V_{sh}^\mu n_l m \delta(V_{sh}^\mu x_\mu - C), \tag{12}$$

where V_{sh}^μ is the shock velocity and $V_{sh}^\mu x_\mu = C$ is the equation for the propagation of the shock front. We shall not consider the mass loading in the present paper.

In order to derive Ohm's law a proper magnetohydrodynamical (MHD) frame should be specified. In contrast with nonrelativistic MHD, choice of such a frame is not unique [8]. Usual procedure of introducing the mass flow velocity encounters difficulties, in particular, in the case of the electron-positron plasma [10]. Since it is natural in the collisionless case to write one of MHD equations as the particle number conservation law, we choose the Wigner-Ekkart velocity [8] as the MHD frame velocity [11] (subscript 0):

$$n_0 U_0^\mu = \sum_s n_s u_s^\mu. \tag{13}$$

In this case a simple transformation gives

$$E^\mu = U_{0\nu} F^{\mu\nu} = \sum_s \frac{1}{n_0 q_s} T_{s,\nu}^{\mu\nu} + \sum_{s,s'} \frac{1}{n_0 q_s} DP_s^\mu, \tag{14}$$

where E^μ is the electric field in the MHD frame, and the above expressions should be substituted for DP_s^μ. Eq. (14) is the generalized relativistic Ohm's law, expressed in the manifestly covariant form. In the general multi-fluid case, it cannot be expressed using only MHD velocity U_0 and current $j^\mu = \sum_s q_s N_s^\mu$, in contrast with the usual nonrelativistic two-species case, where such reduction is possible since the fluid velocity can be identified with the ion velocity, while electrons carry the current.

Further simplification is possible when there occurs a copious production of relativistic (electron-positron) pairs. In this case the electron-positron plasma can be assumed symmetric: $n_+ = n_- = n$. We shall assume complete symmetry and distribution isotropy for simplicity, which means

$$T_s^{\mu\nu} = (\rho_s + p_s) u_s^\mu u_s^\nu - p_s g^{\mu\nu}, \tag{15}$$

where ρ is the energy density and p is the pressure, and $\rho_+ = \rho_- = \rho$ and $p_+ = p_- = p$. In this case the chosen frame is also the quasineutrality frame, zero mass flow frame, and zero momentum flux frame. It is easy to find

$$u_\pm^\mu = \frac{n_0}{2n} U_0^\mu \pm \frac{1}{2en} j^\mu, \tag{16}$$

and the generalized Ohm's law takes the following simple form (such simple form is achieved only in the case of symmetric pair plasma):

$$E^\mu = \frac{1}{n_0} \partial_\nu [\frac{n_0}{2e^2 n^2} (\rho + p)(U_0^\mu j^\nu + U_0^\nu j^\mu)] + \eta j^\mu, \tag{17}$$

where

$$\eta = \eta^{(c)} + \eta^{(a)} + \eta^{(an)} + \eta_\gamma, \tag{18}$$

$$\eta^{(c)} = \frac{nm}{n_0 e^2} \sigma^{(c)} F, \quad \eta^{(a)} = \frac{nm}{2 n_0 e^2} \sigma^{(a)} F, \quad \eta^{(an)} = \frac{\nu^{(an)}}{n n_0 e^2}, \quad \eta_\gamma = \frac{n_\gamma m}{n_0 e^2} \sigma_\gamma. \tag{19}$$

The first term in the right hand side of Eq. (17) is a relativistic generalization of the nonrelativistic inertial term. It is small for slow motions in a dense collisional plasma, but can be substantial and dominate in a dilute almost collisionless plasma [10] typical, for example, for pulsar magnetospheres [4]. In the dense primordial lepton plasma with strong photon background [12] the annihilation resistivity $\eta^{(a)}$ can be neglected. In the opposite case of the negligible photon background the Compton resistivity η_γ is absent.

A priori neglecting the inertial term (as done in Ref. 6) means $j^\mu = 0$ and is meaningless. Instead, we shall consider the MHD limit of large spatial and temporal scales of variations and $|j_\mu| \ll en$. In this case we can ignore in the lowest order the difference between $u_\pm^\mu$ and U_0^μ and Eq. (17) takes the following single-fluid form:

$$E^\mu = \frac{1}{n_0 e^2} \partial_\nu [\frac{p_0 + \rho_0}{n_0} (U_0^\mu j^\nu + U_0^\nu j^\mu)] + \eta j^\mu, , \tag{20}$$

where $p_0 + \rho_0 = 2(p + \rho)$ and $n_0 = 2n$. The usual decomposition of the electromagnetic tensor $F^{\mu\nu} = (E^\mu U_0^\nu - E^\nu U_0^\mu) + \epsilon^{\mu\nu\alpha\beta}(U_{0\alpha} B_\beta)$ (see. e.g. Ref. 5) gives in the MHD limit the following:

$$\epsilon^{\mu\nu\alpha\beta}(U_{0\alpha} B_\beta)_{,\nu} = 4\pi j^\mu, \tag{21}$$

$$\epsilon^{\mu\nu\alpha\beta}(U_{0\alpha} E_\beta)_{,\nu} + (U_0^\mu B^\nu - U_0^\nu B^\mu)_{,\nu} = 0, \tag{22}$$

where $\epsilon^{\mu\nu\alpha\beta}$ is the completely antisymmetric tensor, $\epsilon^{0123} = 1$, and E^μ and B^μ are the electric and magnetic field defined in the MHD frame. These should be completed with the MHD equations of continuity and motion, which take the following form:

$$(n_0 U_0^\mu)_{,\mu} = -\frac{2 n_0 e^2}{m} \eta^{(a)}, \tag{23}$$

$$[(p_0 + \rho_0) U_0^\mu U_0^\nu]_{,\nu} = j_\nu F^{\mu\nu} - n_0^2 e^2 \eta^{(a)} U_0^\mu - n_0 e^2 \eta_\gamma (U_0^\mu - u_\gamma^\mu), \tag{24}$$

Eqs. (20-24) constitute the complete set of extended MHD for relativistic pair plasma. This set can be applied to the description of the bulk flow of the electron-positron wind. It is also the appropriate basis for the analysis of the resistive MHD shocks and reconnection processes in such plasmas. In both last cases the derivative term in Eq. (20) would be of significance, since the spatial gradients become substantial (cf. Ref. 13). Relative importance of the inertial and resistive contributions is, of course, different in different systems. For example, the electron-positron plasma in the pulsar magnetosphere is almost collisionless [4, 7]. Therefore, at large scales it behaves as an ideal relativistic MHD fluid, while at smaller scales the derivative term in Eq. (20) should be taken into account. On the other hand, in the case of the relativistic electron-positron jet with a strong ambient radiation (see, for example, [14]) the resistive contribution in the Ohm's law is determined by the interaction with the photon field and may dominate. Since this term does not depend on the temporal and spatial derivatives, it should be taken into account even when large scale bulk plasma motion is studied. In the equation of motion the friction between the plasma and photon field becomes significant.

More generally, in the MHD frame $U = (1, 0, 0, 0)$ and Eq. (17) can be written as

$$E_i = (\eta + \frac{\rho + p}{2e^e n^2 \tau}) j_i, \tag{25}$$

where we substituted $\partial_t \sim \tau^{-1} \sim v_A/L$, τ and L being the typical time and spatial scale of variations, and $v_A = [B^2/(4\pi(\rho + p) + B^2)]^{1/2}$ is the relativistic Alfven velocity [15], while B is the magnetic field in the MHD frame. One can see that the resistive term dominates when the generalized relativistic magnetic Reynolds number $\mathrm{Re} = (p + \rho)v_A/\eta L n^2 e^2 \ll 1$. In the opposite case of large Re the inertial term dominates.

To conclude, we have derived the general form for the relativistic hydrodynamics of multi-species plasmas, taking into account collisions among the plasma species themselves and with photons. This multi-fluid hydrodynamics describes bulk motions of the plasma species. Generalized Ohm's law has been derived. Such Ohm's law is useful only if the plasma motions are slow in the MHD frame and currents are not strong. The Ohm's law takes especially simple form for symmetric electron-positron plasma, where it is similar to the nonrelativistic resistive expression. The resistivity is determined by Coulomb collisions and annihilation cross-sections in a dilute plasma without photon background, and by Compton scattering cross-section and photon density when the photon background is substantial. The Ohm's law for relativistic pair plasma includes resistive and inertial contributions. We have derived also the fully covariant MHD set of equations for the relativistic electron-positron plasma. These equations may constitute the appropriate basis for the analysis of relativistic pair behavior in rather general conditions, from the ideal MHD bulk flow to the initial stage of resistive reconnection.

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