Quantum chaos, dynamical correlations, and the effect of noise on localization

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Localization in the quantum kicked rotator (QKR) problem leads to nontrivial dynamical correlations that are absent in the classical limit. These correlations are related to the spectral properties of the model. The effect of noise on coherence is studied. Also, its effect on the dynamical correlations is investigated, and thus one is able to determine analytically the diffusion that is induced. If the noise is white, then coherence time is the same as that for the undriven system. For colored noise it is essential to take into account the nature of the dynamical correlations. It follows that a Markovian treatment of the dynamics for a system that is coupled to a low-temperature heat bath is not valid even if the system is classically chaotic. The formalism applies also to the problem of QKR with kicks that are not strictly periodic in time, and also to the quantum kicked particle problem, where a nonperturbative mechanism for destruction of coherence has been found.

I. INTRODUCTION

The quantum kicked rotator (QKR) problem [1] constitutes a prototype example for the suppression of classical chaos due to quantal localization [2]. It is related closely to the studies of Zener dynamics in multilevel systems [3] and the ionization of hydrogen atoms by a microwave electric field [4]. Casati, Chirikov, Izrailev, and Ford [1] have discovered that in the quantal model, in contrast to the classical model, stochasticlike diffusive behavior is followed for only a short time t^*, which has been entitled the "break time." On a time scale larger than t^* the dynamics appears to be quasiperiodic. This behavior is due to the localization of the eigenstates of the one-step unitary operator which generates the time evolution of the system. This localization has become the subject for an extensive study initiated by Fishman, Grempel, and Prange [2]. It is very similar to the localization in the tight-binding Anderson model of solid-state physics, but with a pseudorandom potential [5,6]. This similarity has been further emphasized in later works [7,8]. The initial diffusion rate has been related to the localization length on the basis of a scaling hypothesis [7]. Dittrich and Smilansky [8] have introduced evidence for the applicability of Mott's resonating-state picture [9] to the localization in the QKR model.

A heuristic picture [10] may be used in order to establish a simple relation between the break time t^*, the initial diffusion rate D_0, and the localization length ξ. Berman and Izrailev [11] have tried to refine this heuristic picture in order to find the functional form of the crossover from "diffusion" to "saturation" and to relate it to spectral properties of the system. Further study [12] has revealed that in order to recover (formally) their result one should assume that local level statistics is a well-defined notion. In view of the later work by Dittrich and Smilansky [8] this ansatz does not hold, and the importance of the weighting procedure has been stressed. A different strategy is therefore needed, and has been reported in Ref. [13] and is to be elaborated upon here. In particular we are interested in quantifying the break-time concept and analyzing the effect of Mott's resonating states [9] on the asymptotic form of the crossover. Obviously it is related to the nature of the dynamical correlations that are characteristic of the QKR model and thus also to the spectral properties of this system.

The effect of noise on the dynamics of the QKR constitutes the main subject of this paper. Noise may be induced by fluctuations of an external field. As a heuristic example, consider a particle that is confined to move in a mesoscopic ring and is driven by a periodic kicking (e.g., by an electric field which is switched on and off periodically). A noisy magnetic flux through the ring constitutes a noise source to which the particle is coupled via its momentum coordinate. Noise also may emerge from the interaction with other degrees of freedom. For example, the particle may interact with phonons. The coupling to the bath is then via the position variable. In the latter case it is important to consider not only the noise but also the effect of friction on the dynamics. Noise may be associated with the measurement process [14,15]. Also, the problem of QKR with kicks that are not strictly periodic in time [16,15] may be considered within the formalism to be presented here.

Two aspects of the effect that noise has on the dynamics will be discussed in this paper: (a) destruction of coherence and (b) the diffusion process. We shall introduce a study of the decay process and will determine analytically the average decay rate Γ of a quasienergy eigenstate due to the noise. The coherence time is then $t_c \equiv 1/\Gamma$. Then we shall try to estimate analytically the diffusion coefficient $\mathcal{D}$ in the presence of noise. We shall consider different coupling schemes to the noise source, and the significance of noise autocorrelations will be emphasized.

The first to consider the effect of noise on the dynamics in the QKR problem were Ott, Antonsen, and Hanson [17]. They introduced a heuristic picture to determine

the coherence time t_c and related $\mathcal{D}$ to t_c by assuming that the diffusion process is similar to a random-walk process with hopping probability $1/t_c$ per step. They also introduced a direct estimate of $\mathcal{D}$ using perturbation theory. The implicit assumption that underlies the latter estimate is that the diffusion process is a stochastic process that may be treated within the framework of a Markovian picture. This important work, and subsequent works [14,15] that dealt with the same subject (but with different motivations), left open many issues: (a) The decay process has not been studied; (b) only particular coupling schemes were considered; (c) the noise has been assumed to be white; (d) a formal approach to compute $\mathcal{D}$ has not been presented; (e) the validity of the perturbative treatment has not been established; and (f) the estimates for both $\mathcal{D}$ and t_c involved undetermined prefactors. It should be noted that the formal procedure that should be followed in order to estimate $\mathcal{D}$ is well known from the corresponding classical study by Rechester and coworkers [18]. This point has been recognized also by Toda, Adachi, and Ikeda [15], but a systematic study along these line did not follow.

Dittrich and Graham [19] have introduced a model for the investigation of the damped QKR problem. The rotator has been assumed to be coupled to a zero-temperature bath of harmonic oscillators. The dynamics has been analyzed using the master-equation approach [20], which is based on a Markovian treatment of the dynamics. This is essentially the same assumption that underlies the analysis [17] of the effect of noise on the QKR. However, the relations of the Dittrich-Graham model to the latter study have been demonstrated only much later [21]. It was also shown [21] then that the Dittrich-Graham model is non-Ohmic in the Caldeira-Leggett sense [22]. Consequently two other different models have been introduced [23,21] where the QKR has been assumed to be coupled to an Ohmic bath via its momentum variable [23] or via its position coordinate [21]. The former coupling scheme does not lead to dissipation of energy, while the latter results in damping that is proportional to velocity.

One may rely on the Feynman-Vernon formalism [24] in order to show [21,23] that if friction is ignored, then the heat bath has the same effect as that of a stochastic force. Its autocorrelation function is determined by the bath. For an Ohmic bath the noise is white at high temperatures. At the limit of zero temperature the noise does not vanish; instead it has negative autocorrelations. The effect of friction (which is ignored if the bath is replaced by an equivalent c-number noise source) on the long-time dynamical behavior has been shown to be negligible in the case of nondissipative coupling [23]. In the case of weak dissipative coupling a phenomenological Fokker-Planck treatment of the damping process is sufficient [21]. The latter statement is based on the observation that even at the limit of zero temperature coherence time is much shorter than the relaxation time. For strong dissipative coupling [21] the dynamical behavior is essentially classical since the steady state is reached within few time steps. The present study is not relevant in the latter case.

Noise autocorrelations that arise at low temperature due to the quantum nature of the bath affect significantly the time evolution of *integrable* systems [22,26,25,23]. For example, in the case of either an undriven particle [26,25] or an undriven rotator [23] these negative noise autocorrelations result in the suppression of linear diffusion and instead a logarithmic behavior is found. *Markovian treatment* of the dynamics [25] ignores these noise autocorrelations [25,23,21]. If the system is treated classically and is known to be *chaotic*, then this should not be important—due to the exponential instability of the phase-space trajectories we expect no *memory* for noise autocorrelations. However, if the dynamics is treated within the framework of quantum mechanics, then one may expect manifestations of *dynamical correlations* that are absent in the classical limit. This observation has been reported already in Ref. [13] and is to be elaborated in this paper.

The outline of the paper is as follows. In Sec. II various models that may be treated within the framework of the formalism to be presented are introduced. In Sec. III diffusion and localization in the QKR model are discussed. This discussion leads naturally to Sec. IV. There, spectral properties and associated dynamical correlations that are characteristic of the QKR model are analyzed. The effect of noise on coherence is studied in Sec. V, while its effect on dynamical correlations is investigated in Sec. VI. Thus one is able to determine the diffusion which is induced due to the destruction of coherence. Nonperturbative effects are pointed out. The conclusions are summarized in Sec. VII. Appendix A is devoted to the review of Wigner's picture of the dynamics, Appendix B introduces some useful Fourier transform formulas, and Appendix C deals with the calculation procedure that has been adopted in the numerical analysis.

II. MODELS

In this section we shall motivate the study of different models that may be treated within the formalism which will be presented in later sections. In the QKR-p and QKR-x models in the QKR is assumed to be coupled to a c-number noise source via either its momentum coordinate or its position variable, respectively. In the case of the quantum kicked particle (QKP) problem the position coordinate of the QKR is considered to be an extended one. Finally, two versions of the QKR problem with kicks that are not strictly periodic in time are introduced (QKR-t models).

A. The QKR-p model

For the purpose of motivating this model we shall use a heuristic presentation that oversimplifies the mesoscopic physics of small normal rings. Consider a particle that is confined to move in a one-dimensional ring whose length is L. A noisy magnetic field penetrates the ring and the particle is subject to a periodic kicking by a uniform electric field. The Hamiltonian is

$$\hat{\mathcal{H}}=\frac{1}{2M}[\hat{p}+\Phi(\tau)]^2+K\cos\left[2\pi\frac{\hat{x}}{L}\right]\sum_{n=-\infty}^{\infty}\delta(\tau-nT)\ , \tag{2.1}$$

where M is the mass of the particle, K is the kicking strength, T is its period, and $\Phi(\tau)$ is the (scaled) magnetic flux that penetrates the ring. It is implicit that the dynamics should be averaged over realizations of $\Phi(\tau)$ such that $\langle\Phi(\tau)\rangle=0$. We shall use units such that $M=1$, $T=1$, and $L=2\pi$, so that K is a dimensionless parameter. After quantization an additional dimensionless parameter appears, namely (the scaled) $\hbar$. The time evolution over one time step (propagation kick) is generated by the unitary operator

$$\hat{U}=\exp\left[-\frac{i}{\hbar}K\cos\hat{x}\right]\exp\left[-\frac{i}{\hbar}(\tfrac{1}{2}\hat{p}^2+\hat{\mathcal{H}}_{\rm int})\right]\ , \tag{2.2}$$

where

$$\hat{\mathcal{H}}_{\rm int}=g(t)\hat{p} \tag{2.3}$$

with $g(t)\equiv\int_{t-1}^{t}\Phi(\tau)d\tau$. From now on t denotes an integer time variable. Note that a global phase factor has been omitted in (2.2). The dynamics that is generated by U should be averaged over realizations of the sequence $g(t)$ such that

$$\begin{aligned}&\langle g(t)\rangle=0\ ,\\ &\langle g(t)g(t')\rangle=\phi(t-t')\ ,\end{aligned} \tag{2.4}$$

where the noise correlation function is

$$\phi(t,t')\equiv\int_{t-1}^{t}\int_{t'-1}^{t'}d\tau\,d\tau'\langle\Phi(\tau)\Phi(\tau')\rangle\ . \tag{2.5}$$

The evolution in Heisenberg's picture is generated by the map

$$\begin{aligned}&\hat{x}(t)=\hat{x}(t-1)+\hat{p}(t-1)+g(t)\ ,\\ &\hat{p}(t)=\hat{p}(t-1)+K\sin\hat{x}(t)\ ,\end{aligned} \tag{2.6}$$

which is known as the quantized version of the standard map with noise. Its classical limit has been studied extensively by Chirikov [27] and followers [28]. A quantitative study of the classical diffusion process that takes into account the effect of the noise has been introduced by Rechester and co-workers [18]. It is convenient to represent the quantum state of the system by Wigner's function $\rho(x,p)$. See Appendix A for details. Its time evolution is

$$\rho_t(x,p)=\sum_{p_0}\int dx_0\mathcal{H}(x,p|x_0,p_0)\rho_{t=0}(x_0,p_0)\ . \tag{2.7}$$

For the free propagation

$$\mathcal{H}^{\rm free}(x,p|x_0,p_0)=\delta(p-p_0)\delta(x-(x_0+p_0))\ , \tag{2.8}$$

where δ stands for either Dirac's or Kroneker's delta function according to its type of argument. In particular, if its argument is an angle variable, then the δ stands for the 2π-periodic Dirac delta function. For the kick process

$$\mathcal{H}^{\rm kick}(x,p|x_0,p_0)=\delta(x-x_0)\mathcal{J}_{2(p-p_0)/\hbar}\left[2\frac{K\sin x_0}{\hbar}\right]\ , \tag{2.9}$$

which corresponds to the classical expression

$$\mathcal{H}^{\rm kick}_{\rm classical}(x,p|x_0,p_0)=\delta(x-x_0)\delta(p-(p_0+K\sin x_0))\ . \tag{2.10}$$

$\mathcal{J}$ is a Bessel function of integer order. For the (one-step) noise process

$$\mathcal{H}^{g}(x,p|x_0,p_0)=\delta(p-p_0)\delta(x-(x_0+g))\ . \tag{2.11}$$

Averaging $\mathcal{H}^g$ over realizations of g and denoting by $G(g)$ the probability density function of the random variable g, one obtains

$$\mathcal{H}^{\rm noise}(x,p|x_0,p_0)=\delta(p-p_0)\mathcal{G}(x-x_0)\ , \tag{2.12}$$

where $\mathcal{G}(x-x_0)\equiv\sum_{n=-\infty}^{\infty}G(x-(x_0+2\pi n))$. The one-step kernel is therefore

$$\mathcal{H}^{1\,\rm step}=\mathcal{H}^{\rm kick}\circ\mathcal{H}^{\rm noise}\circ\mathcal{H}^{\rm free} \tag{2.13}$$

where $\circ$ denotes convolution of kernels. The t-step kernel is

$$\mathcal{H}^{t\ \rm steps}=\mathcal{H}^{1\,\rm step}\circ\cdots\circ\mathcal{H}^{1\,\rm step} \tag{2.14}$$

only for white noise. Otherwise the average of realizations of $g(t)$ should be taken *after* the convolution is performed. This implies that if the noise is colored, then a Markovian treatment of the dynamics is *not* exact [25].

B. The QKR-x model

We shall consider now the quantized version of the classical map

$$\begin{aligned}&x'=x+p\ ,\\ &p'=p+K\sin x'+f\ ,\end{aligned} \tag{2.15}$$

where f is a random number such that

$$\begin{aligned}&\langle f\rangle=0\ ,\\ &\langle f^2\rangle=\nu\ .\end{aligned} \tag{2.16}$$

Since the quantized version of this map is less trivial then the QKR-p model, we discuss first a one-step process. Thus there is no explicit reference for the dependence of f on time. Once the coupling scheme to the noise source is clarified we shall turn to discuss the multistep process. The time correlation of the noise will be specified then.

Following Ott, Antonsen, and Hanson [17] we assume that the one-step propagator that generates this map is of the form

$$\hat{U}=\exp\left[-\frac{i}{\hbar}(K\cos\hat{x}+\hat{\mathcal{H}}_{\rm int})\right]\exp\left[-\frac{i}{\hbar}\frac{1}{2}\hat{p}^2\right]\ , \tag{2.17}$$

where $\mathcal{H}_{\rm int}$ is the interaction term with the noise source. The simplest coupling scheme is of course

$$\hat{\mathcal{H}}_{\text{int}}=-f\hat{x} \ . \tag{2.18}$$

If x is considered to be an extended coordinate, then the model is designated a QKP problem, to be discussed later. If x is an angle coordinate, then $\exp[-(i/\hbar)\mathcal{H}_{\text{int}}]$ is a well-defined operator provided we add the restriction

$$f\in\hbar\mathbb{Z} \ , \tag{2.19}$$

where $\mathbb{Z}$ are the integer numbers. It is then possible to define a normalized probability function $\mathcal{F}(f)$ such that the requirements (2.16) are satisfied, namely

$$\begin{aligned}&\sum_{f\in\hbar\mathbb{Z}}\mathcal{F}(f)=1 \ ,\\ &\sum_{f\in\hbar\mathbb{Z}}\mathcal{F}(f)f=0 \ ,\\ &\sum_{f\in\hbar\mathbb{Z}}\mathcal{F}(f)f^2=\nu \ .\end{aligned} \tag{2.20}$$

The noise process, for a particular realization of f, is represented by the operation with the kernel

$$\mathcal{H}^f(x,p\,|x_0,p_0)=\delta(x-x_0)\delta(p-(p_0+f)) \ . \tag{2.21}$$

Averaging over realizations of f one obtains

$$\mathcal{H}^{\text{noise}}(x,p\,|x_0,p_0)=\delta(x-x_0)\mathcal{F}(p-p_0) \ . \tag{2.22}$$

The linear coupling scheme (2.18) with the restriction (2.19), in spite of its simplicity, suffers few disadvantages, which will be discussed later. Therefore one is urged to offer different coupling schemes. Ott, Antonsen, and Hanson [17] have used

$$\hat{\mathcal{H}}_{\text{int}}=\sqrt{2\nu}\cos(\hat{x}+\varphi) \ , \tag{2.23}$$

where φ is a randorn phase. The quantized map is then (2.15) with $\hat{f}=\sqrt{2\nu}\sin(\hat{x}+\varphi)$. The requirement (2.16) is indeed satisfied (the average over realizations of $\hat{f}$ is easily performed in the diagonal x representation). A linear coupling to the noise source has been considered in Ref. [21],

$$\hat{\mathcal{H}}_{\text{int}}=\int d\varphi\, f_\varphi\sqrt{2}\cos(\hat{x}+\varphi) \ , \tag{2.24}$$

where f_φ are Gaussian random variables that satisfy $\langle f_\varphi\rangle=0$ and $\langle f_\varphi f_{\varphi'}\rangle=\nu(1/2\pi)\delta(\varphi-\varphi')$. The quantized map is then (2.15) with $\hat{f}=\int d\varphi\, f_\varphi\sqrt{2}\sin(\hat{x}+\varphi)$. Again, it is easily verified that requirement (2.16) is indeed satisfied.

A surprising point is that the two latter coupling schemes (2.23) and (2.24) lead to the same dynamics, as if it were generated by the c-number "quantized" noise (2.19) with the linear coupling (2.18). To prove this point one should calculate the propagator $\mathcal{H}$ in these particular cases and realize that it constitutes a special form of (2.22). Details of the calculation are given in Appendix A. The final result is that one may use instead of (2.23) the linear coupling scheme (2.18) with the probability function

$$\mathcal{F}(f)=\left[\mathcal{J}_{f/\hbar}\left[\frac{\sqrt{2\nu}}{\hbar}\right]\right]^2 \ . \tag{2.25}$$

In case of (2.24) the same statement applies, but the probability function that should be used is

$$\mathcal{F}(f)=e^{-\nu/\hbar^2}\left[\mathcal{J}_{f/\hbar}\left[\frac{\nu}{\hbar^2}\right]\right] \ . \tag{2.26}$$

The symbols $\mathcal{J}$ and $\mathcal{J}$ denote the ordinary and modified Bessel functions of integer order, respectively. To summarize this paragraph we stress again that though f is an operator if the quantization scheme is either (2.23) or (2.24), yet its effect is the same as if it were a c-number quantized noise with the linear coupling (2.18).

The disadvantages of the simplest coupling scheme (2.18) are as follows: (a) The interaction term (2.18) with the restriction (2.19) seems to be artificial from a physical point of view since it does not correspond to the usual form of interaction terms that are found in most physical circumstances. (b) It is problematic to use perturbation theory even for very weak noise since the typical values of f do not go to zero for weaker noise (namely, the minimal nontrivial value of $|f|$ is $\hbar$, and it does not go to zero in the limit $\nu\to 0$). The coupling scheme (2.23) that has been used by Ott, Antonsen, and Hanson [17] is much more reasonable from a physical point of view. Yet the main motivation for investigating the map (2.15) is the study of the damped-rotator problem. A linear coupling scheme to an Ohmic bath has been introduced [21]. It constitutes the simplest generalization of the damped-particle model that has been introduced by Caldeira and Leggett [22]. One may rely on the Feynman-Vernon formalism [24] in order to show [21] that if friction is ignored, then the heat bath has the same effect as that of a stochastic force. The interaction term is then

$$\hat{\mathcal{H}}_{\text{int}}=\int d\varphi\, f_\varphi(t)\sqrt{2}\cos(\hat{x}+\varphi) \ , \tag{2.27}$$

where $f_\varphi(t)$ satisfied $\langle f_\varphi(t)\rangle=0$ and

$$\langle f_\varphi(t)f_{\varphi'}(t')\rangle=\frac{1}{2\pi}\delta(\varphi-\varphi')\nu(t-t') \ . \tag{2.28}$$

The map that is obtained is the multistep version of (2.15) with

$$\hat{f}(t)=\int d\varphi\, f_\varphi(t)\sqrt{2}\sin(\hat{x}+\varphi) \ . \tag{2.29}$$

This stochastic force satisfies

$$\begin{aligned}&\langle\hat{f}(t)\rangle=0 \ ,\\ &\langle\hat{f}(t)\hat{f}(t')\rangle=\nu(t-t') \ .\end{aligned} \tag{2.30}$$

Its autocorrelation function $\nu(\tau)$ is determined by the bath. Explicit expressions for $\nu(\tau)$ in the case of Ohmic bath are presented in Ref. [25]. At high temperatures the noise is white. At the limit of zero temperature the total area under $\nu(\tau)$ goes to zero, but the noise does not vanish; instead it has negative autocorrelations such that $\nu(\tau)\propto -1/\tau^2$ for $1<\tau$. The friction effect is ignored if the bath is replaced by a c-number noise source. This evokes the questions on whether this strategy is physically meaningful, and how can the friction effect be taken into account if we want to consider long-time behavior of the system. In the case of weak coupling it can be shown

[21] that coherence time is much shorter than relaxation time even at the limit of zero temperature. It follows that a phenomenological Fokker-Planck treatment of the damping process is sufficient. For strong coupling the dynamical behavior is essentially classical since the steady state is reached within a few time steps. The present study is not relevant then.

A simpler version of the QKR-x model is (2.17) with either

$$\hat{\mathcal{H}}_{\rm int}=\tilde{f}(t)\sqrt{2}\sin\hat{x} \tag{2.31}$$

or

$$\hat{\mathcal{H}}_{\rm int}=\tilde{f}(t)\sqrt{2}\cos\hat{x}\ , \tag{2.32}$$

where $\tilde{f}(t)$ satisfies, in any case, $\langle\tilde{f}(t)\rangle=0$ and $\langle\tilde{f}(t)\tilde{f}(t')\rangle=\nu(t-t')$. These coupling schemes are simpler than (2.27) and result (as we shall see) in a very similar dynamical behavior. The map that is generated is of the form (2.15); however, requirement (2.16) is not satisfied since the noise is position dependent.

C. The QKP model

Consider a particle that is free to move in a one-dimensional infinite space and is subject to kicking by a cosine potential. In addition we assume that the particle is coupled linearly to a noise source as in (2.18). The one-step propagator is

$$\hat{U}^{f}=e^{(i/\hbar)f\hat{x}}\exp\left[-\frac{i}{\hbar}K\cos\hat{x}\right]\exp\left[-\frac{i}{\hbar}\frac{1}{2}\hat{p}^{2}\right]\ , \tag{2.33}$$

where $f(t)$ satisfies, upon averaging,

$$\begin{aligned}&\langle f(t)\rangle=0\ ,\\ &\langle f(t)f(t')\rangle=\nu(t-t')\ .\end{aligned} \tag{2.34}$$

This model has been introduced in Ref. [29] and has been entitled the quantum kicked particle model. The motivation for its analysis has been discussed there. It constitute a prototype example for systems where dynamical localization is in a sense nongeneric (there is no dynamical localization if a particle is kicked by a generic potential which is not spatially periodic, unless its position in space is bounded, which is not the case here). This problem is also related to the study of diffusive-ionization of highly excited hydrogen atoms that are subject to a monochromatic microwave electric field in the presence of noise. Turning back to the QKP model, the multistep propagator is $\hat{U}^{t\ \rm steps}=\hat{U}^{f(t)}\cdots\hat{U}^{f(1)}$. It is easily verified that this multistep propagator may be written in the following form:

$$\hat{U}^{t\ \rm steps}=e^{(i/\hbar)g(t)\hat{x}}\hat{U}^{g(t-1)}\cdots\hat{U}^{g(0)}\ , \tag{2.35}$$

where $g(0)=0$, $g(t)\equiv\sum_{t'=1}^{t}f(t')$, and the iterations are with

$$\hat{U}^{g}=\exp\left[-\frac{i}{\hbar}K\cos\hat{x}\right]\exp\left[-\frac{i}{\hbar}\frac{1}{2}(\hat{p}+g)^{2}\right]\ .$$

Thus the dynamics for the QKP model is essentially the same as for a QKR-p problem with

$$\phi(t,t')=\sum_{\tau=1}^{t}\sum_{\tau'=1}^{t'}\nu(\tau-\tau')\ . \tag{2.36}$$

The fundamental difference between this problem and the standard QKR-p problem may be understood in terms of our original heuristic presentation of the model. Namely, consider a particle that is confined to move in a ring and is subject to a magnetic flux that penetrates it. Instead of having a noisy flux we should consider a nonstationary process such that in each step the flux is changed slightly. These changes accumulate. Formally, the variance of the noise grows with time. However, this does not imply that the "noise" becomes more intense as time elapses. The noise is simply nonstationary.

At this stage it is appropriate to note that a very similar approach may be used in order to demonstrate that for weak noise the QKR-p problem is equivalent to the QKR-x problem with the interaction term (2.31) and correlation function

$$\nu(t,t')=\tfrac{1}{2}K^{2}\sum_{\tau=1}^{t}\sum_{\tau'=1}^{t'}\phi(\tau,\tau')\ . \tag{2.37}$$

The same strategy is used. Namely, one begins with the one-step propagator (2.33) but with $e^{(i/\hbar)f(t)\hat{x}}$ replaced by $e^{-(i/\hbar)g(t)\hat{p}}$. Then one observes that the same dynamics is generated by the one-step propagator

$$\hat{U}=\exp\left[-\frac{i}{\hbar}K\cos[\hat{x}+\tilde{f}(t)]\right]\exp\left[-\frac{i}{\hbar}\frac{1}{2}\hat{p}^{2}\right]\ ,$$

where $\tilde{f}(0)=0$ and $\tilde{f}(t)\equiv\sum_{t'=1}^{t}g(t')$. If the condition

$$\lim_{t\to\infty}\nu(t,t)<\infty \tag{2.38}$$

is satisfied, then for sufficiently weak noise one may expand $\cos[x+\tilde{f}(t)]$ up to first order in $\tilde{f}(t)$ to obtain the QKR-x one-step propagator (2.17) with the position-dependent interaction term (2.31) and the correlation function (2.36). For example, if $\phi(t,t')=2\lambda\delta_{t,t'}-\lambda\delta_{|t-t'|,1}$, then the QKR-$p$ model is reduced to a QKR-x problem with white noise whose correlation function, after dropping an insignificant constant component, is $\nu(t-t')=\nu\delta_{t,t'}$ with $\nu=\frac{1}{2}K^{2}\lambda$.

D. The QKR-t models

We consider now a rotator that is kicked in variable time steps. The models to be introduced will be denoted by QKR-t. The one-step propagator is

$$\hat{U}=\exp\left[-\frac{i}{\hbar}K\cos\hat{x}\right]\exp\left[-\frac{i}{\hbar}(\tau_{t}-\tau_{t-1})\tfrac{1}{2}\hat{p}^{2}\right]\ , \tag{2.39}$$

where τ_t is the actual time of the tth kick. Two versions of this model are of particular interest: (1) the time steps are uncorrelated, (2) there is an underlying mechanism for periodic kicking and fluctuations are superimposed. In the case of the first version

$$\tau_{t}-\tau_{t-1}=1+\tilde{g}(t),\quad \text{version (1)} \tag{2.40}$$

with $\langle\tilde{g}(t)\tilde{g}(t')\rangle=\tilde{\sigma}\delta_{t,t'}$, while for the second version

$$\tau_t = t + \tilde{f}(t), \quad \text{version (2)} \tag{2.41}$$

with $\langle \tilde{f}(t)\tilde{f}(t')\rangle = \tilde{v}\delta_{t,t'}$. One may regard the evolution as consisting of free propagations, kicks, and noise operations. The latter are represented by the unitary operator $\exp(-(i/\hbar)\mathcal{H}_{\text{int}})$. In the case of the first version $\hat{\mathcal{H}}_{\text{int}} = \tilde{g}(t)\frac{1}{2}\hat{p}^2$, while in the second version $\hat{\mathcal{H}}_{\text{int}} = [\tilde{f}(t) - \tilde{f}(t-1)]\frac{1}{2}\hat{p}^2$. Locally this problem is essentially the same as the QKR-p problem with inhomogeneous noise, namely, in the case of the first version

$$\phi(t,t') = \langle p^2\rangle \tilde{\sigma}\delta_{t,t} , \tag{2.42}$$

while $\phi(t,t') = \langle p^2\rangle \tilde{v}(2\delta_{t,t'} - \delta_{|t-t'|,1})$ in the second case. However, in the latter version a further reduction via (2.37) is possible to produce an equivalent QKR-x model with white-noise whose correlation function is

$$\nu(t,t') = \tfrac{1}{2}K^2\langle p^2\rangle \tilde{v}\delta_{t,t'} . \tag{2.43}$$

Thus the two versions of the QKR-t problem reduce locally to either a QKR-p or QKR-x model with white noise.

III. DIFFUSION AND LOCALIZATION IN THE QKR MODEL

A. Eigenstates

The evolution of the QKR is given by iterations with the one-step propagator

$$\hat{U} = \exp\left[-\frac{i}{\hbar}K\cos\hat{x}\right]\exp\left[-\frac{i}{\hbar}\frac{1}{2}\hat{p}^2\right], \tag{3.1}$$

with $[\hat{x},\hat{p}] = i\hbar$ and periodic boundary conditions on $[0,2\pi]$ imposed. The matrix elements of U in momentum representation are

$$\langle m|\hat{U}|n\rangle = (-i)^{m-n}\mathcal{J}_{m-n}\left[\frac{K}{\hbar}\right]e^{-i\hbar(1/2)n^2} , \tag{3.2}$$

where $|n\rangle$ and $|m\rangle$ are momentum eigenstates, namely $\hat{p}|n\rangle = \hbar n|n\rangle$, where n is an integer. The matrix U couples $|n\rangle$ mainly to those states $|m\rangle$ that are within the range of a "kick." The number of these states is of the order $2k$ where

$$k = \frac{K}{\hbar} . \tag{3.3}$$

The eigenstates of U are defined via the eigenvalue equation

$$\hat{U}|r\rangle = e^{-i\omega_r}|r\rangle . \tag{3.4}$$

The eigenvalues are $e^{-i\omega_r}$ where ω_r are the quasienergies. For $\hbar = 4\pi(\mathcal{M}/\mathcal{N})$, where $\mathcal{N}$ and $\mathcal{M}$ are relatively prime integers, the matrix (3.2) is invariant under translations $n \to n + \mathcal{N}$ and therefore the eigenstates are then extended Bloch states in the n representation. However, for large $\mathcal{N}$ this periodic structure is nonapparent in any local study of the dynamics. For generic $\hbar$ there is no periodicity. We tend then to make the conjecture that

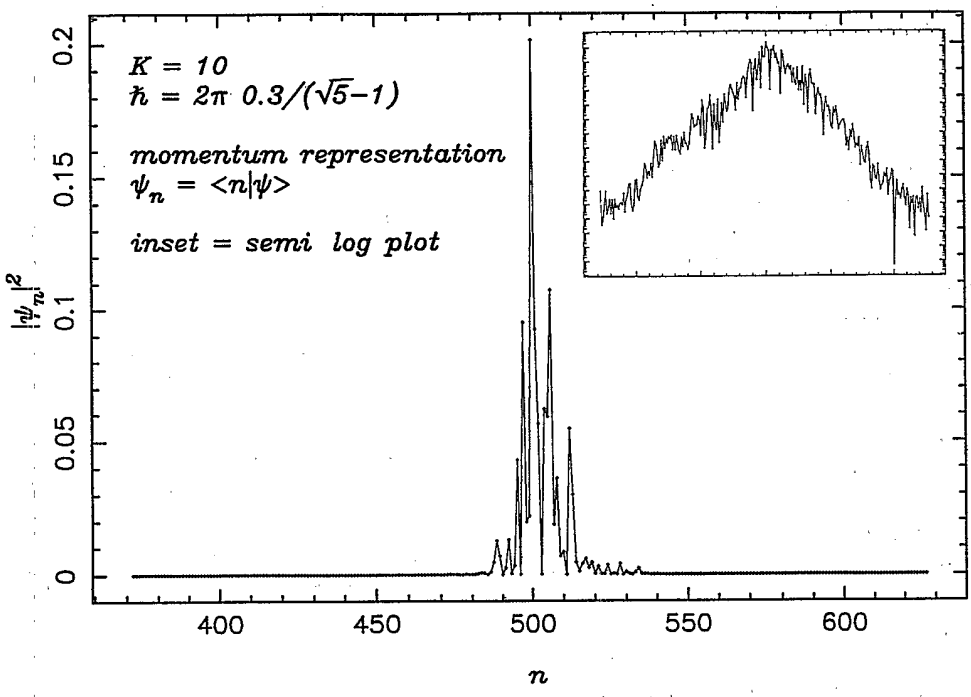

FIG. 1. A typical eigenstate of U that has been found numerically. A truncated 256×256 matrix has been diagonalized.

the eigenstates of U are of the same nature as the eigenstates of a generic banded matrix, namely that these states are localized in n space:

$$|\langle n|r\rangle| \sim e^{-|n-\Lambda_r|/\xi_r} , \tag{3.5}$$

where Λ_r and ξ_r are the localization center and localization length of the eigenstates $|r\rangle$, respectively. The nature of this localization is the subject of extensive study initiated by Fishman, Grempel, and Prange [2]. Figure 1 introduces a typical eigenstate that has been found via direct diagonalization of U. Note the simple exponential structure of localization.

Recently, Dittrich and Smilansky [8] pointed out that Mott's picture of localization [9] is applicable also to the investigation of localization in the QKR problem. Consider the eigenvalue equation (3.4) in momentum representation. An approximation eigenstate basis $|\alpha\rangle$ may be found by dividing U into blocks that are large compared with ξ, ignoring coupling between different blocks, and diagonalizing each block separately. In terms of solid-state physics it means that the solid is assumed to be cut into segments and hopping between these segments is neglected. Let us assume that the states $|\alpha\rangle$ are indeed localized states with a simple localization structure as in (3.5). We now "turn on" the interaction between states of different blocks and try to find the "true" eigenstates $|r\rangle$. Following closely the presentation of Mott's picture by Sivan and Imry [9], we assume that the interaction between two approximate eigenstates $|\alpha\rangle$ and $|\beta\rangle$ is

$$|\langle\alpha|\hat{U}|\beta\rangle| \sim \Delta e^{-|\Lambda_{\alpha\beta}|/\xi} , \tag{3.6}$$

where $\Lambda_{\alpha\beta}=\Lambda_\alpha-\Lambda_\beta$. Mott's heuristic picture does not enable one to fix Δ, but evidently it is of the order $1/\xi$. The interaction (3.6) is significant only if a resonance condition is satisfied, namely

$$|\omega_{\alpha\beta}| \ll \Delta e^{-|\Lambda_{\alpha\beta}|/\xi} , \tag{3.7}$$

where $\omega_{\alpha\beta}\equiv\omega_\alpha-\omega_\beta$ and we use in what follows the convention $-\pi\ll\omega_{\alpha\beta}\ll\pi$. If (3.7) holds then the states $|\alpha\rangle$ and $|\beta\rangle$ are almost degenerate and we may apply the standard approach of degenerate perturbation theory. Due to the interaction (3.6) the (almost) degeneracy is removed. The true eigenstates are

$$\begin{aligned} |r\rangle &= \frac{1}{\sqrt{2}}(|\alpha\rangle+|\beta\rangle) , \\ |s\rangle &= \frac{1}{\sqrt{2}}(|\alpha\rangle-|\beta\rangle) . \end{aligned} \tag{3.8}$$

If the inequality (3.7) is not strongly satisfied then (3.8) holds only approximately. These double-hump states each have two centers of localization and the separation in their quasienergies is

$$|\omega_{rs}| = \Delta e^{-|\Lambda_{\alpha\beta}|/\xi} . \tag{3.9}$$

The dipole matrix element between such states is exceptionally large, namely

$$\langle s|\hat{p}|r\rangle = \tfrac{1}{2}(\langle p\rangle_\alpha-\langle p\rangle_\beta)\equiv\tfrac{1}{2}\hbar\Lambda_{\alpha\beta} . \tag{3.10}$$

It follows that

$$\langle s|\hat{p}|r\rangle \cong \tfrac{1}{2}\hbar\xi\ln\left[\frac{\Delta}{|\omega_{rs}|}\right] . \tag{3.11}$$

Thus, if the quasienergy difference is small then we expect a large dipole matrix element. Figure 2 introduces a pair of double-hump states that have been found via direct diagonalization of U.

We turn now to evaluate the density of double-hump state pairs with respect to quasienergy difference ω_{rs}. Let $|\alpha\rangle$ be an approximate eigenstate and $|\beta\rangle$ some other approximate eigenstate that belongs to a *different* "block." On the average, the density of states $|\beta\rangle$ with respect to $\Lambda_{\beta\alpha}$ and $\omega_{\beta\alpha}$ is

$$\frac{\delta\mathcal{N}}{\delta\omega\delta\Lambda}\equiv\frac{1}{\mathcal{N}}\sum_\alpha\sum_{\beta\,(\neq\alpha)}\delta(\omega-\omega_{\alpha\beta})\delta(\Lambda-\Lambda_{\alpha\beta})=\frac{1}{2\pi} , \tag{3.12}$$

where $\mathcal{N}$ is the dimension of the basis which is used in the computation (formally one should take the limit $\mathcal{N}\to\infty$).

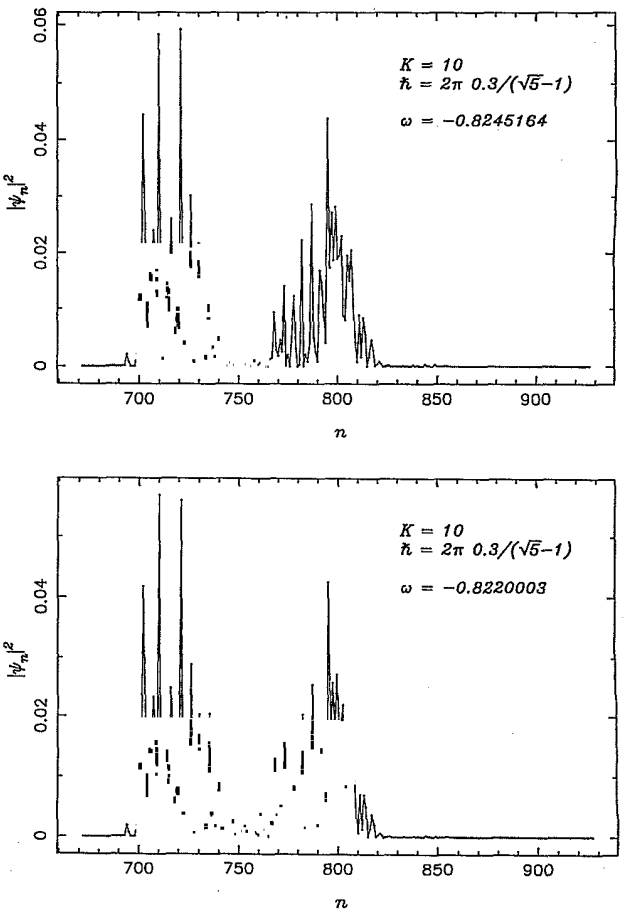

FIG. 2. A typical pair of double-hump states that has been found as in Fig. 1; the parameters are the same. The dipole matrix element between these states is $|\langle s|\hat{n}|r\rangle|\approx 39$.

The result (3.12) follows simply from the observation that there is no interaction between states that belong to different blocks. Expression (3.12) does not hold when Λ is of the order ξ since if $|\alpha\rangle$ and $|\beta\rangle$ belong to the *same* block, one should take into consideration the effect of level repulsion. The number of states $|\beta\rangle$ that are in resonance with $|\alpha\rangle$ such that $\Lambda<|\Lambda_{\alpha\beta}|<\Lambda+\delta\Lambda$ is obtained by integrating (3.12) over the domain that is defined by this restriction and by the resonance condition (3.7). One obtains $\delta\mathcal{N}=4(1/2\pi)\Delta e^{-|\Lambda|/\xi}\delta\Lambda$. The corresponding quasienergy difference ω_{rs} satisfies $\omega<|\omega_{rs}|<\omega+\delta\omega$ where $\omega=\Delta e^{-\Lambda/\xi}$ and $\delta\omega=\Delta e^{\Lambda/\xi}(\delta\Lambda/\xi)$. It follows that the density of double-hump states pairs with respect to the quasienergy difference $\delta\omega$ is

$$\left.\frac{\delta\mathcal{N}}{\delta\omega}\right|_{\text{res}}\equiv\frac{1}{\mathcal{N}}\sum_r\sum_s\delta(\omega-\omega_{sr})=\frac{\xi}{\pi}\ . \tag{3.13}$$

The apparently missing factor of 2 is due to the possibility of ω being either positive or negative within $[-\pi,\pi]$.

B. Dynamics

The one-step map that is generated by the one-step propagator if one adopts Heisenberg's picture of the dynamics is simply

$$\begin{aligned}\hat{x}(t)&=\hat{x}(t-1)+\hat{p}(t-1)\ ,\\ \hat{p}(t)&=\hat{p}(t-1)+K\sin\hat{x}(t)\ .\end{aligned} \tag{3.14}$$

It is the quantized version of the classical standard map. The latter has been studied extensively by Chirikov [27] and followers [28]. As the value of K is increased the classical dynamics follow the Kolmogorov-Arnold-Moser (KAM) scenario. For $K_c<K$ ($K_c\approx0.9716$), the last KAM trajectories that bound diffusion in momentum are already destroyed. We assume from now on that $1\ll K$. Almost all the classical trajectories that are generated by the standard map are chaotic, i.e., reveal a sensitive dependence for change in initial conditions. A small change ε in initial conditions results in exponential deviation which grows like $\varepsilon e^{\gamma t}$ where γ is known as the Lyapunov exponent. The inverse of γ constitutes a time scale for the loss of "memory." For the standard map [28]

$$t_L\equiv\frac{1}{\gamma}=\frac{1}{\ln(K/2)}\ . \tag{3.15}$$

For $1\ll K$ memory is lost on a time scale of order unity (smaller time scales are meaningless). The trajectory, once projected onto momentum space, is very similar to a random-walk process. From the map (3.14) it follows that

$$p(t)-p(0)=\sum_{\tau=1}^{t}K\sin x(\tau)\ . \tag{3.16}$$

Therefore

$$E(t)=K^2\sum_{\tau'=1}^{t}\sum_{\tau''=1}^{t}C_s(\tau'-\tau'')\ , \tag{3.17}$$

where the dispersion (energy growth) function is

$$E(t)\equiv\langle\langle[p(t)-p(0)]^2\rangle\rangle\ , \tag{3.18}$$

and the sine-correlation function, which depends only on the absolute value of the time difference, is

$$C_s(\tau'-\tau'')\equiv\langle\langle\sin x(\tau')\sin x(\tau'')\rangle\rangle\ . \tag{3.19}$$

The notation $\langle\langle\ \rangle\rangle$ stands for the uniform statistical average over all initial conditions, namely

$$\langle\langle\mathcal{O}\rangle\rangle\equiv\frac{\frac{1}{2\pi\hbar}\int\int dx\,dp\,\mathcal{O}(x,p)}{\frac{1}{2\pi\hbar}\int\int dx\,dp}\ . \tag{3.20}$$

Since the standard map is invariant for 2π translations both in x and p, it follows that it is sufficient to perform this average over the rectangular $[0,2\pi]\times[0,2\pi]$.

Due to the chaotic nature of the dynamics, dynamical correlations decay exponentially on a time scale t_L. Rechester *et al.* [18] calculated explicitly the first few correlations:

$$\begin{aligned}C_s(0)&=\tfrac{1}{2}\ ,\\ C_s(1)&=0\ ,\\ C_s(2)&=-\tfrac{1}{2}\mathcal{J}_2(K)\ .\end{aligned} \tag{3.21}$$

It follows that $E(t)$ is a monotonic function that grows diffusively. The diffusion rate is

$$D(t)=E(t+1)-E(t)=K^2\sum_{\tau=-t}^{t}C_s(\tau)\ . \tag{3.22}$$

Its initial value is $D(t=0)=\frac{1}{2}K^2$, while its asymptotic value is

$$D_{\text{cl}}(K)=D(t=\infty)\cong\tfrac{1}{2}K^2[1-2\mathcal{J}_2(K)]\ . \tag{3.23}$$

For $5\leq K$ it is already a very good approximation.

The quantum-mechanical time evolution is generated by the one-step propagator U. We shall consider in this paper mainly the generic behavior that is generated by this propagator. Nongeneric dynamical features are (a) quantum resonances and (b) classical diffusion. Quantum resonances appear if $\hbar$ has a nongeneric value. If $\hbar=4\pi(\mathcal{M}/\mathcal{N})$, where $\mathcal{M}$ and $\mathcal{N}$ are integers, then the eigenstates of U are extended in momentum space and the dispersion function $E(t)$ may have an asymptotic quadratic behavior. This resonant behavior is nongeneric.

The second non-generic dynamical behavior is classical diffusion. It lasts a time t_E where t_E is Ehrenfest's correspondence time that is determined as follows. The larger classical structures of the standard map have momentum scale 2π. An initial simple distribution of points in phase space will develop structure of finer momentum scale $2\pi e^{-\gamma t}$ as time elapses. When this structure is of the order $\hbar$ classical to quantal correspondence is broken. This leads to

$$t_E=\frac{1}{\gamma}\ln\left[\frac{2\pi}{\hbar}\right]=\left[\ln\left[\frac{2\pi}{\hbar}\right]\right]t_L\ . \tag{3.24}$$

In most of the numerical experiments one uses relatively large values of $\hbar$ such that t_E is of the order 1. Therefore classical diffusive behavior is not apparent in such experiments. The initial diffusive behavior that is observed *does not correspond to the classical diffusive behavior* and is to be discussed in what follows. A discussion of the (semi)classical diffusive behavior which is apparent in the limit $\hbar \to 0$ will not be presented here since it requires further study.

Consider a rotator that is prepared initially in a momentum eigenstate $|p\rangle$. By iterating its state with U it is found that for short time a stochasticlike diffusive behavior is followed, but on a larger time scale a quasiperiodic behavior becomes apparent. This is simply because $|p\rangle$ may be approximated to be a superposition of roughly ξ quasienergy eigenstates, whose quasienergies are distributed in the interval $[-\pi,\pi]$ with typical spacing $\Delta\omega \sim 2\pi/\xi$. One may define a break time $t^* \sim 2\pi/\Delta\omega \sim \xi$. On a time scale small compared with t^* localization does not manifest itself. Since each kick couples k momentum eigenstates, successive iterations lead to stochasticlike diffusion with the initial rate $d_0 \approx \frac{1}{2}k^2$. (The momentum here is measured in natural units, otherwise this value should be multiplied by $\hbar^2$.) However, on a time scale larger than t^* the quasiperiodic behavior becomes apparent. A general argument [10] that is based on a scaling hypothesis [7] relates the initial diffusion rate to the localization length ξ of the quasienergy eigenstates, namely $d_0 \approx (1/\alpha)\xi$ with $\alpha = \frac{1}{2}$ for the QKR. In our system of units this relation will be written in the form

$$\xi = \alpha \frac{D_0}{\hbar^2} , \tag{3.25}$$

where D_0 is the initial diffusion rate in p space within the first few iterations. It should be emphasized that semiclassical considerations are not required in the derivation of this result. One should be aware of the unessential notion of "classical diffusion" that appears frequently in this context.

In order to quantify the qualitative picture of the dynamics we should adopt a statistical approach as in the classical analysis. It is natural to define the energy function by averaging over initial conditions, namely

$$E(t) \equiv \frac{1}{\mathcal{N}} \sum_p \langle p | [\hat{p}(t) - p]^2 | p \rangle . \tag{3.26}$$

Here the average is taken over initial states $|p\rangle$. However, this definition may be written in a way that emphasizes that in the semiclassical limit it reduces to the classical definition. We shall use the notation

$$E(t) \equiv \langle\langle [\hat{p}(t) - \hat{p}(0)]^2 \rangle\rangle . \tag{3.27}$$

From now on $\langle\langle\ \rangle\rangle$ denotes quantal statistical average, namely

$$\langle\langle \hat{\mathcal{O}} \rangle\rangle \equiv \frac{1}{\mathcal{N}} \mathrm{tr}(\hat{\mathcal{O}}) , \tag{3.28}$$

where $\mathcal{N}$ denotes the dimension of the basis that is used to compute the trace. In the semiclassical limit this definition indeed reduces to (3.18) since the trace operation then reduces to integration over phase space cells.

Figure 3 presents the results of numerical simulations. The initial state of the rotator has been iterated using the one-step propagator

$$\hat{U} = \exp[-iT(\hat{n})]\exp(-ik\cos\hat{x}) , \tag{3.29}$$

where $T(n) = \frac{1}{2}\hbar n^2$, $n = 0, \pm 1, \pm 2, \ldots$, and $p = \hbar n$. The solid curve illustrates the time evolution of $\langle (p - p_0)^2 \rangle$ for a rotator that has been prepared initially in a zero-momentum eigenstate ($p_0 = 0$). The truncation of the basis was performed accordingly (symmetrical with respect to the initial momentum p_0). After averaging over 1000 initial conditions the energy curve looks quite smooth (not shown in the figure). After averaging over 10 000 initial conditions (dashed line) the energy curve becomes somewhat smoother, but if one observes carefully, it is revealed that there is still a slow "oscillatory" structure that is stable against the averaging procedure. This structure is specific for the parameter we use. By changing $\hbar$ but keeping k fixed we may alter this structure without affecting much the other features of the curve that are determined by $\xi \approx \frac{1}{4}k^2$. Thus in order to obtain numerically the generic functional form of $E(t)$ we should average not only over initial conditions but also over different parametric realizations of the model. Actually, a much more effective procedure to obtain this generic behavior exists. The diffusion process is not of classical nature and does not depend on either the features or (even) the existence of the underlying classical model. Therefore the kinetic term $T(n)$ may be any generic function or even a realization of a random sequence of phases. The dotted curve has been obtained by averaging over 10 000 realizations of $T(n)$ while k is kept fixed. In order to overlay the obtained plot of $\langle\langle (n - n_0)^2 \rangle\rangle$ on the same figure we scaled it by $\hbar^2$, where $\hbar$ is the same

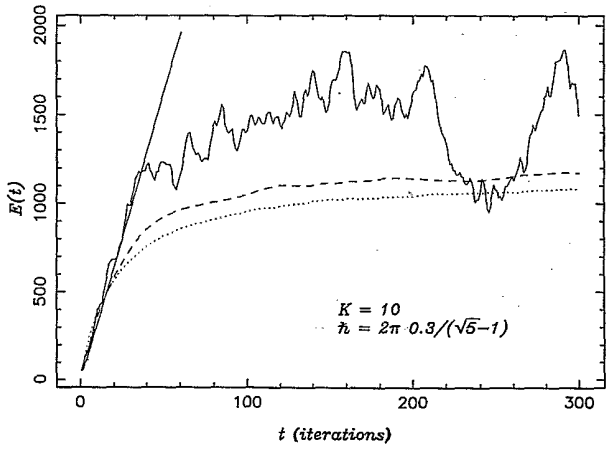

FIG. 3. The dispersion $\langle (p - p_0)^2 \rangle$ as a function of time. Solid line, the initial preparation is $p_0 = 0$; dashed line, average over 10 000 initial momenta p_0 has been taken; dotted line, average over 10 000 realizations of the kinetic term has been performed; straight line, classical behavior.

value used in the QKR simulations.

Using the Heisenberg picture of the dynamics it is obvious that relations (3.16), (3.17), and (3.22) hold also quantum mechanically. The few first correlations (3.21) have been calculated by Shepelyansky [10] and it was found that a very good approximation is to replace K by $[(2/\hbar)\sin(\hbar/2)]K$. Thus

$$D_0 \approx D_{\rm cl}\left[\left[\frac{2}{\hbar}\sin\frac{\hbar}{2}\right]K\right] . \tag{3.30}$$

It is obvious, however, that unlike the classical case, the other correlations cannot be neglected since, due to localization,

$$\lim_{t\to\infty} D(t) = \sum_{\tau=-\infty}^{\infty} C_s(\tau) = 0 . \tag{3.31}$$

It follows that $C_s(\tau)$ should have a negative tail that compensates exactly the contribution of the first few correlations which lead to (3.30).

IV. DYNAMICAL CORRELATIONS AND SPECTRAL FUNCTIONS

The spectral function of a dynamical variable $\hat{X}$ is defined as follows:

$$C_X(\omega) \equiv \frac{1}{\mathcal{N}} \sum_{r,s}{}' |\langle s|\hat{X}|r\rangle|^2 2\pi\delta(\omega-\omega_{rs}) , \tag{4.1}$$

where the prime denotes omission of the diagonal terms in the summation. This spectral function is the weighted density of states $|s\rangle$ with respect to their quasienergy difference from a reference state $|r\rangle$, averaged over the reference state $|r\rangle$. In particular we shall consider the following spectral functions: $C_p(\omega)$, $C_s(\omega)\equiv C_{\sin x}(\omega)$, and $C_c(\omega)\equiv C_{\cos x}(\omega)$. The Fourier transform of $C_X(\omega)$ will be denoted by $C_X(\tau)$. One easily obtains

$$C_X(\tau) = \frac{1}{\mathcal{N}} \sum_{r,s}{}' \langle r|\hat{X}(\tau)|s\rangle\langle s|\hat{X}(0)|r\rangle . \tag{4.2}$$

This result may be written in a more elegant fashion which stresses that $C_X(\tau)$ is actually the proper definition for the autocorrelation function of the dynamical variable $\hat{X}$,

$$C_X(\tau) = \langle\langle \hat{X}(\tau)\hat{X}(0)\rangle\rangle - \langle\langle \hat{X}\rangle^2\rangle . \tag{4.3}$$

The notation $\langle\langle\ \rangle\rangle$ stands for the quantum statistical average (3.28) and we use also the notation

$$\langle\langle \hat{X}\rangle^2\rangle \equiv \frac{1}{\mathcal{N}} \sum_{r} |\langle r|\hat{X}|r\rangle|^2 . \tag{4.4}$$

We shall be interested in particular in the autocorrelation functions of the dynamical variables $\hat{p}$, $\sin\hat{x}$, and $\cos\hat{x}$, which will be denoted by $C_p(\tau)$, $C_s(\tau)$, and $C_c(\tau)$, respectively. Note that due to the symmetry of U under reflections ($x\to -x$) it follows that $\langle\langle \sin x\rangle^2\rangle = 0$ and therefore the present definition (4.3) for $C_s(\tau)$ agrees with (3.19).

There is a transparent relationship between the energy function $E(t)$ and the momentum autocorrelation function $C_p(\tau)$,

$$\begin{aligned} E(t) &= \langle\langle [p(t)-p(0)]^2\rangle\rangle \\ &= 2[\langle\langle p^2\rangle\rangle - \langle\langle p(t)p(0)\rangle\rangle] \\ &= 2[C_p(0)-C_p(t)] , \end{aligned} \tag{4.5}$$

where we used the cyclic property of the trace operation. We shall see later that $C_p(t)$ decays to zero for $t\to\infty$, therefore $E(\infty)=2C_p(\tau=0)$ is the saturation energy. It follows from the definition (4.3) that $C_p(\tau=0)$ is simply the average dispersion of the localized states, namely

$$C_p(\tau=0) \equiv \frac{1}{\mathcal{N}} \sum_{r} \langle (p-\langle p\rangle_r)^2\rangle_r . \tag{4.6}$$

If all the states have a simple exponential structure with the same localization length $\xi=\xi_0$ then their dispersion is $\frac{1}{2}(\hbar\xi_0)^2$ and therefore

$$E(\infty) = 2C_p(0) \approx (\hbar\xi_0)^2 . \tag{4.7}$$

Actually the assumption that leads to (4.7) is not correct. Indeed, all the states decay exponentially *asymptotically,* yet they may possess large plateau regions and/or multihump structures. As a result expression (4.7) underestimates $C_p(\tau=0)$. Numerical studies indicate [10] that a quite good approximation is to substitute $\xi_0=2\xi$ in (4.7), where ξ is the localization length which is determined by the asymptotic behavior. We turn now to find the global behavior of $C_p(t)$ by inspection of its Fourier transform $C_p(\omega)$.

The energy function $E(t)$ is a monotonic ascending function that saturates on a time scale t^*. It follows that $C_p(t)$ is a positive decreasing function that decays to zero on a time scale t^*. Therefore the spectral function $C_p(\omega)$ is peaked within a frequency range $2\pi/t^*$ around $\omega=0$. This feature of $C_p(\omega)$ reveals that the matrix $\langle s|\hat{p}|r\rangle$ is, in a sense, banded. Namely, if one uses $e^{i\omega_r}$ and $e^{i\omega_s}$ to order the rows and columns of $\langle s|\hat{p}|r\rangle$ then one finds that the off-diagonal matrix elements decay as one goes away from the diagonal. This result depends only on the observation that U is a generic banded matrix in the p representation whose eigenstates are localized in this representation. We have not used any other special features of U in order to derive this result. Our claim may be rephrased as follows: If a unitary operator $\hat{U}$ is generically banded in the $\hat{n}$ representation, then $\hat{n}$ is banded in the $\hat{U}$ representation. A similar statement has been introduced [30] concerning pairs of Hermitian operators using *semiclassical* considerations.

In the absence of dynamical correlations $C_s(\tau=0)=\frac{1}{2}$ leads to $C_s(\omega)=\frac{1}{2}$. It had been argued at the end of Sec. III that this result does not hold, since due to localization $C_s(\tau)$ has a negative tail that compensates exactly the trivial correlation $C_s(\tau=0)=\frac{1}{2}$. Namely, (3.31) leads to

$$\lim_{\omega\to 0} C_s(\omega) = 0 . \tag{4.8}$$

The negative tail of $C_s(\tau)$ ranges on a time scale t^* and therefore all the nontrivial correlations of $C_s(\tau)$ are of or-

der $1/t^*$. Evidently they affect mainly the behavior of $C_s(\omega)$ within a frequency range $2\pi/t^*$ around $\omega=0$. Outside this frequency range their contribution to $C_s(\omega)$ is of the order $1/t^*$, namely

$$|C_s(\omega)-\tfrac{1}{2}|<\mathcal{O}\left[\frac{1}{t^*}\right] \quad \text{for } \frac{\pi}{t^*}<|\omega| \ . \tag{4.9}$$

From the relations (3.17) and (4.5) it follows that

$$K^2C_s(\tau)=2C_p(\tau)-C_p(\tau+1)-C_p(\tau-1) \ , \tag{4.10}$$

which leads to

$$K^2C_s(\omega)=2(1-\cos\omega)C_p(\omega) \ . \tag{4.11}$$

Using (4.9) in the relation (4.11) it follows that

$$C_p(\omega)\approx\frac{1}{2(1-\cos\omega)}\tfrac{1}{2}K^2 \quad \text{for } \frac{\pi}{t^*}<|\omega| \ . \tag{4.12}$$

Assuming that all the states have momentum dispersion which is roughly $\hbar\xi_0$, it follows that $|\langle r|p|s\rangle|^2$ is at most of the order $\frac{1}{2}(\hbar\xi_0)^2$. Since there are roughly ξ_0 eigenstates that overlap with any reference state one obtains

$$C_p(\omega)\le\xi_0\tfrac{1}{2}(\hbar\xi_0)^2=\tfrac{1}{4}K^2\xi_0^2 \ , \tag{4.13}$$

where we have used the definition (4.1) and the relation (3.25) has been applied in the last step. Finally, since (4.7) holds, it follows that $C_p(\omega)$ has to satisfy the sum rule

$$\frac{1}{2\pi}\int C_p(\omega)d\omega\approx\tfrac{1}{2}(\hbar\xi_0)^2 \ . \tag{4.14}$$

It is obvious that the simplest functional form that is consistent with (4.12)–(4.14) is

$$C_p(\omega)|_{\text{nonres}}=\frac{1}{2(1-\cos\omega)+(1/t^*)^2}\tfrac{1}{2}K^2 \ , \tag{4.15}$$

where $t^*\approx\xi_0\approx2\xi$. Note that we assume all the time $1\ll K$ and therefore $1\ll t^*$ holds. This function is roughly a Lorentzian. A Fourier transform (Appendix B) leads, after simplification, to an exponential behavior, namely

$$C_p(\tau)|_{\text{nonres}}=\tfrac{1}{4}K^2t^*e^{-|\tau|/t^*} \ . \tag{4.16}$$

Equation (4.15) evidently does not hold for $\omega\ll1/t^*$ since due to local repulsion of levels we should expect $C_p(\omega)|_{\text{nonres}}$ to vanish when $\omega\to0$. However, in this regime $C_p(\omega)$ is dominated by the singular contribution of the resonant term to be discussed now.

We turn to evaluate the contribution of the double-hump states to the spectral function $C_p(\omega)$. This contribution is singular since $|\langle s|p|r\rangle|$ is not bounded by $\hbar\xi$. Using results (3.11) and (3.13) we find that

$$C_p(\omega)|_{\text{res}}=\tfrac{1}{2}\hbar^2\xi^3\ln^2\left|\frac{\Delta}{\omega}\right| \quad \text{for } |\omega|<\Delta \tag{4.17}$$

and $C_p(\omega)|_{\text{res}}=0$ for $\Delta<|\omega|$. The Fourier transform can be calculated analytically (Appendix B), and leads to

$$C_p(\tau)|_{\text{res}}\cong\tfrac{1}{2}\hbar^2\xi^3\frac{1}{\tau}\ln|\Delta\tau| \quad \text{for } \xi<|\tau| \tag{4.18}$$

and is approximately constant for $\tau<\xi$. Obviously for $\tau<\xi$ the nonresonant part of $C_p(\tau)$ dominates. We note that, using the relation (4.11), one obtains $C_s(\omega)|_{\text{res}}\cong\frac{1}{8}\xi^2\omega^2\ln^2|\Delta/\omega|$ for $|\omega|<\Delta$. Thus, in spite of the singular contribution of the double-hump states to $C_p(\omega)$, the requirement (4.8) that is necessary for dynamical localization effect is satisfied.

We found that the spectral function $C_p(\omega)$ reveals apparent level attraction while $C_s(\omega)$ reveals apparent level repulsion. We use the term "apparent" to emphasize the significance of the weighting procedure. The global spectrum is, of course, of Poisson type [31,8]. Finally, the results obtained in this section do not depend on the functional form of the kinetic term $T(n)$ that appears in (3.29). The kinetic term may be any generic function or even a realization of a random sequence of phases.

We turn back to study the functional form of the dynamical crossover to localization in order to test our predictions. From relations (4.16), (4.5), and (3.22) it follows that, for short time of order t^*,

$$D(t)\approx D_0e^{-t/t^*} \ , \tag{4.19}$$

with $D_0\approx\frac{1}{2}K^2$ and $t^*\approx2\xi$. This result (after a Fourier transform) bear some resemblance to Drude's formula for the frequency-dependent conductivity. On a time scale larger than t^* there is a crossover to power-law behavior

$$D(t)\approx cD_0\left[\frac{t^*}{t}\right]^{1+\beta} \ , \tag{4.20}$$

where $c\approx\frac{1}{8}\ln(t/t^*)$ and $\beta=1$. We intentionally absorbed the logarithmic dependence in the prefactor since it varies very slowly and cannot be detected in actual numerical investigation.

Figure 4 presents the numerical result for $D(t)$ that is

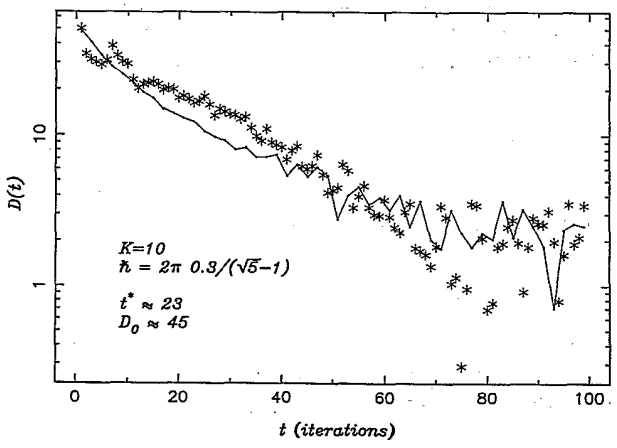

FIG. 4. The diffusion rate $D(t)$ as a function of time. The stars correspond to the dashed line of Fig. 3, while the dots (joined by a line) correspond to the dotted line of Fig. 3.

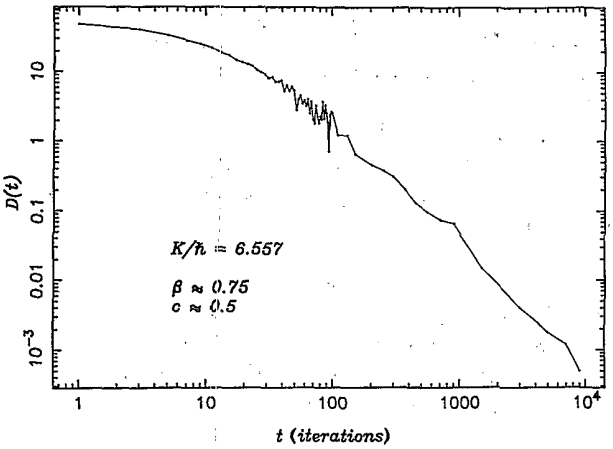

FIG. 5. The same plot as in Fig. 4, but on a log-log scale.

extracted on the basis of Fig. 3. The stars and the dots correspond to the dashed and dotted lines of Fig. 3, respectively. The curve that joins the dots is to guide the eye. Standard fitting procedure to (4.19) yields $D_0=45$ and $t^*=23$. Figure 5 is a log-log plot of the same results on a longer time scale. The "stars" are not presented since the oscillatory structure that we observed when we looked on the dashed line of Fig. 3 does not permit a reliable evaluation of $D(t)$ within this range. A fitting to the power-law behavior (4.20) yields $c\approx 0.5$ and $\beta\approx 0.75$. The error that may be in the numerical determination of β is ± 0.5. We cannot explain the deviation from the expected value $\beta=1$. It may be due to a long transient behavior. Therefore, for the purpose of later analysis, we shall assume

$$C_p(\tau)|_{\rm res}=\frac{c}{2\beta}D_0 t^*\left|\frac{\tau}{t^*}\right|^{-\beta} \quad \text{for } t^*<|\tau| \tag{4.21}$$

rather than (4.18). If β is indeed less than 1, it implies that the singular behavior of $C_p(\omega)$ around $\omega=0$ is worse than logarithmic divergence.

V. DESTRUCTION OF COHERENCE BY NOISE

A. Heuristic considerations

Following Ref. [23] we consider the first destruction of coherence in the QKR-p model in the absence of kicks. Consider a rotator that is prepared initially in a pure state that constitutes a superposition $(1/\sqrt{2})(|p_1\rangle+|p_2\rangle)$. Wigner's function is

$$\rho_{t=0}(x,p)=\tfrac{1}{2}\rho^{(1)}_{t=0}+\tfrac{1}{2}\rho^{(2)}_{t=0}+\rho^{\rm int}_{t=0}\ , \tag{5.1}$$

where

$$\rho^{(i)}_{t=0}(x,p)=\frac{1}{2\pi}\delta_{p,p_i},\quad i=1,2 \tag{5.2}$$

and

$$\rho^{\rm int}_{t=0}(x,p)=\frac{1}{2\pi}\delta_{p,(p_1+p_2)/2}\cos\left[\frac{x}{\delta x_c}\right], \tag{5.3}$$

with $\delta x_c=\hbar/|p_2-p_1|$. More generally, a superposition of momentum eigenstates which is localized in momentum within some range $\Delta p=\hbar\xi_0$ results in an interference part with a variation on the spatial scale $\delta x_c=1/\xi_0$. Iteration with (2.8) and (2.12) affects only the interference part. Due to the noise the interference part is smeared on a spatial scale $\delta x(t)$, which is given by

$$\delta x^2(t)=\sum_{\tau'=1}^{t}\sum_{\tau''=1}^{t}\phi(\tau',\tau'')\ , \tag{5.4}$$

where $\phi(\tau',\tau'')$ is defined as in (2.4). For Gaussian noise one obtains [23]

$$\rho^{\rm int}_t(x,p)\approx\exp\left[-\frac{1}{2}\left[\frac{\delta x(t)}{\delta x_c}\right]^2\right]\rho^{\rm int}_{t=0}(x-pt,p)\ . \tag{5.5}$$

A natural definition for the coherence time is therefore the time when $\delta x(t)$ becomes of the order δx_c. If $\sum_{-\infty}^{\infty}\phi(\tau)\neq 0$ then after a transient the interference part decays exponentially with a decay rate which is roughly

$$\Gamma\approx\tfrac{1}{2}\xi_0^2\sum_{\tau=-\infty}^{\infty}\phi(\tau)\ . \tag{5.6}$$

The prototype example is white noise that has the autocorrelation function $\phi(t-t')=\sigma\delta_{t,t'}$. The coherence time is then

$$t_c\equiv\frac{1}{\Gamma}=\left[\frac{1}{\xi_0}\right]^2\frac{2}{\sigma}$$

for an undriven rotator and σ noise . (5.7)

A completely different behavior is found if $\sum_{-\infty}^{\infty}\phi(\tau)=0$, which characterizes zero-temperature noise. Then we find nonexponential anomalous decay behavior. An extreme example is λ noise, namely $\phi(t-t')=2\lambda\delta_{t,t'}-\lambda\delta_{|t-t'|,1}$. If λ is small $[\lambda\ll(1/\xi_0)^2]$ then coherence is not destroyed

$$t_c=\infty \quad \text{for an undriven rotator and } \lambda \text{ noise .} \tag{5.8}$$

A less extreme example is η noise that has the autocorrelation function $\phi(t-t')=(2\pi/6)\eta\delta_{t,t'}-(\eta/\pi)[1/(t-t')^2](1-\delta_{t,t'})$. The noise-induced spatial diffusion is logarithmic [23], namely $\delta x^2(t)\cong(2/\pi)\eta\ln(t/0.2)$. The interference then decays as a power law rather than exponentially. The coherence time is therefore exponentially long, namely

$$t_c=0.2\exp\left[\frac{\pi}{2}\left[\frac{1}{\xi_0}\right]^2\frac{1}{\eta}\right]$$

for an undriven rotator and η noise . (5.9)

We may try to extend this approach to deal with the QKR-x model. It is, however, difficult to introduce a simple formulation of the decay process as in the QKR-p

model. We therefore proceed heuristically. Following Ott, Antonsen, and Hanson [17] we argue that coherence time in the QKR-x model is simply the time it takes for the noise to "mix" neighboring momenta. The diffusion that is induced by the noise is $\delta p^2(t)=\nu t$ and the equation $\delta p(t)\sim\hbar$ leads to

$$t_c=\hbar^2\frac{1}{\nu} \quad \text{for an undriven rotator and } \nu \text{ noise} . \tag{5.10}$$

A faster mechanism for destruction of coherence is found in the QKP problem. Here $\rho(x,p)$ is defined on $\mathbb{R}\times\mathbb{R}$ where $\mathbb{R}$ are the real numbers. Since momentum is not quantized, the noise-induced diffusion in momentum is associated with spatial spreading. As in the QKR-p model the interference part of Wigner's function is smeared, but on a spatial scale $\delta x(t)\sim\delta p(t)t$. The time it takes to spread over a spatial scale $\delta x\sim 1/\xi_0$ and as a result to destroy coherence, is found via the equation $\nu t^3\sim(1/\xi_0)^2$. Thus

$$t_c=\left[\left[\frac{1}{\xi_0}\right]^2\frac{1}{\nu}\right]^{1/3} \quad \text{for an undriven particle and } \nu \text{ noise} , \tag{5.11}$$

which is much shorter than (5.10).

It is evident that coherence time in the presence of kicks is not necessarily the same as in the case of an undriven system. If the system is treated classically, and is known to be *chaotic*, then noise autocorrelations are of little importance. Due to the exponential instability of the phase-space trajectories, we expect no *memory* for noise autocorrelations. However, if the dynamics is treated within the framework of quantum mechanics, then one may expect manifestation of the long-range dynamical correlations, which has been discussed in Sec. IV. To make this point less abstract we note first a result to be derived later in this section: For white-noise expressions, (5.10) and (5.7) with $\xi_0=2\xi$ also apply in the presence of kicks. Consider now the colored λ noise. The coherence time for the undriven system is $t_c=\infty$. However, we have shown (Sec. II C) that the QKR-p model with λ noise is essentially equivalent to the QKR-x model with white ν noise where $\nu=\frac{1}{2}K^2\lambda$. It follows then from (5.10), after using (3.25), that $t_c=(1/\xi)(1/2\lambda)$ for a kicked rotator and λ noise. Thus, for λ noise, coherence time is not infinite, as suggested by an inspection of the undriven system. On the other hand, coherence time is not the same as for white noise (that has the same variance), as suggested by an inspection of the classical limit, but rather a factor 2ξ larger.

B. Perturbation theory

We are therefore urged to develop a systematic formalism to overcome the natural limitations of the heuristic picture that has been presented up to now. The decay probability $P(t)$ of a quasienergy eigenstate as a function of time may be calculated using leading-order perturbation theory. We consider for simplicity a process that is represented by the unitary operator

$$\hat{U}^f\equiv e^{(i/\hbar)f\hat{X}}\exp\left[-\frac{i}{\hbar}K\cos\hat{x}\right]\exp\left[-\frac{i}{\hbar}\frac{1}{2}\hat{p}^2\right] , \tag{5.12}$$

where $\hat{X}$ is any dynamical variable of the system (e.g., $\hat{p}$, $\sin\hat{x}$, $\cos\hat{x}$). The transition probability from energy eigenstate $|r\rangle$ to energy eigenstate $|s\rangle$ after time t is

$$\text{Prob}(s|r)=|\langle s|\hat{U}^{f(t)}\cdots\hat{U}^{f(1)}|r\rangle|^2 .$$

A first-order expansion of U^f is $\hat{U}^f\approx[1+(i/\hbar)f\hat{X}]\hat{U}^0$. Thus, up to leading order

$$\text{Prob}(s|r)=\left|\sum_{\tau=1}^{t}e^{-i\omega_s(t-\tau)}\left\langle s\left|\frac{i}{\hbar}f(\tau)\hat{X}\right|r\right\rangle e^{-i\omega_r\tau}\right|^2$$
$$=\frac{1}{\hbar^2}\sum_{\tau'=1}^{t}\sum_{\tau''=1}^{t}e^{-i\omega_{sr}(\tau'-\tau'')}|\langle s|\hat{X}|r\rangle|^2 f(\tau')f(\tau'') . \tag{5.13}$$

This result should be averaged over realizations of $f(t)$. This average results in the replacement $f(\tau')f(\tau'')\to\phi(\tau',\tau'')$. The decay probability is obtained by summing over the final state $|s\rangle$ and averaging over the initial state $|r\rangle$

$$P(t)=\frac{1}{\mathcal{N}}\sum_r\sum_s{}' \text{Prob}(s|r) . \tag{5.14}$$

Using the Fourier transformed version of the definition (4.1), and performing a change of the summation variables to

$$\tau\equiv\tau'-\tau'' ,$$
$$t=\max(\tau',\tau'') , \tag{5.15}$$

one obtains that the decay probability as a function of time is

$$P(t)=\frac{1}{\hbar^2}\sum_{t'=0}^{t-1}\sum_{\tau=-t'}^{t'}C_X(\tau)\phi(\tau;t') , \tag{5.16}$$

while its time derivative $\dot{P}(t)\equiv P(t+1)-P(t)$ is

$$\dot{P}(t)=\frac{1}{\hbar^2}\sum_{\tau=-t}^{t}C_X(\tau)\phi(\tau;t) . \tag{5.17}$$

The notation $\phi(\tau;t)\equiv\phi(\tau',\tau'')$ for the autocorrelation function of the noise has been used. For stationary noise $\phi(\tau',\tau'')$ depends only on the difference $|\tau'-\tau''|$ and therefore $\phi(\tau;t)$ is independent of t. Equation (5.16) for $P(t)$ is the central result of this section. We turn to apply

it to study the decay of coherence in the QKR-p, QKR-x and QKP models, which have been introduced in Sec. II. These results will be compared with those of Sec. V A.

We consider first the destruction of coherence in the QKR models to be distinguished from the QKP problem. The noise is assumed to be stationary. The behavior of the decay probability after a short transient is

$$P(t)=\Gamma t \ , \tag{5.18}$$

where the decay rate is given by

$$\Gamma=\frac{1}{\hbar^2}\sum_{\tau=-\infty}^{\infty} C_X(\tau)\phi(\tau) \ . \tag{5.19}$$

The coherence time is defined as usual in the case of a decay process, namely

$$t_c\equiv\frac{1}{\Gamma} \ . \tag{5.20}$$

For future purposes we note that from (5.13) it follows that the transition rate from state $|s\rangle$ to state $|r\rangle$ is

$$W(r|s)=\frac{1}{\hbar^2}|\langle s|\hat{X}|r\rangle|^2\phi(\omega_{rs}) \ , \tag{5.21}$$

where $\phi(\omega)$ is the noise fluctuations spectrum, i.e., the Fourier transform of $\phi(\tau)$. Evidently, the sum rule

$$\Gamma=\frac{1}{\mathcal{N}}\sum_r{\sum_s}' W(r|s) \tag{5.22}$$

is satisfied.

Considering the QKR-p model, we find that $\Gamma=(1/\hbar^2)\sum_\tau C_p(\tau)\phi(\tau)$, rather than $\Gamma=\frac{1}{2}\xi_0^2\sum_\tau\phi(\tau)$, which applies if the kicks are absent [Eq. (5.6)]. Substituting (4.7) one finds that for white noise Γ is the same for the kicked rotator as for the undriven system, namely

$$\Gamma=\tfrac{1}{2}(t^*)^2\sigma \quad \text{for a kicked rotator and } \sigma \text{ noise} \ , \tag{5.23}$$

which agrees with (5.7). However, if the noise is colored, then it is not correct to ignore the effect of kicking on the dynamical correlations. In particular, zero-temperature noise for which $\sum_{-\infty}^{\infty}\phi(\tau)=0$ does not lead to $\Gamma=0$ and hence the anomalous dependence of t_c on noise intensity as in (5.9) and (5.8) is not found. Rather, for η noise one obtains

$$\Gamma\cong t^*(\text{const}+\ln t^*)\frac{\eta}{\pi}$$

$$\text{for a kicked rotator and } \eta \text{ noise} \ , \tag{5.24}$$

while for λ noise

$$\Gamma=t^*\lambda \quad \text{for a kicked rotator and } \lambda \text{ noise} \ , \tag{5.25}$$

in accordance with the *ad hoc* result that had been derived at the end of Sec. V A. Additional examples for computation of Γ are given in Appendix C.

We turn now to discuss the QKR-x problem. If we used (2.23) as a coupling scheme, then a similar derivation to that, which leads to (5.19), yields

$$\Gamma=\frac{1}{\hbar^2}\nu \quad \text{for a kicked rotator and } \nu \text{ noise} \ , \tag{5.26}$$

which is consistent with our heuristic result (5.10). For the coupling scheme (2.27) it has been shown [21], using the same formalism, that formula (5.19) should be replaced by

$$\Gamma=\frac{1}{\hbar^2}\sum_{\tau=-\infty}^{\infty}[C_s(\tau)+C_c(\tau)]\nu(\tau) \ . \tag{5.27}$$

Again, for white noise (5.26) is recovered. This result is also recovered if we assume either (2.31) or (2.32) to be the coupling term and apply (5.19). Unlike the QKR-p model here negative autocorrelations of zero-temperature noise enhance rather than suppress the decay process [21]. Therefore a Markovian treatment of the dynamics underestimates the effect of zero-temperature noise on coherence. Additional examples for computation of Γ for this model are given in Appendix C.

In conclusion, we found that the heuristic results for coherence time in the QKR models that ignore the effect of kicking on dynamical correlations are recovered by formal calculation if the noise is white. For colored noise it is essential to take into account that dynamical correlations decay. If this decay were on the short time scale t_E as in the classical (chaotic) model, then a good approximation would be to treat the colored noise as if it were white noise. However, since the quantal correlations range over a time scale t^*, such a Markovian approximation is not legitimate; instead the formalism that has been presented here should be applied.

We now turn to discuss the QKP problem. We have seen (Sec. II C) that this model is equivalent to the QKR-p model with noise whose correlation function is given by (2.36). White ν noise leads to $\phi(\tau',\tau'')=\nu\min(\tau',\tau'')$, or in terms of the "new" variables (5.15)

$$\phi(\tau;t)=(t-|\tau|)\nu \ , \tag{5.28}$$

which is a nonstationary noise. Therefore (5.18) does not hold and one should go back to (5.16) in order to analyze the decay process. Substituting (5.28) one obtains

$$\dot{P}(t)=\frac{1}{\hbar^2}\nu\sum_{\tau=-t}^{t}C_p(\tau)(t-|\tau|) \ . \tag{5.29}$$

Unlike the QKR problem, here the decay is anomalous, and it is impossible to define a (constant) decay rate. If we substitute $C_p(\tau)=\frac{1}{2}D_0t^*e^{-|\tau|/t^*}$, then we obtain

$$P(t)=\frac{1}{\hbar^2}\nu D_0(t^*)^2\{\tfrac{1}{2}t^2-t^*[t-t^*(1-e^{-t/t^*})]\} \ , \tag{5.30}$$

which is roughly a quadratic behavior. This behavior does not last long. If we take into account the resonant contribution (4.21) to $C_p(t)$ we obtain that on long times (compared with t^*, but not too long since we are applying leading-order perturbation theory) the decay is

$$P(t)\propto\nu\xi^{2+\beta}t^{3-\beta} \quad \text{for } t^*\ll t\ll t_c \ . \tag{5.31}$$

In the QKR model we have defined $t_c=1/\Gamma$ so that $P(t)=t/t_c$. It is then natural to assume that $P(t)$ is a function of the scaled variable t/t_c, e.g., the usual ex-

ponential decay behavior $1-P(t)=e^{-t/t_c}$. Similarly, in the QKP problem it is natural to assume that after a short transient $P(t)$ is a function of a scaled variable t/t_c. The perturbative result (5.31) suggests then that

$$P(t)=\left[\frac{t}{t_c}\right]^{3-\beta} \quad \text{for } t^* \ll t \ll t_c \ . \tag{5.32}$$

Comparing with (5.31) it follows that

$$t_c \approx \left[\left[\frac{1}{\xi}\right]^{2+\beta}\frac{1}{\nu}\right]^{1/(3-\beta)}$$

for a kicked particle and ν noise . (5.33)

This should be compared with the result $t_c=[(1/\xi^2)1/\nu]^{1/3}$ for undriven QKP. Evidently, the parameter β quantifies the deviation from complete dynamical correlations which characterize the undriven system. $\beta=1$ represents a marginal case since then expression (5.30) is of the same order also on the long time scale.

VI. DIFFUSION DUE TO DESTRUCTION OF COHERENCE

In the absence of noise, stochasticlike diffusion in momentum is suppressed due to localization. However, if noise is present, coherence is destroyed, and the dynamics gains a stochastic feature. Several simulations to illustrate the time evolution of the QKR in the presence of noise are presented in Fig. 6. The diffusion coefficient is defined as follows:

$$\mathcal{D}=\lim_{t\to\infty}\frac{\langle\langle[p_H(t)-p_H(0)]^2\rangle\rangle}{t} \ , \tag{6.1}$$

where the Heisenberg picture is used and an average over realizations of the noise is implicit. Note that operators in the interaction picture are denoted without a subscript.

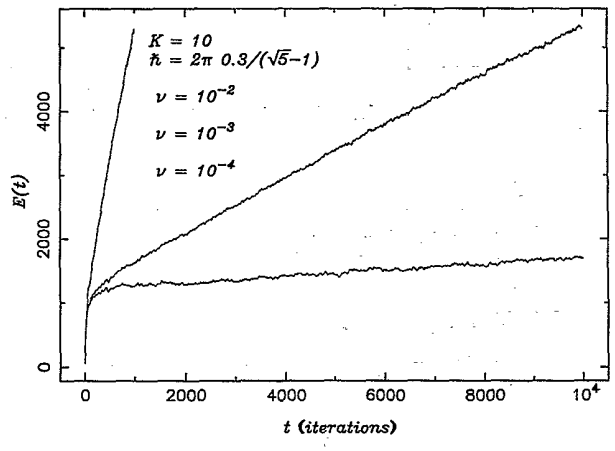

FIG. 6. Typical simulation of the time evolution of a QKR in the presence of noise. Here the coupling scheme is (2.23). The *dispersion* function $\langle(p-p_0)^2\rangle$ is plotted. The average has been taken over 100 initial conditions and noise realizations.

In the absence of noise $\mathcal{D}=0$. In the QKR-p model, a direct generalization of Eq. (3.22) leads to

$$\mathcal{D}=K^2\sum_{\tau=-\infty}^{\infty}C(\tau) \ , \tag{6.2}$$

where $C(\tau)$ is the appropriate generalization of $C_s(\tau)$ if noise is present, namely

$$C(\tau)\equiv\langle\langle \sin x_H(\tau)\sin x_H(0)\rangle\rangle \ . \tag{6.3}$$

In the QKR-x model $\mathcal{D}=K^2\sum_{\tau=-\infty}^{\infty}C(\tau)+O(\nu)$. We shall see that for weak noise ($t^* \ll t_c$) the inequality $\nu \ll \mathcal{D}$ is satisfied. It means that diffusion is mainly due to the destruction of coherence. The $O(\nu)$ term may be ignored and formula (6.2) can be used then. This observation applies also when the equivalence of the QKP problem to the QKR-p problem is discussed, namely the last operation in (2.35) can be ignored if the noise is weak.

If the noise is strong so that coherence is destroyed on a time scale of order unity (one iteration), then $C(\tau-\tau')=\frac{1}{2}\delta_{\tau,\tau'}$. The diffusion is then a stochastic process with $\mathcal{D}=\frac{1}{2}K^2$. In the QKR-$x$ and QKP problems, if the noise is very strong (i.e., ν becomes of the order $\frac{1}{2}K^2$ or larger), then it enhances stochastic diffusion with a coefficient $\mathcal{D}=\frac{1}{2}K^2+\nu$. Thus in the QKR-$x$ problems there are three regimes: weak noise, strong noise, and very strong noise. The strong-noise regime in the QKR-p, QKR-x, and QKP problems (including the very-strong-noise regime in the latter cases) is not very interesting since a stochastic picture of the diffusion process that ignores dynamical correlations applies there. The weak-noise regime, in contrast, is of much interest since coherence is not destroyed completely and localization affects the dynamical behavior.

One may try to use the heuristic diffusion picture that is implicit in the work by Ott, Antonsen, and Hanson [17] in order to estimate $\mathcal{D}$. It is argued that for weak noise ($t^* \ll t_c$) the diffusion process in momentum space is similar to a random walk on a grid with spacing $\hbar\tilde{\xi}_0$ and hopping probability $1/t_c$. The diffusion coefficient is therefore

$$\mathcal{D}\approx(\hbar\tilde{\xi}_0)^2\frac{1}{t_c} \ , \tag{6.4}$$

where $\tilde{\xi}_0$ is of order ξ. The heuristic approach alone is not sufficient in order to fix the ratio $\tilde{\xi}_0/\xi$. A somewhat more careful way to derive this result may be presented in case of QKR (to be distinguished from QKP). Here we follow again argumentation which is implicit in Ref. [17]. Let us assume that diffusion in the quasienergy basis is a stochastic process that may be treated within the framework of a Markovian picture. The coefficient of diffusion is then given by

$$\mathcal{D}\approx\frac{1}{\mathcal{N}}\sum_{s,r}{}'W(s|r)(\langle p\rangle_s-\langle p\rangle_r)^2 \ . \tag{6.5}$$

We make a further approximation and assume that $W(s|r)$, which is given by (5.21), is non-negligible only if $|\langle p_s\rangle-\langle p\rangle_r|$ is of the order $\hbar\tilde{\xi}_0$. We obtain then $\mathcal{D}\approx(\hbar\tilde{\xi}_0)^2(1/\mathcal{N})\sum'_{s,r}W(s|r)$ and thus via (5.22) we recov-

er the heuristic result (6.4). Again the precise value of $\tilde{\xi}_0$ is not fixed by this approach.

We shall try now to adopt a formal approach in order to compute $\mathcal{D}$. We shall use leading-order perturbation theory, in order to estimate $C(\tau)$, and (6.2). In particular, we are interested in finding out which approximations should be made in order to rederive (6.4) without relying on a heuristic picture of the diffusion process. Thus we will be able to fix the value of the prefactor in (6.4) that has not been determined by the heuristic approach. It is convenient to use the interaction picture for the calculation, namely

$$C(t)=\langle\langle U_I^{-1}\sin x(t)U_I\sin x(0)\rangle\rangle \ , \tag{6.6}$$

where

$$U_I=e^{(i/\hbar)f(t)\hat{X}(t)}\cdots e^{(i/\hbar)f(1)\hat{X}(1)} \ .$$

A systematic leading-order calculation of $C(\tau)$ requires the expansion of U_I up to second order in the coupling, namely,

$$\hat{U}_I\approx\hat{I}+\sum_{\tau=1}^{t}\left[\frac{i}{\hbar}f(\tau)\hat{X}(\tau)-\frac{1}{2\hbar^2}f(\tau)^2\hat{X}(\tau)^2\right] - \sum_{1\le\tau''<\tau\le t}\sum\frac{1}{\hbar^2}f(\tau')f(\tau'')\hat{X}(\tau')\hat{X}(\tau'') \ . \tag{6.7}$$

Substitution into (6.6) and careful assemblage of the terms yields

$$C(t)=\langle\langle\sin x(t)\sin x(0)\rangle\rangle - \frac{1}{\hbar^2}\sum_{\tau',\tau''}^{t}\mathcal{C}_X(\tau',\tau'';t)\phi(\tau',\tau'') \ , \tag{6.8}$$

where for $\tau''<\tau'$ the definition of $\mathcal{C}_X$ is

$$\mathcal{C}_X(\tau',\tau'';t) \equiv\mathrm{Re}[\langle\langle\sin x(t)X(\tau')X(\tau'')\sin x(0)\rangle\rangle] -\mathrm{Re}(\langle\langle X(\tau')\sin x(t)X(\tau'')\sin x(0)\rangle\rangle) \ , \tag{6.9}$$

while for $\tau'<\tau''$ it is $\mathcal{C}_X(\tau',\tau'';t)\equiv\mathcal{C}_X(\tau'',\tau';t)$. We shall see soon that if it is desired to rederive the heuristic formula (6.4), it seems essential to assume the factorization of this four-point correlation function, namely

$$\mathcal{C}_X(\tau',\tau'';t)\approx C_X(\tau'-\tau'')C_s(t) \ . \tag{6.10}$$

Substituting in (6.8) and using (5.16) one obtains

$$C(t)\approx[1-P(t)]C_s(t) \ . \tag{6.11}$$

Namely, due to the noise, dynamical correlations are suppressed by a factor that expresses the decay of coherence. Taking into account Eq. (3.31) one finds via (6.2) that $\mathcal{D}=2K^2\sum_{\tau=1}^{\infty}P(\tau)[-C_s(\tau)]$. Using the discrete version of integration by parts one obtains

$$\mathcal{D}=\sum_{t=0}^{\infty}\dot{P}(t)D(t) \ . \tag{6.12}$$

Expression (6.12) constitutes the main result of this section. In the next paragraph we shall demonstrate that for the QKR model with weak stationary noise (6.4) is recovered. Later we shall apply it to study the diffusion in the QKP problem.

In the QKR models $P(t)=\Gamma$ for $t\ll t_c$, which is the time regime on which perturbation theory is valid. On a time scale that is of order t_c or larger the behavior of $P(t)$ is not known. However, if the noise is weak ($t^*\ll t_c$), then the latter information is not needed since the sum (6.12) is dominated then by the short-time terms whose number is of order t^*. One may substitute then without a big error expression (4.19) and $\dot{P}(t)=\Gamma$, and sum over t from $t=0$ up to infinity. The result is

$$\mathcal{D}=D_0t^*\Gamma=\frac{t^*}{t_c}D_0 \ . \tag{6.13}$$

This formula agrees with the heuristic result (6.4) provided $\tilde{\xi}_0=2\xi$ is substituted there. If one takes (4.20) into account then D_0 of (6.13) should be replaced by $E(\infty)/t^*$, which cannot be distinguished from D_0 in actual numerical experiments that are presented later in this section.

Perturbation theory enables one to find the behavior of $C(t)$ for time $t\ll t_c$. In order to find the global behavior of $C(t)$ we should go beyond perturbation theory. The simplest extrapolation scheme is (a) to assume that relation (6.11) holds also on long times and (b) to assume that the perturbative expression (5.16) for $P(t)$ holds for $t<t_c$ while $P(t)=1$ for $t_c<t$. Evidently, if the results of the calculations depend on this extrapolation scheme, then we should not be surprised if we fail to predict the right behavior. There is a simple indication for nonperturbative behavior. If only the short-time terms with $t\ll t_c$ contribute appreciably to the sum (6.12), then evidently $\mathcal{D}$ should be proportional to the intensity of the noise. Therefore, if $\mathcal{D}$ is *not* proportional to the intensity of the noise, then the behavior is necessarily nonperturbative.

In the QKR problem if $t^*\ll t_c$, then perturbation theory is sufficient in order to estimate $\mathcal{D}$. However, if t_c is of the order t^*, then we find ourselves in a nonperturbative regime. Let us estimate the width of this nonperturbative regime in few special cases. In the QKR-p model coherence is destroyed within one time step provided $(1/\xi)^2<\delta x(t=1)$, where $\delta x(t)$ is given by (5.4). It follows that $(1/\xi)^2<\phi(0)$ is the condition for being in the strong-noise (stochastic) regime. The condition for being in the weak-noise (perturbative) regime is $t^*\ll t_c$. For σ noise [Eq. (5.7)] it leads to $\sigma\ll(1/\xi)^3$, while for λ noise [Eq. (5.8)] it leads to $2\lambda\ll(1/\xi)^2$. Thus for σ noise we expect a nonperturbative regime within $(1/\xi)^3<\sigma<(1/\xi)^2$, while for λ noise there should not be such a regime. In the latter case we expect quite sharp crossover from the weak-noise (perturbative) regime to the strong-noise (stochastic) regime. The same behavior is expected in the QKR-x model with white ν noise since we have demonstrated (Sec. II C) that this model and the QKR-p model with λ noise are essentially equivalent.

Figures 7–10 present the results of numerical simulations. In Fig. 7 the diffusion coefficient $\mathcal{D}$ has been determined by simulations for the QKR-x and QKR-p models. (The QKP model is to be discussed later on in this sec-

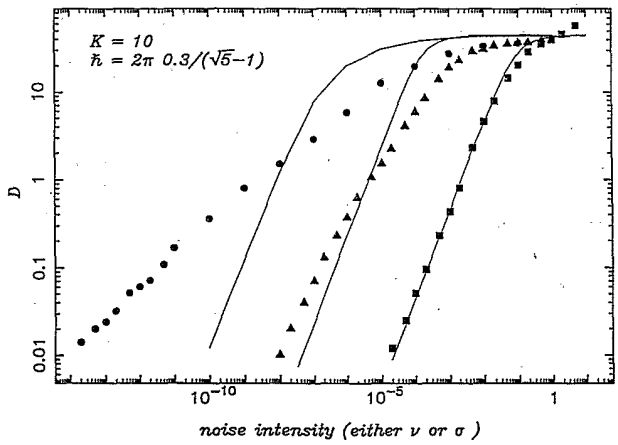

FIG. 7. The diffusion coefficient $\mathcal{D}$ has been determined by simulations for the QKR-x model (■), the QKR-p model (▲), and the QKP model (●). The noise is white. The curves represent the result of a perturbative calculation (Appendix C) with no fitting parameters.

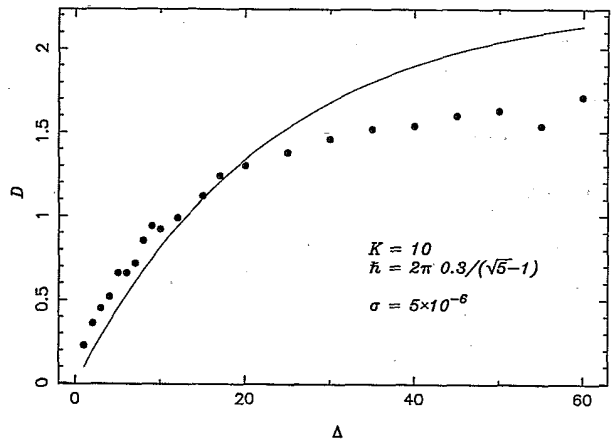

FIG. 9. The same as in Fig. 8, but with fainter noise. The agreement with the theory is much better.

tion.) The noise is white and $\mathcal{D}$ is plotted as a function of its intensity. Note the wide nonperturbative regime in case of the QKR-p model. The curves represent the results of analytical calculations (Appendix C). In Figs. 8 and 9 the results of numerical experiments on the QKR-p model with colored noise are presented. For fainter noise the agreement with the analytical calculation become better. Note that no fitting parameters are involved in these calculations. Figure 10 presents the result of additional numerical experiments with colored noise. Here the QKR-x model is simulated. The interaction term is either (2.31) or (2.32). We do not have analytical results in the case of the latter coupling scheme. From the numerical experiment it emerges that $C_c(\tau)$ is probably very similar to $C_s(\tau)$.

We now turn to discuss the QKP problem. Here the behavior of the terms in the sum (6.12) that satisfy $t^* \ll t \ll t_c$ is

$$\dot{P}(t)D(t)=(3-\beta)cD_0\frac{(t^*)^{1+\beta}}{(t_c)^{3-\beta}}t^{1-2\beta}, \tag{6.14}$$

where we have used (5.32) and (4.20). This behavior indicates that most of the contribution to $\mathcal{D}$ in (6.12) comes from the long-time terms with t, which is of the order t_c. This observation is supported by the comparison of the numerical results (Fig. 7, solid circles) with the perturbative estimate that takes into account only the short-range part of the dynamical correlations (Fig. 7, smooth curve). Unlike the QKR models, there is no agreement even for very weak noise. It follows that knowledge of correlations on a time scale comparable with t_c is required in order to estimate $\mathcal{D}$. Thus, unlike the QKR case, even in the weak-noise regime ($t^* \ll t_c$), the behavior is nonperturbative. If we use the simplest extrapolation scheme

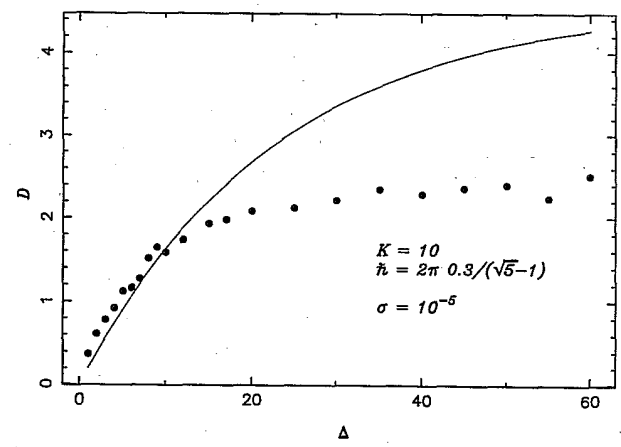

FIG. 8. The diffusion coefficient $\mathcal{D}$ for the QKR-p model with the colored noise (C1). The curve represents the result of a perturbative calculation. Equations (C2) and (6.13) have been used with no fitting parameters.

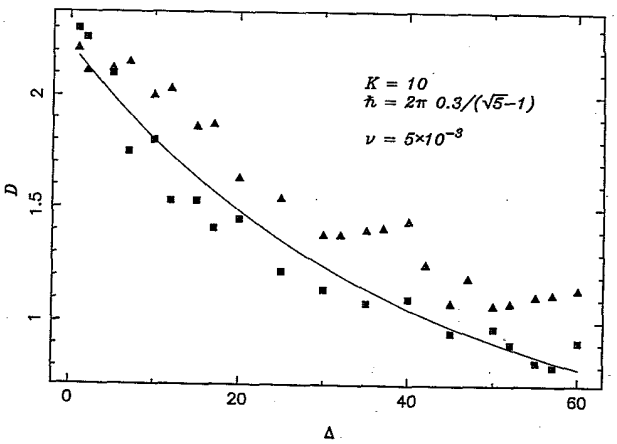

FIG. 10. The diffusion coefficient $\mathcal{D}$ for the QKR-x model with the colored noise (C3). For the (■) the interaction term is (2.31), while for the (▲) the interaction term is (2.32). The curve is a plot of Eq. (6.13) with (C4).

that has been introduced previously (see the paragraph after 6.13), then we obtain

$$\mathcal{D}=c'\left[\frac{t^*}{t_c}\right]^{1+\beta}D_0\ , \tag{6.15}$$

where $c'\equiv(3-\beta)/2(1-\beta)$ is a prefactor of order unity. Upon substitution of (5.33) it follows that $\mathcal{D}\propto\nu^\alpha$ with $\alpha=(1+\beta)/(3-\beta)\approx 0.78$. The observed behavior (Fig. 7) is $0.35<\alpha<0.38$, which is very close to $\alpha\approx\frac{1}{3}$. One may be tempted to use a naive heuristic approach in order to find α, namely to substitute expression (5.11), which applies to an undriven particle, into (6.4). One obtains then

$$\mathcal{D}\approx D_0\xi^{5/3}\nu^{1/3}\ , \tag{6.16}$$

which coincides with (6.15) provided $\beta=0$. The deviation from this value is due to the nontrivial nature of the dynamical correlations. However, we see that perturbation theory fails to quantify this effect. The observed value of α is much closer to the naive result $\alpha=\frac{1}{3}$ than to the perturbative one. The latter overestimates the deviation.

VII. SUMMARY AND CONCLUSIONS

It has been shown that there are simple relations between (a) the functional form of the crossover from diffusion to localization, (b) correlation functions of certain dynamical variables, and (c) spectral properties of the system. These relations enable one to introduce a qualitative study of dynamical correlations that are characteristic of a classical chaotic system that exhibits quantal localization. Classically, dynamical correlations decay exponentially on a very short time scale t_L which is the inverse of Lyapunov instability exponent. Quantum mechanically these correlations decay exponentially [Eq. (4.16)] on a relatively long time scale t^*. Due to the existence of pairs of resonating states the exponential decay is replaced on a longer time scale by a power-law decay [Eq. (4.21)].

An extensive discussion has been devoted to motivate and introduce different models where the effect of noise on the dynamics is of great interest. In the QKR-p model the rotator is coupled to a noise source via its momentum coordinate. This is, for example, the coupling in the case of an electron that is confined to move in a mesoscopic ring and is subject to a noisy magnetic flux that penetrates the ring. The QKR-x model is motivated by the "Ohmic model," which has been introduced in order to study the damped-rotator problem. The interaction with the noise source is then via the position coordinate. Such is the interaction, for example, in the case of electron that moves in a ring and is subject to interaction with impurities. The problematic features of the quantization procedure in the case of the QKR-x model have been discussed. This discussion leads to the introduction of the QKP model where the position coordinate of the standard QKR model is considered to be an extended coordinate. We demonstrated that the QKP problem with white noise may be reduced to a QKR-p problem with nonstationary noise. The problem of QKR with kicks that are not strictly periodic in time has been reduced to either the QKR-p model or the QKR-x model with white noise, depending on the nature of the deviations from strict time periodicity.

Using first Wigner's picture of the dynamics we were able to discuss the destruction of coherence in the case of undriven systems (either a rotator or a particle which is subject to noise). In order to study the behavior of the decay probability $P(t)$ in the case of kicked system we had to use perturbation theory. The formalism leads to Eq. (5.16). Assuming that $P(t)$ is a function of a scaled variable t/t_c, we were able to determine the coherence time t_c as it is implied by perturbation theory. In the QKR models with stationary noise $P(t)=\Gamma t$, where Γ is the decay rate and hence $t_c=1/\Gamma$. We introduced an explicit expression for Γ [Eq. (5.19)]. It emerges that for white noise t_c is the same as for an undriven rotator since Γ does not depend then on the nature of the dynamical correlations. The latter are important if the noise is not white. The autocorrelations of colored noise affect significantly Γ. However, the nature of the decay process is not altered even if the noise is a zero-temperature noise. This is quite different from the case of integrable system where negative noise autocorrelations may suppress the exponential decay process. To be specific, this claim was demonstrated in the case of λ noise and η noise, where it was found that (5.25) and (5.24) hold respectively, in the presence of kicks, while in the absence of kicks (5.8) and (5.9) apply. In the case of the QKR-x model it has been found that zero-temperature noise autocorrelations enhance rather than suppress the decay process. Evidently, a Markovian treatment of the dynamics fails to predict such effects. The destruction of coherence in the QKP problem is quite different in nature from that in the corresponding QKR-x model. The anomalous decay in the QKP problem is due to a noise-induced spreading mechanism for the destruction of coherence, which leads to a very short coherence time compared with the corresponding time scale in the QKR-x problem.

Next we discussed the effect of noise on dynamical correlations. This enables one to find the quantal diffusion coefficient $\mathcal{D}$ in the presence of noise [Eq. (6.2)]. If the noise is strong so that coherence is destroyed completely, then diffusion is a stochastic process. In both the QKR-x and QKP problems (but not in the QKR-p model) diffusion is enhanced by very strong noise. If the noise is weak, then the diffusion is due to the partial destruction of dynamical correlations [Eq. (6.11)]. Equation (6.12) for $\mathcal{D}$ then applies. One should distinguish in the latter case between perturbative and nonperturbative contributions to $\mathcal{D}$. If the noise is weak enough ($t^*\ll t_c$), the latter are negligible in the QKR models but not in the QKP model. This leads to a linear dependence of $\mathcal{D}$ on noise intensity in the case of the QKR models but

not in the case of QKP model (Fig. 7). Note, however, that also in the QKR-p model there is a wide intermediate nonperturbative noise regime (Fig. 7). The relation introduced by Ott, Antonsen, and Hanson, which relates $\mathcal{D}$ to the coherence time t_c within the QKR's perturbative noise regime, is recovered by the formal calculation [Eq. (6.13)]. Originally this relation was based on either a heuristic picture or a perturbative Markovian treatment of the diffusion process. Our formal approach enables one also to fix in the formula a prefactor that has not been determined in its original version. The dependence of $\mathcal{D}$ on noise autocorrelations, which is implied by the theory, has been demonstrated by numerical simulations to support our analytical estimates.

Some open questions should be pointed out. A study of dynamical correlations in the semiclassical limit ($\hbar \ll 1$) should be carried out. In this paper it has been assumed that $t_E \leq 1$ rather than $1 \ll t_E \ll t^*$. As far as I know this regime has not previously been explored. It also has been assumed that $1 \ll K$. Different results [32] are expected to be found if $K \approx K_c$. The effect of noise has been treated within the framework of perturbation theory. Nonperturbative effects were pointed out. It is important to find a formalism that enables one to go beyond perturbation theory in order to quantify these effects.

Note added. I thank B. Chirikov for informing me about a new phenomenological approach toward the dynamical crossover that has been reported in recent unpublished work. His results are consistent with those of the present paper.

ACKNOWLEDGMENTS

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APPENDIX A: WIGNER'S PICTURE

The quantum state of a particle in one dimension may be represented by the probability matrix in the x representation

$$\rho(x,r) \equiv \langle x''|\hat{\rho}|x'\rangle \ , \tag{A1}$$

where

$$x = \tfrac{1}{2}(x''+x') \ , \qquad r = x''-x' \ . \tag{A2}$$

An alternative representation is Wigner's function

$$\rho(x,p) \equiv \int_{-\infty}^{\infty} dr\, \rho(x,r) e^{-(i/\hbar)pr} \ , \tag{A3}$$

which is a phase-space quasidistribution, i.e., a normalized real function on $\mathbb{R}\times\mathbb{R}$, where $\mathbb{R}$ are the real numbers. The normalization is

$$\frac{1}{2\pi\hbar}\int\int dx\, dp\, \rho(x,p) = 1 \ . \tag{A4}$$

For a rotator this definition should be modified. Since $\rho(x,r)$ satisfies then periodic boundary conditions on the rectangular $[0,2\pi]\times[-2\pi,2\pi]$, the proper definition takes the form

$$\rho(x,p) \equiv \frac{1}{4\pi}\int_{-2\pi}^{2\pi} dr\, \rho(x,r) e^{-(i/\hbar)pr} \ , \tag{A5}$$

which is a normalized real function on the domain $[0,2\pi]\times\hbar\mathbb{Z}$. The normalization is

$$\sum_{p\in\hbar\mathbb{Z}} \int_0^{2\pi} dx\, \rho(x,p) = 1 \ . \tag{A6}$$

Note that for half integer p,

$$\int_0^{2\pi} dx\, \rho(x,p) = 0 \ . \tag{A7}$$

Wigner's function for a rotator that is prepared in momentum eigenstate $|p\rangle$ is

$$\rho(x,p) = \frac{1}{2\pi}\delta_{p,p_0} \ . \tag{A8}$$

If the rotator is prepared in a superposition of momentum eigenstates, then $\rho(x,p)$ is a weighted sum of function of the type (A8) plus an interference part whose average is zero.

The time evolution of the probability matrix is given by

$$\rho(x,r) = \int\int \mathcal{H}(x,r|x_0,r_0)\rho(x_0,r_0)dx_0 dr_0 \ , \tag{A9}$$

where

$$\mathcal{H}(x,r|x_0,r_0) = \langle x''|\hat{U}|x_0''\rangle\langle x'|\hat{U}|x_0'\rangle^* \ . \tag{A10}$$

The transformed version of (A9) is

$$\rho_t(x,p) = \sum_{p_0}\int dx_0 \mathcal{H}(x,p|x_0,p_0)\rho(x_0,p_0) \ , \tag{A11}$$

where the propagator of Wigner's function is

$$\mathcal{H}(x,p|x_0,p_0) \equiv \frac{1}{4\pi}\int\int dr\, dr_0 e^{-(i/\hbar)pr} \times \mathcal{H}(x,r|x_0,r_0) e^{(i/\hbar)p_0 r_0} \ . \tag{A12}$$

If the Hamiltonian is such that the classical equations of motion are linear in the dynamical variables, then $\mathcal{H}(x,p|x_0,p_0)$ is identical to its classical limit. For free propagation $\hat{U} = \exp[-(i/\hbar)\frac{1}{2}p^2 t]$ and

$$\mathcal{H}^{\text{free}}(x,p|x_0,p_0)=\delta(p-p_0)\delta(x-(x+p_0t)) . \quad \text{(A13)}$$

$$\mathcal{H}(x,p|x_0,p_0)=\delta(x-x_0)\mathcal{F}(p|p_0) . \quad \text{(A14)}$$

For a kick process Wigner's evolution kernel is of the form

Note that $\mathcal{F}$ may depend on the position variable. In particular for $\hat{U}=\exp[-(i/\hbar)K\cos\hat{x}]$ one obtains

$$\begin{aligned}\mathcal{F}&=\frac{1}{4\pi}\int_{-2\pi}^{2\pi}dr\exp\left[-\frac{i}{\hbar}K\left[\cos(x+\tfrac{1}{2}r)-\cos(x-\tfrac{1}{2}r)\right]-i\frac{p-p_0}{\hbar}r\right]\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\exp\left[-\frac{i}{\hbar}K[\cos(x-\theta)-\cos(x+\theta)]+i\left[2\frac{p-p_0}{\hbar}\right]\theta\right]\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\exp\left[-i\left[2\frac{K\sin x}{\hbar}\right]\sin\theta+i\left[2\frac{p-p_0}{\hbar}\right]\theta\right]\\&=\mathcal{J}_{2(p-p_0)/\hbar}\left[2\frac{K\sin x}{\hbar}\right], \quad \text{(A15)}\end{aligned}$$

which corresponds to the classical propagator

$$\mathcal{F}=\delta((p-p_0)-K\sin x) . \quad \text{(A16)}$$

For $\hat{U}=\exp[-(i/\hbar)\sqrt{2\nu}\cos(x+\varphi)]$ one obtains

$$\mathcal{F}^{\varphi}=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\exp\left[-\frac{i}{\hbar}\sqrt{2\nu}[\cos(x+\varphi-\theta)-\cos(x+\varphi+\theta)]+i\left[2\frac{p-p_0}{\hbar}\right]\theta\right] . \quad \text{(A17)}$$

Averaging over realization of the random phase φ one obtains

$$\begin{aligned}\mathcal{F}&=\left[\frac{1}{2\pi}\right]^2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}d\theta\,d\varphi\exp\left[-\frac{i}{\hbar}\sqrt{2\nu}[\cos(\varphi-\theta)-\cos(\varphi+\theta)]+i\left[2\frac{p-p_0}{\hbar}\right]\theta\right]\\&=\left[\frac{1}{2\pi}\right]^2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}du\,dv\exp\left[-\frac{i}{\hbar}\sqrt{2\nu}(\cos v-\cos u)+i\frac{p-p_0}{\hbar}(u-v)\right]\\&\equiv\left[\mathcal{J}_{(p-p_0)/\hbar}\left[\frac{\sqrt{2\nu}}{\hbar}\right]\right]^2, \quad \text{(A18)}\end{aligned}$$

which is a stochastic kernel unlike (A15).

For $\hat{U}=\exp[-(i/\hbar)\int d\varphi\, f_{\varphi}\cos(\hat{x}+\varphi)]$ one easily finds

$$\mathcal{F}^{f}=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\exp\left[-i\left[2\frac{\int d\varphi\, f_{\varphi}\sqrt{2}\sin(x+\varphi)}{\hbar}\right]\sin\theta+i\left[2\frac{p-p_0}{\hbar}\right]\theta\right] . \quad \text{(A19)}$$

Assuming f_{φ} to be Gaussian distributed, averaging over its realizations, and using the well-known Gaussian integral formula, one obtains

$$\begin{aligned}\mathcal{F}&=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\exp\left[-\tfrac{1}{2}\nu\frac{1}{2\pi}\int d\varphi\left[\frac{2}{\hbar}\sqrt{2}\sin(x+\varphi)\sin\theta\right]^2+i\left[2\frac{p-p_0}{\hbar}\right]\theta\right]\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\exp\left[-\frac{\nu}{\hbar^2}2\sin^2\theta+i\left[\frac{p-p_0}{\hbar}\right]2\theta\right]\\&=e^{-\nu/\hbar^2}\left[\mathcal{J}_{(p-p_0)/\hbar}\left[\frac{\nu}{\hbar^2}\right]\right], \quad \text{(A20)}\end{aligned}$$

which is again a stochastic kernel.

APPENDIX B: FOURIER TRANSFORM FORMULAS

The Fourier transform of $C(\tau)=e^{-\gamma|\tau|}$ is

$$C(\omega)=\sum_{\tau=-\infty}^{\infty} e^{-\gamma|\tau|}e^{i\omega\tau}=\frac{\sinh\gamma}{\cosh\gamma-\cos\omega} . \tag{B1}$$

For $\gamma \ll 1$ a good approximation is

$$C(\omega)=\frac{2\gamma}{2(1-\cos\omega)+\gamma^2} . \tag{B2}$$

The (inverse) Fourier transform of $C(\omega)=\Theta(\Delta-|\omega|)\ln^2|\Delta/\omega|$, where Θ denotes the unit step function and $\Delta \ll \pi$, is

$$\begin{aligned} C(\tau)&=\frac{1}{2\pi}\int_{-\pi}^{\pi} C(\omega)e^{i\omega\tau}d\omega \\ &=\frac{1}{2\pi}\int_0^{\Delta}\ln^2\left|\frac{\Delta}{\omega}\right|\cos(\omega\tau)d\omega \\ &=\frac{2}{\pi}\int_0^{\Delta}\left[-\frac{1}{\omega}\ln\left[\frac{\omega}{\Delta}\right]\right]\left[\frac{1}{\tau}\sin(\omega\tau)\right]d\omega \\ &=\frac{1}{\tau}F(\Delta\tau) , \end{aligned} \tag{B3}$$

where $F(\lambda)\equiv(2/\pi)\int_0^1(-\ln x)[\sin(\lambda x)/x]dx$. The derivative of $F(\lambda)$ is

$$\begin{aligned} F'(\lambda)&=\frac{2}{\pi}\int_0^1(-\ln x)\cos(\lambda x)dx \\ &=\frac{1}{\lambda}\frac{2}{\pi}\int_0^{\lambda}\frac{\sin z}{z}dz\equiv\frac{1}{\lambda}\frac{2}{\pi}\mathrm{si}(\lambda) . \end{aligned} \tag{B4}$$

For our purpose a quite good approximation for the sine integral is $\mathrm{si}(\lambda)\approx\lambda$ for $\lambda<\pi/2$, while $\mathrm{si}(\lambda)\approx\pi/2$ otherwise. It follows that $F(\lambda)\approx(2/\pi)\lambda$ in the former regime, while for $\pi/2<\lambda$ it is a logarithmic function of λ.

APPENDIX C: CALCULATION PROCEDURE

All the calculations to determine analytically Γ and $\mathcal{D}$ for the numerical experiments were performed as follows. The basic parameters were $K=10$ and $\hbar=2\pi(0.3/\sqrt{5}-1)$. We used (4.19) with $D_0=45$ and $t^*=23$. The corresponding expressions for the correlation functions were $C_p(\tau)=\frac{1}{2}D_0t^*e^{-|\tau|/t^*}$ and $C_s(\tau\neq0)=-\frac{1}{2}(D_0/K^2t^*)e^{-|\tau|/t^*}$. The long-time behavior of these correlation functions was not required since in our numerical experiments the noise autocorrelations range over a time scale of the order t^*, where the exponential behavior dominates.

The decay rate Γ has been calculated in several special cases using Eq. (5.19). For the QKR-p model with white noise $\phi(t,t')=\sigma\delta_{t,t'}$, and one obtains $\Gamma=(1/\hbar^2)\frac{1}{2}D_0t^*\sigma$, while for the colored generalized λ noise

$$\phi(t,t')=2\lambda\delta_{t,t'}-\lambda\delta_{|t-t'|,\Delta} , \tag{C1}$$

and one obtains

$$\Gamma=\frac{1}{\hbar^2}D_0t^*(1-e^{-\Delta/t^*})\lambda . \tag{C2}$$

For the QKR-x model with white noise $\nu(t,t')=\nu\delta_{t,t'}$, and one obtains $\Gamma=1/\hbar^2\nu$. For the interaction term (2.31) with

$$\nu(t,t')=\nu\max\left[1-\left|\frac{t-t'}{\Delta}\right|,0\right] , \tag{C3}$$

one obtains

$$\Gamma\cong\frac{1}{\hbar^2}\frac{t^*}{\Delta}(1-e^{-\Delta/t^*})\nu . \tag{C4}$$

The diffusion coefficient $\mathcal{D}$ has been determined via (6.12). The extrapolation scheme presented in Sec. VI has been assumed. Thus

$$P(t)=\max\left[\frac{1}{\hbar^2}\sum_{t'=0}^{t-1}\sum_{\tau=-t'}^{t'}C_X(\tau)\phi(\tau;t),1\right] . \tag{C5}$$

In the case of the QKR models one obtains

$$\mathcal{D}=(1-e^{t_c/t^*})\frac{t^*}{t_c}D_0 , \tag{C6}$$

where $t_c\equiv1/\Gamma$. Note that for $t^*\ll t_c$ this expression reduces to the perturbative result (6.13). A different extrapolation procedure in the case of the QKR models is to assume an exponential decay. Thus

$$P(t)=1-\exp\left[-\frac{1}{\hbar^2}\sum_{t'=0}^{t-1}\sum_{\tau=-t'}^{t'}C_X(\tau)\phi(\tau;t)\right] . \tag{C7}$$

One obtains then

$$\mathcal{D}=\frac{1}{1+t_c/t^*}D_0 . \tag{C8}$$

Again, for weak noise this expression reduces to the perturbative result (6.13).

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