Counting statistics in multiple path geometries and the fluctuations of the integrated current in a quantum stirring device

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Abstract

The amount Q of particles that are transported via a path of motion is characterized by its expectation value $\langle Q \rangle$ and by its variance $\mathrm{Var}(Q)$. We analyze what happens if a particle has two optional paths available to get from one site to another site, and in particular what is $\mathrm{Var}(Q)$ for the current which is induced in a quantum stirring device. It turns out that coherent splitting and the stirring effect are intimately related and cannot be understood within the framework of the prevailing probabilistic theory.

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1. Introduction

The possibility to induce DC currents by periodic (AC) modulation of the potential is familiar from the context of electronic devices. If an open geometry is concerned, it is known as 'quantum pumping' [1–6], while for closed geometry [7, 8] we use the term 'quantum stirring' [9, 10]. Of recent interest is the possibility to stir condensed Bose particles that are confined in an optical trap and are described by a three-site Bose–Hubbard Hamiltonian [11–13]. While in a parallel study [14] we explore the role of interactions in this stirring process, in the present work we would like to explore a new aspect of the problem that has to do with *counting statistics*. For simplicity, we neglect the interactions and accordingly the problem reduces to 'one-particle physics'.

It is well established [15–24] that counting statistics in the context of shot noise studies provides information on the fluctuations of the occupation and on the random probabilistic nature of the quantum-mechanical transmission/reflection process. In fact these two effects combine together. The prototype example is barrier crossing. If the average channel occupation is f and the transmission probability is p, then the emerging number of particles Q is proportional to fp, while the variance (per-particle) is $\mathrm{Var}(\mathcal{Q}) = (1 - fp)fp$. Furthermore, the results which are derived using classical methods (Master equation; Boltzmann–Langevin) are consistent with the quantum-mechanical calculation (Scattering approach; Green function
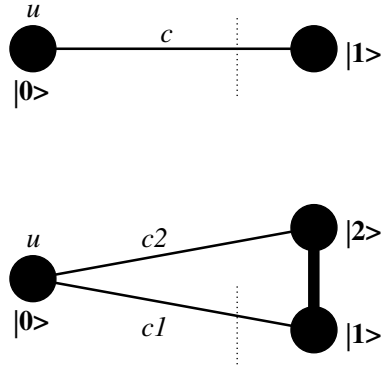
Figure 1. Toy models that are analyzed in the paper: a particle in a two-site system (upper illustration) and a particle in a three-site system (lower illustration). Initially the particle is prepared in the $|0\rangle$ site where it has a potential energy u. The hopping amplitudes per unit time (the cs) are also indicated. In the case of a three-site system, the time units are chosen such that the hopping amplitude per unit time between $|1\rangle$ and $|2\rangle$ equals unity, while the other amplitudes are assumed to be small ($|c_1|, |c_2| \ll 1$). The current is measured through the dotted section.

techniques), and the quantum-classical crossover is related to the statistics of the transmission coefficients as reflected by the Fano factor.

1.1. Scope

The purpose of this paper is to argue that the above common wisdom does not apply in the context of quantum stirring. In order to demonstrate this point we analyze the prototype three-site system of figure 1. We measure the current $\mathcal{I}$ that flows through a section in the c_1 bond, and define the splitting ratio as $\lambda = c_1/(c_1 + c_2)$. The integrated current over a time period is denoted as $\mathcal{Q}$. If the passage probability from left to right is p we do not get for its variance $\mathrm{Var}(\mathcal{Q}) = (1 - \lambda p)\lambda p$ as implied by the naive probabilistic considerations, but rather $\mathrm{Var}(\mathcal{Q}) = \lambda^2(1 - p)p$. Then we turn to analyze a full pumping cycle that consists of two sequential Landau–Zener adiabatic passages. During the first half of the cycle $\lambda = \lambda_\circlearrowright$ while during the second half of the cycle $\lambda = \lambda_\circlearrowleft$. If $\lambda_\circlearrowright \neq \lambda_\circlearrowleft$ then it follows that the net integrated current is $\langle \mathcal{Q} \rangle \neq 0$, and we ask what is the variance $\mathrm{Var}(\mathcal{Q})$.

1.2. Observations

There are some qualitative observations that are associated with our results and we would like to enumerate them in advance:

(1) Coherent splitting of a wavepacket does not generate a noisy current in the system.
(2) The 'fractional charge' of a partial wavepacket is determined by the splitting ratio, and is physically meaningful.
(3) The splitting ratio can be greater than unity or negative. This has the interpretation of having an induced circulating current in the system.
(4) The splitting ratio concept allows an easier better understanding of quantum stirring, in comparison with the complicated Kubo formalism [7].
(5) The fluctuations in $\mathcal{Q}$ reflect the non-adiabaticity of the driving cycle.
(6) The fluctuations of the integrated current grow with time as t and not as $\sqrt{t}$.

(7) Interference appear differently in the calculation of counting statistics when compared
with the calculation of occupation statistics.

2. Definitions

The current operator $\mathcal{I}$ is a conventional observable in quantum mechanics. For a single-mode
ring the current through a section $r = r_0$ can be expressed using the position and the velocity
operators

$$\mathcal{I} = \hat{v}\delta(\hat{r} - r_0)|_{\text{symmetrized}}. \tag{1}$$

We denote by Q the total number of particles that go through the specified section, and
accordingly define a counting operator,

$$\mathcal{Q} = \int_0^t \mathcal{I}(t')\,\mathrm{d}t'. \tag{2}$$

Calligraphic letters are used to distinguish the $\mathcal{I}$ and the $\mathcal{Q}$ operators, and $\mathcal{I}(t) \equiv U(t)^\dagger \mathcal{I} U(t)$,
where $U(t)$ is the evolution operator. Formally we can diagonalize $\mathcal{Q}$, find its Q eigenvalues,
and the associated $|Q\rangle$ eigenstates. The *initial* state of the system can be expanded in
the Q representation, and accordingly the *final* probability distribution of Q at time t is
$\mathrm{P}(Q) = |\langle Q|\Psi\rangle|^2 = |\Psi_Q|^2$. This distribution can be characterized by its moments. Of
particular interest are the expectation value and the variance:

$$\langle \mathcal{Q} \rangle = \sum_Q |\Psi_Q|^2 Q = \langle \Psi|\mathcal{Q}(t)|\Psi\rangle \tag{3}$$

$$\mathrm{Var}(Q) = \langle \mathcal{Q}^2 \rangle - \langle \mathcal{Q} \rangle^2. \tag{4}$$

The naive definition of $\mathrm{P}(Q)$ that we have introduced above follows [25] which was later
criticized. A more careful analysis [26, 27] of the continuous measurement scheme implies
that the full counting statistics is characterized by the following physically measurable quasi-
distribution:

$$\mathrm{P}_0(Q) = \frac{1}{2\pi}\int \langle [\mathcal{T}e^{-i\frac{r}{2}\mathcal{Q}}]^\dagger [\mathcal{T}e^{+i\frac{r}{2}\mathcal{Q}}]\rangle e^{-iQr}\,\mathrm{d}r. \tag{5}$$

The naive mathematical definition is obtained if we ignore time ordering,

$$\mathrm{P}(Q) = \frac{1}{2\pi}\int \langle e^{+ir\mathcal{Q}}\rangle e^{-iQr}\,\mathrm{d}r = \langle \delta(Q - \mathcal{Q})\rangle. \tag{6}$$

It is easily verified that $\mathrm{P}_0(Q)$ has the same first and second moments as $\mathrm{P}(Q)$. Therefore,
for the purpose of variance analysis, we refer below to the latter (naive) definition. For more
details about *full* counting statistics in the context of closed geometries, see [28].

3. Modeling

We consider the simple models that are illustrated in figure 1. The Hamiltonian of the three-site
system is

$$\mathcal{H} = \begin{pmatrix} u & c_1 & c_2 \\ c_1 & 0 & 1 \\ c_2 & 1 & 0 \end{pmatrix}. \tag{7}$$
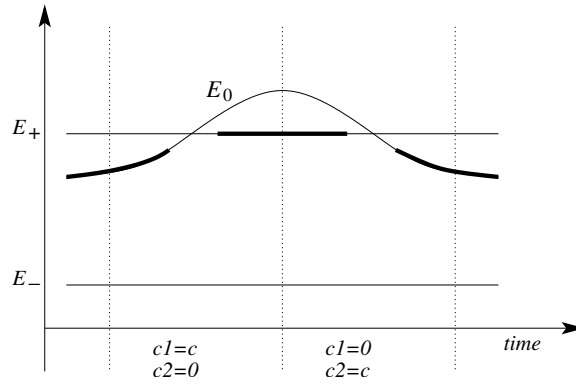
Figure 2. The adiabatic levels of the three-site Hamiltonian during one period of a pumping cycle. In the absence of coupling ($c_1 = c_2 = 0$) the $E_0 = u(t)$ level intersects the symmetric $E_+ = 1$ level. With nonzero coupling these intersections become avoided crossings, and the particle follows adiabatically the thickened lines. For presentation purpose we indicate that either c_1 or c_2 equal zero ('blocked'), but in the general analysis we allow any splitting ratio, including the possibility $c_1 = c_2$ of having the same amplitude to take either of the two paths.

Without loss of generality, we choose the units of time such as to have the hopping amplitude per unit time between $|1\rangle$ and $|2\rangle$ equal 1. For $c_1 = c_2 = 0$ there are two energy levels $E_\pm = \pm 1$ that correspond to the states $|1\rangle \pm |2\rangle$, and $E_0 = u$ that corresponds to $|0\rangle$.

The prototype scenario that we consider later is as follows. The particle is initially prepared in the left site, and the potential u is slowly raised from $u < 1$ to $u > 1$. This induces an adiabatic crossing from the left site to the right side. For completeness we note that in the standard Bose–Hubbard Hamiltonian all the hopping amplitudes are negative, and accordingly the symmetric level E_+ is in fact the ground state of the double well, to which the condensed particles are transported from the $E_0 = u$ level.

More generally we consider later a quantum stirring scenario. The transport is measured via the $0 \mapsto 1$ bond, and accordingly the current operator is represented by the matrix

$$\mathcal{I} = \begin{pmatrix} 0 & \mathrm{i}c_1 & 0 \\ -\mathrm{i}c_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

Assuming control over the couplings c_1 and c_2 we can design a pumping cycle with nonzero net transport $\langle \mathcal{Q} \rangle \neq 0$. See figure 2 for illustration.

4. The two-level approximation

Let us assume a driving cycle of period t_p, such that in any moment $|c_1|, |c_2| \ll 1$ and $u \sim 1$, as in figure 2. In the strict adiabatic limit the particle stays in the same adiabatic level with no leakage to the other levels. But if the driving is not strictly adiabatic there is some leakage. Below we introduce the conditions for neglecting the leakage to E_-. Accordingly the dynamics is very well described within the framework of a two-level approximation:

$$\mathcal{H} = \begin{pmatrix} u & c \\ c & 1 \end{pmatrix}, \qquad \mathcal{I} = \lambda \begin{pmatrix} 0 & \mathrm{i}c \\ -\mathrm{i}c & 0 \end{pmatrix}, \tag{9}$$

where

$$c \equiv \frac{1}{\sqrt{2}}(c_1 + c_2) \equiv \text{effective coupling} \tag{10}$$

and

$$\lambda \equiv \frac{c_1}{c_1 + c_2} \equiv \text{splitting ratio.} \tag{11}$$

If we had only two sites as in the upper illustration of figure 1, then λ would not emerge.

One can estimate the transition probability from an initial adiabatic level E_n to some other adiabatic level E_m by writing the Hamiltonian in the adiabatic basis: if one changes a parameter X the coupling between the adiabatic levels is $\dot{X}[\mathrm{i}(\partial\mathcal{H}/\partial X)_{nm}/(E_n - E_m)]$. Then using leading order perturbation theory with respect to the driving rate $(1/t_\mathrm{p} \propto \dot{X})$, and assuming *smooth* temporal variation of the potentials, one typically obtains (see, e.g., equation (31))

$$P = \left| \int_{\text{cycle}} \frac{1}{t_\mathrm{p}} f\left(\frac{t}{t_\mathrm{p}}\right) \mathrm{e}^{\mathrm{i}\Phi(t)} \, \mathrm{d}t \right|^2 \sim \mathrm{e}^{-\Omega t_p}, \tag{12}$$

where $f()$ provides the time dependence of the coupling between the adiabatic levels, and the dynamical phase $\Phi(t)$ is obtained by integrating over $E_n(X(t)) - E_m(X(t))$. The parameter Ω characterizes the rate of change of the dynamics phase. In the absence of avoided crossings Ω is simply the mean-level spacing and we label the result as P_{FGR}. This notation implies that we deal with an off-resonant Fermi golden rule (FGR) transition. But if we have avoided the crossing of the E_0 and the E_+ levels, then the predominant contribution to the integral comes from a time interval $t_{\text{LZ}} = c/\dot{u}$, during which $\dot{\Phi} \sim c$. Thus for the so-called Landau–Zener transition we effectively have in equation (12) the replacements $\Omega \mapsto c$ and $t_\mathrm{p} \mapsto c/\dot{u}$, and we get $P_{\text{LZ}} \gg P_{\text{FGR}}$. The exact result including the correct prefactors is [29–31]

$$P_{\text{LZ}} = \exp\left[-2\pi \frac{c^2}{\dot{u}}\right], \tag{13}$$

which is known as the Landau–Zener transition probability. It is important to realize that the first-order calculation reproduces correctly the singular exponential dependence of P on the rate ($\dot{u}$) of the driving.

In this paper, we assume an adiabatic stirring process and analyze it within the framework of a two-level approximation. This means that two inequalities have to be satisfied,

$$P_{\text{FGR}} \ll P_{\text{LZ}} \ll 1. \tag{14}$$

From the above discussion it should be clear that for smooth temporal variation of the potential, the first inequality is automatically satisfied simply because $c \ll 1$ is much smaller compared with the mean-level spacing, and it can be further improved if we care for separation of time scales $t_\mathrm{p} \gg t_{\text{LZ}}$.

5. Single-path crossing

Consider a two-site system as in the upper illustration of figure 1. Initially the particle is prepared in the left site. Then after some time we measure the probability p to find the particle in the right side. In our terminology the probability p characterizes the *occupation statistics* at the end of the dynamical scenario. We assume nothing about the dynamical scenario, except of being coherent. This means that the dynamics is generated by a 2×2 Hamiltonian that can be possibly time dependent. We ask whether the *counting statistics* is related to p. Within a classical probabilistic point of view the answer is very simple: one would expect to measure $Q = 1$ with probability $\mathrm{P}(1) = p$, and $Q = 0$ with probability $\mathrm{P}(0) = 1 - p$. Accordingly one would expect to have $\langle \mathcal{Q}^k \rangle = p$, where $k = 0, 1, 2, \ldots$. In particular, the expectation value would be $\langle \mathcal{Q} \rangle = p$ and the variance would be $\text{Var}(\mathcal{Q}) = (1 - p)p$.

In the quantum-mechanical context this innocent reasoning is wrong. As discussed in [28] the eigenvalues of the operator $\mathcal{Q}$ are $Q_\pm = \pm\sqrt{p}$ with probabilities $p_\pm = \frac{1}{2}(1 \pm \sqrt{p})$. Accordingly the kth moment is

$$\langle \mathcal{Q}^k \rangle = p_+ Q_+^k + p_- Q_-^k = p^{\lfloor \frac{k+1}{2} \rfloor}, \tag{15}$$

where $k = 0, 1, 2, 3, \ldots$ and $\lfloor \cdots \rfloor$ stands for the integer part (i.e. rounded downwards). Still this result coincides with the corresponding classical result for $k = 1, 2$, which we call *restricted* quantum-classical correspondence. The purpose of the following sections is to explore what happens to the relation between *counting statistics* and *occupation statistics* once multiple path geometries are involved, and specifically to evaluate the first two moments for a quantum stirring process.

6. Double-path crossing

Consider a three-site system as in the lower illustration of figure 1. Initially the particle is prepared in the left site. Then we raise slowly the potential from $u < 1$ to $u > 1$. At the end of the process there is some probability p to find the particle in the right side. Our interest is in the counting statistics of the current that flows during this process through one bond, say the $0 \mapsto 1$ bond. Since we have a double path geometry we expect a splitting ratio that reflects the relative coupling strength $|c_1|^2/|c_2|^2$. Using a probabilistic point of view the splitting ratio should combine with the transition probability ($p \mapsto \lambda p$), leading to

$$\text{Var}(\mathcal{Q}) = (1 - \lambda p)\lambda p. \tag{16}$$

Thus for a complete transition ($p = 1$) with two equally probable paths ($\lambda = 1/2$) we expects to measure the maximal variance $\text{Var}(\mathcal{Q}) = 1/4$.

In the quantum-mechanical context this innocent reasoning is wrong. By inspection of equation (9) we see that the results for the double-path crossing can be obtained from the results for the single-path crossing using λ as scaling factor. Namely,

$$\langle \mathcal{Q} \rangle = \lambda p \tag{17}$$

$$\text{Var}(\mathcal{Q}) = \lambda^2(1 - p)p. \tag{18}$$

The splitting ratio reflects so-to-say the relative weight of c_1 in the net hopping amplitude $c_1 + c_2$. For $c_1 = c_2$ we have $\lambda = 1/2$, which would imply an *exact splitting* of the wavepacket into two equal pieces. In particular, we realize that for $p = 100\%$ transfer efficiency the variance in such a case would not be 1/4 but zero. The value 1/4 would correspond to a probabilistic (rather than exact) splitting of the wavepacket. We may say that the correct description of the transition from the left site to the right side is not 'the particle has an equal probability to go either via the first or via the second path', but rather 'the particle goes simultaneously via the two paths'. Both phrasings are equivalent if we have in mind expectation values, but only the latter phrasing has the correct connotation once counting statistics is considered.

Naively we expect that a fraction $0 < \lambda < 1$ would be transported via the $0 \mapsto 1$ bond, while the complementary fraction $0 < (1 - \lambda) < 1$ would be transported via the $0 \mapsto 2$ bond. But if c_1 and c_2 have opposite signs then (say) λ becomes larger than unity, while $(1 - \lambda)$ is negative. This reflects that the driving induces a *circulating current* within the ring, and illuminates the fallacy of the classical peristaltic point of view which we discuss below.

7. The peristaltic picture

Typically the driving is periodic, and Q is defined as the amount of particles that are transported per period. The feasibility to have a nonzero $\langle Q \rangle$ (nonzero 'DC' current) due to periodic ('AC') driving is known in the context of open geometry as 'quantum pumping' [3, 4]. We use the term 'quantum stirring' [10, 8] in order to describe the analogous effect with regard to a closed device. During an adiabatic pumping cycle a conventional two barrier quantum device takes an electron from the left lead and ejects it to the right lead. Hence the pumped charge per cycle for a leaky pump is naively expected to be $\langle Q \rangle < 1$. This naive result is indeed valid if the pump is operated in an open geometry between two reservoirs.

Inspired by the peristaltic picture we assume control over the on-site potential u and the coupling constants (c_1 and c_2) which are like 'valves'. In the first half of the cycle $c_2 = 0$ and u is raised across $u \sim 1$, so as to have an adiabatic passage from the left side to the right side via the $0 \mapsto 1$ bond. In the second half of the cycle $c_1 = 0$ and u is lowered so as to have an adiabatic passage from the right side back to the left side via the $0 \mapsto 2$ bond. The net effect is to pump one particle per cycle.

8. Quantum stirring

We can use the results that have been obtained for a double-path adiabatic passage in order to illuminate the fallacy of the peristaltic picture once quantum stirring in a closed geometry is considered. If during the first half of the cycle $\lambda = \lambda_\circlearrowleft$, and during the second half of the cycle $\lambda = \lambda_\circlearrowright$ then in leading approximation (neglecting non-adiabatic effects) we get

$$\langle Q \rangle = \lambda_\circlearrowleft - \lambda_\circlearrowright \quad \text{[per cycle]}. \tag{19}$$

An optional way to derive this result is to make a full three-level calculation using the Kubo formula [7]. Here we have bypassed the 'long derivation' by making a reduction to an 'adiabatic passage' problem.

It is correct that for a simple minded cycle, where either c_1 or c_2 are zero at each stage, we get an agreement with the peristaltic picture. But in general this is a fallacy. In fact the essence of quantum stirring is the circulating current which is induced by the driving. Contrary to the naive expectation we can get $Q \gg 1$ per cycle. This happens if c_1 and c_2 are roughly opposite in sign and hence $|\lambda| \gg 1$. This also can happen if c_1 and c_2 have the same sign: if we had considered an adiabatic passage at $u \sim -1$ from $|0\rangle$ to the antisymmetric state $|1\rangle - |2\rangle$ then it would be like replacing c_2 by $-c_2$. In general, it is better to say that $Q \gg 1$ reflects a huge circulating current which is induced if the pumping cycle encircles a degeneracy [7].

To avoid confusion it should be emphasized that we are not talking here about persistent currents which are zero-order conservative effect, but about 'linear response' which is a first-order effect that might have in general both geometric and dissipative aspects. It is important to remember that the amount of pumped charge per cycle is well defined in the $\dot{u} \to 0$ adiabatic limit, implying that it does not depend on the actual duration of the cycle.

9. Fluctuations

Having determined $\langle Q \rangle$ per cycle we would like to find out what is the variance $\text{Var}(Q)$. The straightforward calculation procedure is to write the current operator in the Heisenberg picture

$$\mathcal{I}(t)_{nm} = \langle n|U(t)^\dagger \mathcal{I} U(t)|m\rangle \tag{20}$$

and then to integrate it over time so as to get

$$\mathcal{Q}_{nm} \equiv \begin{pmatrix} +Q_\parallel & iQ_\perp \\ -iQ_\perp^* & -Q_\parallel \end{pmatrix}. \tag{21}$$

The first two moments $\langle \mathcal{Q} \rangle$ and $\langle \mathcal{Q}^2 \rangle$ are obtained from this matrix, leading to the identifications

$$\langle \mathcal{Q} \rangle = Q_\parallel \tag{22}$$

$$\text{Var}(\mathcal{Q}) = |Q_\perp|^2. \tag{23}$$

For a single-path Landau–Zener crossing in a two-site system it has been argued in [28] that the first two moments should be the same as in the classical calculation. This is the *restricted* quantum-classical correspondence that has been mentioned in section 5. Consequently one deduces that

$$Q_\parallel = 1 - P_{\text{LZ}} \tag{24}$$

$$Q_\perp = \sqrt{(1 - P_{\text{LZ}})P_{\text{LZ}}} \times \text{PhaseFactor}. \tag{25}$$

But we have a multiple path geometry, for which restricted quantum-classical correspondence cannot be established. Therefore we have to carry out the straightforward calculation, which is much more complicated. In practice, in order to make the calculation manageable, we can rely on the adiabatic approximation scheme. Within this framework the evolution operator is

$$U(t) \approx \sum_n |n(t)\rangle \exp\left[-i \int_{t_0}^t E_n(t')\,dt'\right] \langle n(t_0)| \tag{26}$$

and accordingly the time-dependent current operator is:

$$\mathcal{I}(t)_{nm} \approx \langle n(t)|\mathcal{I}|m(t)\rangle \times \exp\left[i \int_{t_0}^t E_{nm}(t')\,dt'\right]$$

$$\equiv \lambda \begin{pmatrix} \cdots & ic\,e^{i\Phi(t)} \\ -ic\,e^{-i\Phi(t)} & \cdots \end{pmatrix}. \tag{27}$$

If we use the zero-order adiabatic states (in $\dot{u}$) we get for the diagonal terms *zero*, because the zero-order adiabatic states are time-reversal symmetric and hence support zero current. If we use the first-order adiabatic states we get for the diagonal terms a nonzero result with $Q_\parallel = \lambda_\circlearrowleft - \lambda_\circlearrowright$. The details of this 'Kubo' calculation are not included here because this result is *a priori* expected on the basis of the much *simpler* analysis of the previous section.

As emphasized in the previous paragraph, in the case of multiple path geometry $Q_\perp$ is not related to $Q_\parallel$, and therefore an actual calculation should be carried out. The good news is that we get from equation (27) a nonzero leading order result already in the zero-order approximation,

$$Q_\perp = \int_{-\infty}^\infty \lambda c\,e^{i\Phi(t)}\,dt, \tag{28}$$

where

$$\Phi(t) = \int^t \sqrt{(u-1)^2 + (2c)^2}\,dt'. \tag{29}$$

For a single-path Landau–Zener transition in a two-site system one substitutes $\lambda = 1$ and $u = \dot{u}t$. Then it is possible to demonstrate, see [28], that the outcome of the integral is $\sim\sqrt{P_{\text{LZ}}}$ in agreement with equation (25). But we have a multiple path geometry for which the result

is not known *a priori*, so we have to stick to the integral and see what comes out. If we had only one Landau Zener crossing we would get $\sqrt{P_{LZ}}$ multiplied by the splitting ratio λ. A full pumping cycle is a sequence of two Landau Zener crossing, one at $t = t_1$ and the second at $t = t_2$. Therefore the integral is a sum of two terms, and we get

$$\text{Var}(\mathcal{Q}) = |Q_\perp|^2 = |\lambda_\circlearrowleft \, e^{i\varphi_1} + \lambda_\circlearrowright \, e^{i\varphi_2}|^2 P_{LZ}, \tag{30}$$

where $\varphi_1 \equiv \Phi(t_1)$ and $\varphi_2 \equiv \Phi(t_2)$. The result depends on the phase difference $\varphi = \varphi_2 - \varphi_1$, which is determined by the time separation of the two crossings.

For the sake of comparison one should realize that the probability p to have remnants of the particle in the right side is determined by the coherent addition of the two Landau–Zener transitions [32],

$$p = \left| \frac{1}{2} \int_{-\infty}^{\infty} \frac{\dot{u}/2c}{1 + (u/2c)^2} e^{i\Phi(t)} \, dt \right|^2 = |e^{i\varphi_1} - e^{i\varphi_2}|^2 P_{LZ}. \tag{31}$$

Here the interference is with opposite sign, because $\dot{u}$ in the integrand changes sign. Thus we see in a tangible way why due to interference $Q_\perp$ is in general not trivially related to $Q_\parallel$.

10. Long-time limit

From the above analysis it follows that $\langle \mathcal{Q} \rangle$ grows linearly with the number of cycles. An equivalent statement is that the eigenvalues $Q_\pm$ of the $\mathcal{Q}$ operator grow linearly with the number of cycles. It follows that for a general preparation also the spreading $\sqrt{\text{Var}(\mathcal{Q})}$ grows linearly with time. Needless to say that the probabilistic point of view would predict $\propto \sqrt{t}$ growth of the spreading, on the basis of the central limit theorem. Thus we have here again another manifestation of the way in which quantum coherent behavior differs from its classical stochastic analog. If we have good control over the preparation we can select the initial state to be a Floque eigenstate of the quantum evolution operator. For such preparation the linear growth of the spreading is avoided, and it oscillates around a residual value.

11. Fractional 'charge'

The derivation of equation (15) is based on the observation that the eigenvalues of the counting operator are $Q = \pm\sqrt{p}$. Exactly the same fractional value has surfaced in the pioneering publication about counting statistics [25], where the authors had interpreted it as an effective 'fractional charge'. Their observation was immersed in complicated diagrammatic calculations involving a many-body system of Fermions in an open geometry. In fact their result has been largely ignored once realized [26] that the naive definition of P(Q) is of no physical relevance, while $P_0(Q)$ gives no indication for 'quantized' fractional charges.

It is therefore amusing to realize that a similar idea might emerge in the present context. From the above analysis of coherent splitting it follows that for $p = 100\%$ transfer efficiency it is feasible to a 'fractional charge' $\langle \mathcal{Q} \rangle = \lambda$ with (formally) zero dispersion. In fact the measured fraction can be greater than unity which we can call 'mega charge' or it can be negative. The observation of a 'mega' charge in this context simply reflects the presence of an induced circulating current, which is the essence of the quantum stirring effect.

In any case it should be clear that the notion of 'fractional charge' in the context of coherent quantum stirring is possibly misleading, and we have raised it merely for argumentative purpose.

12. Summary

Counting statistics in closed geometries is not of classical nature. Even in the simplest problem of a coherent transition between two sites, the full counting statistics comes out different compared with the probabilistic expectation. Still, the variance comes out the same which can be regarded as an example of restricted quantum-classical correspondence. In contrast to that multiple path geometries require further reasoning, because there is no simple way to deduce the counting statistics.

One observes that the correct description of a quantum passage in a double path geometry is not 'the particle has an equal probability to go either via the first or via the second path', but rather 'the particle goes simultaneously via the two paths'. Both phrasings are equivalent if we have in mind expectation values, but only the latter has the correct connotation once counting statistics is considered, leading to equation (18) rather than equation (16). The coherent splitting of a quantum particle is 'exact' rather than probabilistic. Furthermore, in a double-path adiabatic passage one may find that (say) 170% of the particle goes via one path, while −70% goes via the second path. This reflects the emergence of a circulating current which is induced by the driving.

The analysis of adiabatic passage has opened the way to figure out what is the counting statistics in the quantum stirring problem. We argue that both the average and the spreading of $\mathcal{Q}$ grow linearly in time, where the rate of the former characterizes the pumping cycle, while the rate of latter depends on the quality of the preparation.

The result equation (30) that we have obtained for Var($\mathcal{Q}$) is exiting because it demonstrates how interference gets into the counting statistics calculation once multiple path geometries are concerned. Unlike in the calculation of the transition probability, the interference is with a different sign, and consequently the *counting statistics* becomes unrelated to the *occupation statistics*. This should be contrasted with the single-path crossing problem where non-trivial topology is not involved, and consequently the two types of statistics are *a priori* related.

We believe that the analysis of counting statistics in closed geometries that posses non-trivial topology not only opens an interesting direction in the study of quantum stirring, but also unmasks some essential physics of the counting statistics problem in general.

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