The conductance of a multi-mode ballistic ring: beyond Landauer and Kubo

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Abstract. – The Landauer conductance of a two terminal device equals to the number of open modes in the weak scattering limit. What is the corresponding result if we close the system into a ring? Is it still bounded by the number of open modes? Or is it unbounded as in the semi-classical (Drude) analysis? It turns out that the calculation of the mesoscopic conductance is similar to solving a percolation problem. The "percolation" is in energy space rather than in real space. The non-universal structures and the sparsity of the perturbation matrix cannot be ignored.

The theory for the conductance of *closed* mesoscopic rings has attracted a lot of interest [1–6]. In a typical experiment [7] a collection of mesoscopic rings are driven by a time dependent magnetic flux $\Phi(t)$ which creates an electro-motive-force (EMF) $-\dot{\Phi}$ in each ring. Assuming that Ohm's law applies, the induced current is $I = -G\dot{\Phi}$ and consequently Joule's law gives

$$\text{Rate of energy absorption} \quad = \quad G\,\dot{\Phi}^2 \tag{1}$$

where G is called the conductance. For diffusive rings the Kubo formula leads to the Drude formula for G. A major challenge in past studies was to calculate the weak localization corrections to the Drude result, taking into account the level statistics and the type of occupation [6]. It should be clear that these corrections do not challenge the leading order Kubo-Drude result.

It is just natural to ask what is the conductance if the mean free path ℓ increases, so that we have a ballistic ring as in Fig. 1, where the total transmission is $g_T \sim 1$. To be more precise, we assume that the mean free path $\ell \approx L/(1-g_T)$ is much larger than the perimeter L of the ring. In such circumstances "quantum chaos" considerations become important. Surprisingly this question has not been addressed so far [8], and it turns out that the answer requires considerations that go well beyond the traditional framework. Following [9] we argue that the calculation of the energy absorption in Eq.(1) is somewhat similar to solving a percolation problem. The "percolation" is in energy space rather than in real space. This idea was further elaborated in [10] using a resistor network analogy (Fig. 2). As in the standard derivation of the Kubo formula it is assumed that the leading mechanism for absorption is Fermi-golden-rule transitions. These are proportional to the squared matrix elements $|\mathcal{I}_{nm}|^2$ of the current operator. Still, the theory of [9] does not lead to the Kubo formula. This is because the rate of absorption depends crucially on the possibility to make *connected* sequences of transitions, and it is greatly reduced by the presence of bottlenecks. It is implied that both the structure of the $|\mathcal{I}_{nm}|^2$ band profile and its sparsity play a major role in the calculation of G.

The outline of this Letter is as follows: (a) We define a model example for which the analysis is carried out; (b) We make a distinction between the Landauer, the Drude and the actual mesoscopic conductance. (c) We calculate the matrix elements of the current operator; (d) We define an "averaging" procedure that allows the calculation of G. The result of the calculation is contrasted with that of the conventional Kubo approach.

We regard the ballistic ring (Fig.1) as a set of $\mathcal{M}$ open modes, and a small scattering region that is characterized by its total transmission g_T. To be specific we adopt a convenient network model where all the bonds $(a = 1, 2, ..., \mathcal{M})$ have similar length $L_a \sim L$. The scattering is described by

$$\boldsymbol{S} = \begin{pmatrix} \epsilon \exp\left(i\, 2\pi \frac{a\, b}{\mathcal{M}}\right) & \sqrt{1-\mathcal{M}\epsilon^2}\delta_{a,b} \\ \sqrt{1-\mathcal{M}\epsilon^2}\delta_{a,b} & -\epsilon \exp\left(-i\, 2\pi \frac{a\, b}{\mathcal{M}}\right) \end{pmatrix} \tag{2}$$

The transitions probability matrix $\boldsymbol{g}$ is obtained by squaring the absolute values of the $\boldsymbol{S}$ matrix elements. It is composed of a reflection matrix $[\boldsymbol{g}^R]_{a,b} = \epsilon^2$ and a transmission matrix $[\boldsymbol{g}^T]_{a,b} = (1 - \mathcal{M}\epsilon^2)\delta_{a,b}$. The total transmission is $g_T = 1 - \mathcal{M}\epsilon^2$. If the system were open as in Fig.1c. then its Landauer conductance would be

$$G_{\text{Landauer}} = \frac{e^2}{2\pi\hbar} \sum_{a,b} [\boldsymbol{g}^T]_{a,b} = \frac{e^2}{2\pi\hbar} \mathcal{M} g_T \tag{3}$$

If we had a closed ring and we could assume that there is no quantum interference within the bonds, then we could use the multimode conductance formula of Ref. [8]

$$G_{\text{Drude}} \;=\; \frac{e^2}{2\pi\hbar} \sum_{a,b} \left[2\boldsymbol{g}^T/(1-\boldsymbol{g}^T+\boldsymbol{g}^R)\right]_{a,b} \;=\; \frac{e^2}{2\pi\hbar} \mathcal{M} \frac{g_T}{1-g_T} \tag{4}$$

The first expression can be derived in various ways: Boltzmann picture formalism; semiclassical Kubo formalism; or quantum Kubo calculation that employs a diagonal approximation. In order to get the specific result for our network model we had to invert the matrix $(1-\boldsymbol{g}^T+\boldsymbol{g}^R)$. We see that in the limit $g_T \to 1$ the semiclassical G_{Drude} is unbounded, while G_{Landauer} is bounded by the number of open modes.

Our objective is to find the conductance of the closed ring in circumstances such that the motion inside the ring is essentially coherent (quantum interference within the bonds is not ignored): As in the traditional linear response theory (LRT) it is assumed that the level broadening Γ is larger compared with the mean level spacing, but otherwise very small semiclassically. On the other hand, in contrast to LRT, we assume "mesoscopic circumstance", meaning that the environmentally-induced relaxation is very slow compared with the EMF-induced rate of transitions. An extensive discussion of these conditions can be found in [9]. The calculation of G is done using the formula

$$G \;=\; \pi\hbar\, \varrho_F^2 \times \langle\langle |\mathcal{I}_{nm}|^2 \rangle\rangle \;=\; \frac{e^2}{2\pi\hbar} \times 2\mathcal{M}^2 \langle\langle |I_{nm}|^2 \rangle\rangle \;\equiv\; \frac{e^2}{2\pi\hbar} \times 2\mathcal{M}^2 \mathrm{g} \tag{5}$$

where ϱ_F is the density of states at the Fermi energy, and $\mathcal{I}_{nm}$ are the matrix elements of the current operator. For our network system $\varrho_F = \mathcal{M}L/(\pi\hbar v_F)$, where v_F is the Fermi velocity. Furthermore it is convenient to write $\mathcal{I}_{nm} = -i(ev_F/L)I_{nm}$ so as to deal with real dimensionless quantities, leading to the second expression. Eq.(5) would be the Kubo formula if $\langle\langle ... \rangle\rangle$ stood for a simple algebraic average. But in view of the percolation-like nature of the energy absorption process, the definition of $\langle\langle ... \rangle\rangle$ involves a more complicated "averaging" procedure that will be discussed and developed later.

The eigenstates: Our model system, in the absence of driving, is time reversal symmetric. Consequently the unperturbed eigenfunctions can be chosen as real

$$|\psi\rangle = \sum_a A_a \sin(kx + \varphi_a) \otimes |a\rangle. \tag{6}$$

For a given g_T we can find numerically the eigenvalues and the eigenstates, thus obtaining a table $(k_n, \varphi_a^{(n)}, A_a^{(n)})$ with $n =$ level index. In the limit of small ϵ it is not difficult to derive the expressions

$$k_n \;\approx\; \left(2\pi \times \text{integer} \pm \frac{1}{\sqrt{\mathcal{M}}}\,\epsilon\right)\frac{1}{L_a} \tag{7}$$

$$\varphi_a^{(n)} \;\approx\; -\frac{\pi a^2}{\mathcal{M}} - \frac{1}{2}kL_a + \begin{cases} \pi/4 \\ 3\pi/4 \end{cases} \tag{8}$$

The numerical results over the whole range of g_T values are presented in Fig. 3. By normalization we have $\sum_a (L_a/2)A_a^2 \approx 1$. The degree of ergodicity of a wavefunctions is characterized by the participation ratio:

$$\text{PR} \equiv \left[\sum_a \left(\frac{L_a}{2}A_a^2\right)^2\right]^{-1} \approx 1 + \frac{1}{3}(1-g_T)\mathcal{M} \tag{9}$$

The approximation in the last equality is based on the following observations: By definition we have $\text{PR} \approx 1$ for a wavefunction which is localized on one bond, while $\text{PR} \sim \mathcal{M}$ for an ergodic wavefunction. In the trivial regime $(1-g_T) \ll 1/\mathcal{M}$ the eigenstates are like those of a reflectionless ring, with $\text{PR} \sim 1$. Once $(1-g_T)$ becomes larger compared with $1/\mathcal{M}$, first order perturbation theory breaks down, and the mixing of the levels is described by a Wigner Lorentzian. The analysis is completely analogous to that of the single mode case in Ref. [9], and leads to $\text{PR} \propto (1-g_T)\mathcal{M}$. This is confirmed by the numerical analysis (Fig. 4). In practice we have found that the proportionality constant is roughly 1/3. Our interest is focused in the *non-trivial* ballistic regime

$$1/\mathcal{M} \;\ll\; (1-g_T) \;\ll\; 1 \tag{10}$$

where we have strong mixing of levels ($\text{PR} \gg 1$), but still the mean free path $\ell \approx L/(1-g_T)$ is very large compared with the ring's perimeter ($\ell \gg L$). It is important to realize that in this regime we do not have "quantum chaos" ergodicity. Rather we have $\text{PR} \ll \mathcal{M}$ meaning that the wavefunctions occupy only a small fraction of the classically accessible phase space.

The matrix elements: The current operator $\mathcal{I}$ is the symmetrized version of $e\hat{v}\delta(\hat{x}-x_0)$, where $\hat{v}$ and $\hat{x}$ are the velocity and the position operators respectively. The section through which the current is measured is arbitrary and we simply take $x_0 = +0$. Given a set of eigenstates, it is straightforward to calculate the matrix elements of the current operator (Fig. 5), and to get insight into their statistical properties (e.g. Fig. 6). The scaled matrix elements are

$$I_{nm} \approx \sum_a \frac{L_a}{2} A_a^{(n)} A_a^{(m)} \sin(\varphi_a^{(n)} - \varphi_a^{(m)}) \tag{11}$$

Needless to say that small PR of wavefunctions implies sparsity of I_{nm}. It is also worthwhile to point out that there are several extreme cases that allow simple estimates: The case where

n and m are localized on different bonds leading to $|I_{nm}|^2 = 0$; The case where n and m are nearly degenerate states localized on the same wire leading to $|I_{nm}|^2 = 1$; The case where n and m are ergodic and uncorrelated leading to $|I_{nm}|^2 \approx 1/(2\mathcal{M})$; Irrespective of this, it is clear that by normalization the maximal value that can be obtained is $|I_{nm}|^2 = 1$.

Landauer? Drude? From Eq.(5) and the above discussion we deduce that

$$G\Big|_{\rm ergodic} = \frac{e^2}{2\pi\hbar}\mathcal{M} \tag{12}$$

$$G\Big|_{\rm maximal} = \frac{e^2}{2\pi\hbar}2\mathcal{M}^2 \tag{13}$$

The first expression suggests agreement with the Landauer result if we had complete "quantum chaos" ergodicity, while Eq.(13) implies a necessary condition for a correspondence with the semiclassical result Eq.(4):

$$\frac{1}{1-g_T} \ll \mathcal{M} \tag{14}$$

This can be rephrased by saying that the ballistic time $t_{cl} = (1-g_T)^{-1} \times (L/v_F)$ should be much smaller compared with the Heisenberg time $t_H = \mathcal{M} \times (L/v_F)$. In fact it has been argued [9], on the basis of a diagonal approximation, that semiclassical correspondence is indeed realized in the 'Kubo calculation". By "Kubo calculation" we mean Eq.(5) with algebraic average over the near diagonal matrix elements of $|I_{nm}|^2$. The Kubo calculation might have a physical validity in the presence of a strong relaxation process that suppresses the quantum nature of the dynamics. See [9] for a detailed discussion of this point.

The FGR picture: The Hamiltonian in the adiabatic basis is $\mathcal{H} \mapsto E_n\delta_{nm} + \dot{\Phi}W_{nm}$ where $W_{nm} = i\hbar\mathcal{I}_{nm}/(E_n - E_m)$, and $-\dot{\Phi}$ is the EMF. The FGR transition rate between level n and level m is proportional to $|W_{nm}|^2$ multiplied by a broadened $\delta(E_n - E_m)$ which we call $F()$. The effective broadening of the levels reflects either the power spectrum or the non-adiabaticity of the driving. After trivial scaling the dimensionless transition rates are

$$\mathsf{g}_{nm} = \frac{|I_{nm}|^2}{(n-m)^2}\,\frac{1}{\gamma}F\left(\frac{n-m}{\gamma}\right) \tag{15}$$

The dimensionless broadening parameter γ is identical with Γ/Δ of Ref. [9, 11] and with $\hbar\omega_0/\Delta$ of Ref. [10], where Δ is the mean level spacing. There is an implicit approximation in Eq.(15), namely $(E_n - E_m)/\Delta \approx (n-m)$, that underestimates the exceptionally large couplings between pairs of almost degenerated levels. But this is not going to be reflected in the energy absorption rate, since the latter is indifferent(!) to large sparse values.

The calculation of the conductance: Given the transition rates g_{nm} we want to calculate the rate of energy absorption and hence G as defined by Eq.(1). It is most convenient to exploit the "resistor network" analogy of Ref. [10](Fig. 2). Within this framework g of Eq.(5) is simply the *resistivity* of the network. The practical numerical procedure is as follows: **(i)** Cut an N site segment out of the network. **(ii)** Define a vector $\boldsymbol{J}_n (n = 1..N)$ whose elements are all zero except the first and the last that equal $\boldsymbol{J}_1 = +J$ and $\boldsymbol{J}_N = -J$. **(iii)** Solve the Kirchhoff equation $\boldsymbol{J}_n = \sum_m \mathsf{g}_{nm}(\boldsymbol{V}_n - \boldsymbol{V}_m)$ for the vector $\boldsymbol{V}$. **(iv)** Find the overall resistance of the truncated network $\mathsf{g}_N = J/(\boldsymbol{V}_N - \boldsymbol{V}_1)$. And finally: **(v)** Define the resistivity as $\mathsf{g}^{-1} = \mathsf{g}_N^{-1}/N$. For a locally homogeneous network it has been argued in [9] that $\mathsf{g} \approx \langle\langle (1/2)\sum_m (m-n)^2 \mathsf{g}_{nm}\rangle\rangle$, where the sum over m reflects the addition of resistors in parallel, and the harmonic average $\langle\langle ...\rangle\rangle$ reflects the addition of resistors in series. This

expression can be written simply as $\mathsf{g} = \langle\langle |I_{nm}|^2 \rangle\rangle$, with the implicit understanding that the harmonic average is taken over the near diagonal elements of the γ-smoothed $|I_{nm}|^2$ matrix.

Numerical results: The results of the calculation are presented in Figs. 7-8. The calculation has been done numerically using the resistor network procedure that has been explained in the previous paragraph. We did not to rely on the harmonic average approximation because there are prominent structures (notably the strongly coupled nearly degenerate levels) that make it a-priori questionable. However from the numerics it turns out (not displayed) that the harmonic average is doing quite well. We mention this fact because it gives an insight for the numerical results which are displayed in Fig. 7.

Our numerical results suggest that *typically* $G < G_{\text{Landauer}}$. For an optimal value of γ, such that G is maximal, it seems that we still have $G \lesssim G_{\text{Landauer}}$. It is too difficult to figure out the numerical prefactor which is involved in the latter inequality (Fig. 8). Our conjecture is that this inequality is true *in general* (disregarding the prefactor which is of order 1). We did not find a mathematical argument to establish this conjecture, except the very simple case of a single mode ballistic ring [9] where the calculations of G can be done analytically.

Conclusions: In this Letter we have studied the mesoscopic conductance of a ballistic ring with mean free path $\ell \gg L$. The specific calculation has been done for a network model, but all its main ingredients are completely *generic*. Ballistic rings with $\ell \gg L$ are not typical "quantum chaos" systems. Their eigenfunctions are not ergodic over the whole accessible phase space, and therefore the perturbation matrix $\mathcal{I}_{nm}$ is highly structured and sparse. Consequently the Kubo formula is no longer valid, and one has to adopt an appropriate "resistor network" procedure in order to calculate the true mesoscopic conductance. *However*, it should be emphasized that if there is either a very effective relaxation or decoherence process, then the theory that we have discussed does not apply. In the presence of strong environmental influence one can justify, depending on the *circumstances* [8,9], either the use of the traditional Kubo-Drude result, or the use of the Landauer result.

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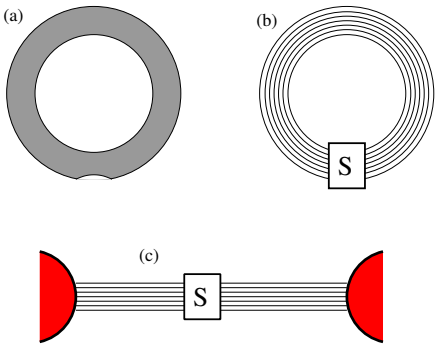

Fig.1: **(a)** A billiard example for a ballistic ring. The annular region supports $\mathcal{M}$ open modes. The electrons are scattered by a small bump. **(b)** A network model of a ballistic ring. In the numerics the lengths of the $\mathcal{M}$ bonds ($0.9 < L_a < 1.1$) are chosen in random. The scattering is described by an S matrix. **(c)** The associated open (leads) geometry which is used in order to define the S matrix and the Landauer conductance.

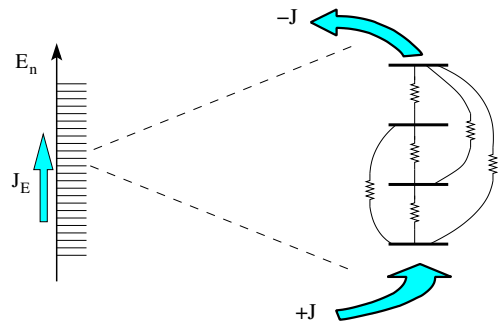

Fig.2: The EMF induces diffusion of probability in energy space, and hence absorption of energy. Within the framework of the Fermi golden rule picture the flow of the probability in the multi level system is analogous to the flow of a current via a resistor network. The resistance of each "resistors" corresponds to g_{nm}^{-1}. The inverse of the diffusion coefficient is re-interpreted as the resistivity of the network. On the right we display a truncated segment, where $+J$ is the current injected from one end of the network, while $-J$ is the same current extracted from the other end. The injected current to all the other nodes is zero.

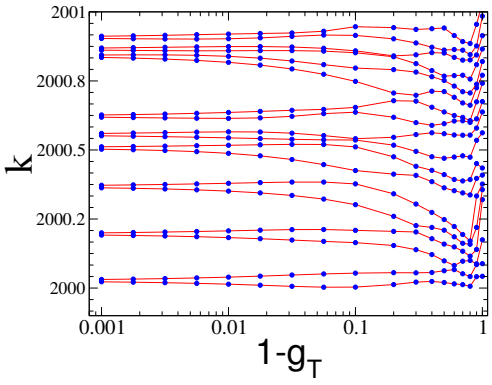

Fig.3: The eigenvalues k_n within a small energy window around $k \sim 2000$ are shown as a function of $1-g_T$. We consider here a network model with $\mathcal{M} = 50$ bonds. The length of each bond was chosen in random within $0.9 < L_a < 1.1$.

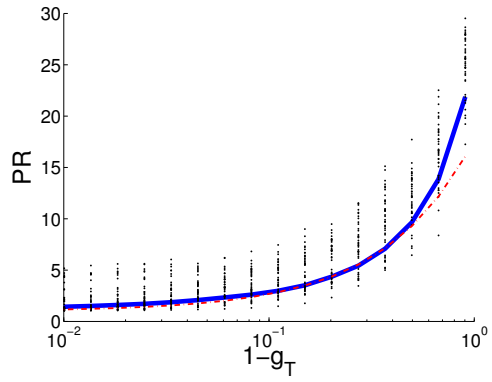

Fig.4: For each value of g_T we calculate the participation ratio (PR) for all the eigenstates. We display (as symbols) the minimum value, the maximum value, and a set of randomly chosen representative values. The solid line is the average PR, while the dotted line is the formula $\mathrm{PR} \approx 1 + \frac{1}{3}(1-g_T)\mathcal{M}$.

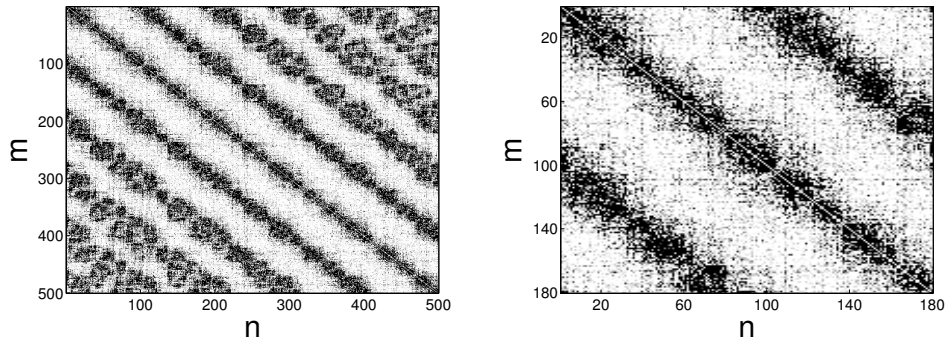

Fig.5: The image of the perturbation matrix $|I_{nm}|^2$ for $g_T = 0.9$. The right panel is a zoomed image. If we chose larger $1 - g_T$ more elements would become non-negligible, and the matrix would become less structured and less sparse.

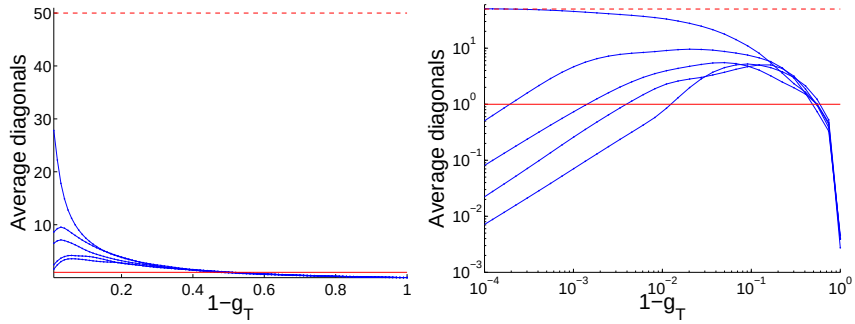

Fig.6: The n-averaged value of $2\mathcal{M}^2|I_{n,n+r}|^2$ as a function of $1 - g_T$ for $r = 1, 2, 3, 4, 5$. The ergodic value $\mathcal{M}$ and half the maximal value $\mathcal{M}^2$ are indicated by horizontal dotted lines. The left panel is normal scale, while the right panel is log-log scale.

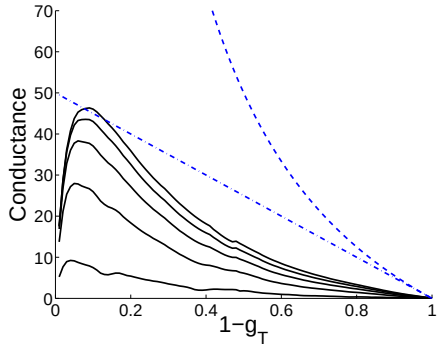

Fig.7: The mesoscopic conductance G in units of $e^2/(2\pi\hbar)$ as a function of $1-g_T$ for $\gamma = 1, 2, 3, 4, 5$. Note again that the total number of open modes in our numerics is $\mathcal{M} = 50$. The dotted line is G_{Landauer} while the dashed line is G_{Drude}.

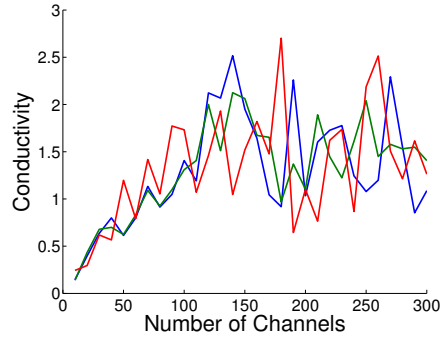

Fig.8: The mesosocopic conductance divided by the number of open modes for $\mathcal{M}$ up to 300. Here $\gamma = 3$ and $g_T = 0.8$. The different curves are calculated with segments of length $N = 30, 50, 70$, so as to provide an estimate for the numerical error.